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An Integrated Approach for Multi-Level Sample Size Determination*

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An Integrated Approach for Multi-Level Sample Size Determination

§1. Introduction

Inspection procedures involving the sampling of items in a population (to determine their quality) often require steps of increasingly sensitive measurements, with correspondingly smaller sample sizes; these are referred to as "multilevel" sampling schemes. In the case of nuclear safeguards inspections verifying that there has been no diversion of Special Nuclear Material (SNM), these procedures have been examined often (see [1], [2] and [3]) and increasingly complex algorithms have been developed to implement them [4]. While the underlying notions are implicit in these papers, the general expositions are not clearly coordinated and integrated.

Our aim in this paper is to provide an integrated approach, and, in so doing, to describe a systematic, consistent method that proceeds logically from level to level with increasing accuracy. We emphasize that the methods discussed are generally consistent with those presented in the above references, and yield comparable results when the error models are the same. However, because of its systematic, integrated approach our proposed method elucidates the conceptual understanding of what goes on, and, in many cases, simplifies the calculations.

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In nuclear safeguards inspections, an important aspect of verifying nuclear items to detect any possible diversion of nuclear fissile materials is the sampling of such items at various levels of sensitivity. The first step usually is sampling by "attributes" involving measurements of relatively low accuracy, followed by further levels of sampling involving greater accuracy. This process is discussed in some detail in the references given above; also, the nomenclature is described.

Here, we outline a coordinated step-by-step procedure for achieving such multilevel sampling, and we develop the relationships between the accuracy of measurement and the sample size required at each stage, i.e., at the various levels. The logic of the underlying procedures is carefully elucidated; the calculations involved and their implications, are clearly described, and the process is put in a form that allows systematic generalization.

The procedures and logic of the development are summarized in §3, and described briefly in §5.

§2. Basic Equations

2.1 Notations

The notation is as follows:

- x: amount of nuclear material in an item, in kg.
- N: total number of items in the population.
- G: detection goal quantity, in kg, of (special) nuclear material.

γ_i : defect fraction at the i -th level, i.e., the fractional amount of material diverted.
 $i = 1$ is considered as the "gross" defect, when an entire item has been diverted;
 $\gamma_1 = 1$.

m_i : number of defected items at the i -th level when the total defect is G ,
 $m_i = G/\gamma_i x$.

σ_i : relative standard deviation of the measurement to detect the i -th level defect. In general, it is the standard deviation of an operator-inspector difference, but, to simplify the following discussions, the operator's measurement error is considered negligible.

n_i : number of items sampled for the i -th level measurement.

2.2 Basic Calculation

Suppose that n items are sampled for measurement. The probability of not including any defect in the sample, i.e. the non-sampling probability is

$$\beta_s = \prod_{j=1}^m \left(1 - \frac{n}{N-j+1} \right), \quad (1)$$

where the subscripts i for n_i and m_i have been dropped temporarily.

We note that, by symmetry (see sub-section 4.1, equations (19) and (20)), n and m can be exchanged. In addition,

$$\beta_s \leq \left(1 - \frac{n}{N} \right)^m \quad (2)$$

Thus, setting

$$\frac{n}{N} = 1 - \beta_G^{\frac{1}{m}}, \quad (3)$$

where $1 - \beta_G$ is a "detection goal" (probability of detection of a defect), then

$$\beta_s \leq \beta_G.$$

In other words, the probability of including at least one defect in the sample is greater than the stipulated detection goal, $1 - \beta_G$.

A better approximation to expression (3) can be used¹, viz

$$\beta_s \leq \left(1 - \frac{n}{N - \frac{m-1}{2}} \right)^m \quad (4)$$

instead of equation (2), in which case the sample size is reduced to

$$\frac{n}{N} = \left(1 - \beta_G^{\frac{1}{m}} \right) \left(1 - \frac{m-1}{2N} \right). \quad (5)$$

Again, this guarantees the same probability of including at least one defect in the sample.

To determine the overall detection capability, the capability of the inspector's measurement also must be considered. A defect is "detected" when the measured nuclear amount of an item differs from (for simplification, we will use "is smaller than") the declared value by a "rejection limit." To control the role of false alarms, this rejection limit is usually set at some multiple, r , of the standard deviation σ of the

¹ See, for example, Reference [3]

measurement used. In other words, when an item with a declared nuclear content of x is measured by an inspector, then, if the measured amount is less than $(1-r\sigma)x$, the item is classified as a defect, subject to further examination.

If $F(j)$ is the probability that, given j defects in the sample, none are detected by the inspector's measurement, then the overall non-detection probability β satisfies

$$\beta = \sum_{j=0}^n \frac{\binom{m}{j} \binom{N-m}{n-j}}{\binom{N}{n}} F(j) \quad (6)$$

This is the basic relation which underlies our further considerations, and shows how both the combinatorial and measurement characteristics play a role in the detection process.

When a fraction γ of the material of an item has been diverted, the probability that the item is classified as a defect when it is measured with an instrument with standard deviation σ is

$$\Phi\left(\frac{\gamma - r\sigma}{(1-\gamma)\sigma}\right) \quad (7)$$

where $\Phi(z)$ denotes the value of the cumulative normal distribution² at z ,

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^z e^{-\frac{u^2}{2}} du .$$

² Because the material quantities are all finite, the cumulative distribution used should be truncated at both ends. However, from the conceptual point of view, it is more convenient to use the normal distribution; from the practical viewpoint, it makes very little difference.

Therefore, the probability that a sampled defective item will not be so classified is therefore

$$1 - \Phi\left(\frac{\gamma - r\sigma}{(1-\gamma)\sigma}\right) = \Phi\left(\frac{r\sigma - \gamma}{(1-\gamma)\sigma}\right).$$

Combining both sampling and measurement probabilities, the overall non-detection probability for a defect of m items is now

$$\begin{aligned} \beta &= \sum_{j=0}^n \frac{\binom{m}{j} \binom{N-m}{n-j}}{\binom{N}{n}} F(j) = \beta_s + \sum_{j=1}^n \frac{\binom{m}{j} \binom{N-m}{n-j}}{\binom{N}{n}} F(j) \\ &\leq \beta_s + F(1) \sum_{j=1}^n \frac{\binom{m}{j} \binom{N-m}{n-j}}{\binom{N}{n}} \\ &= \beta_s + (1 - \beta_s) F(1) \end{aligned} \quad (8)$$

since it is clear that $F(1) \geq F(j)$ for $j \geq 1$. If there are several (more than one) defected items in the sample, the measurement non-detection probability will clearly be less than or equal to

$$F(1) = \Phi\left(\frac{r\sigma - \gamma}{(1-\gamma)\sigma}\right),$$

the non-detection probability for a single item.

The overall detection probability then satisfies

$$1 - \beta \geq (1 - \beta_s)(1 - F(1)) = (1 - \beta_s)\Phi\left(\frac{\gamma - r\sigma}{(1 - \gamma)\sigma}\right) \quad (9)$$

If the expression (3) or (5) is used for the sample size n , the detection probability satisfies

$$1 - \beta \geq (1 - \beta_G)\Phi\left(\frac{\gamma - r\sigma}{(1 - \gamma)\sigma}\right) \quad (10)$$

§3. Multilevel Measurements

If the detector can detect the defect, so that

$$\Phi\left(\frac{\gamma - r\sigma}{(1 - \gamma)\sigma}\right)$$

is very close to 1, the stipulated detection goal $(1 - \beta_G)$ can be achieved (from expression (10)). As the defect fraction of each item decreases, the number of items that must be defected must increase to obtain the goal quantity, G . Thus, the probability of including a defect in the sample increases, but this may not be enough to compensate for the decrease in the instrument's detection probability

$$\Phi\left(\frac{\gamma - r\sigma}{(1 - \gamma)\sigma}\right).$$

However, as long as this quantity is not smaller than the detection goal $(1 - \beta_G)$, the sample sizes can be increased so that

$$(1 - \beta_s)\Phi\left(\frac{\gamma - r\sigma}{(1 - \gamma)\sigma}\right) \geq 1 - \beta_G,$$

and the detection goal can be achieved.

The detection goal also can be achieved by using a value of β_S satisfying

$$\beta_s \leq 1 - \frac{1 - \beta_G}{\Phi\left(\frac{\gamma - r\sigma}{(1-\gamma)\sigma}\right)} \quad (11)$$

in the sample size formulae, (3) or (5), in place of β_G . Although such a strategy is not currently used by the IAEA, it would be useful if more accurate methods of measurement do not exist or are very difficult to apply. Optimization of resources, including more intensive sampling and using a more accurate (and hence, more costly) method of measurement need to be considered to achieve greater cost-effectiveness (discussed next).

Instead of increasing the sample sizes, a more accurate method of measurement could be used to mitigate the problem. Considering β as a function of γ , and assuming that equation (3) or (5) was used to determine the first-level sample size with non-detection probability of β_1 (currently, the IAEA sets $\beta_G = \beta_1$)³, the non-detection probability defective items, when the defect fraction is γ , and the total defect is G kg., satisfies (from Eq. (9))

$$\beta(1, \gamma) \leq 1 - \left(1 - \beta_1^{\frac{1}{x}}\right) \Phi\left(\frac{\gamma - r\sigma_1}{(1-\gamma)\sigma_1}\right)$$

(The subscript 1 for σ and β has been added to indicate that they are related to the first-

³ Satisfying the relation (11), and $m=G/x$.

level measurement. To simplify our later discussion

$$\Phi\left(\frac{\gamma - r\sigma_1}{(1-\gamma)\sigma_1}\right)$$

will be denoted by Φ_1 in the remainder of this paper.)

Additional items can now be sampled and measured with more accurate instruments having a smaller standard deviation $\sigma_2 \leq \sigma_1$ of measurement. The number of items to be measured with this more accurate instrument then can be calculated. Using an analogous notation as before, with the new subscript 2, (see, Eq. (8)).

$$\beta(2, \gamma) \leq \beta_2 + (1 - \beta_2) F_2(1) = F_2(1) + (1 - F_2(1))\beta_2$$

where $F_2(1) = 1 - \Phi_2$ is the probability that a defective item will not be identified by a measurement using a detector with a standard deviation of σ_2 , and β_2 is the probability of not including a defect in the second set of samples with size n_2 , (from Eq. (2)).

$$\beta_2 \leq \left(1 - \frac{n_2}{N}\right)^{\frac{G}{K}}$$

Assuming independence, to achieve the overall detection probability of $1 - \beta_G$, the second-level measurement must satisfy

$$\beta_t = \beta(1, \gamma) \beta(2, \gamma)^2 \beta_G$$

Using formulae (2) and (8), the requirement on n_2 now becomes

$$\left(\left(1 - \frac{n_2}{N}\right)^{\frac{G}{K}} \Phi_2 + (1 - \Phi_2) \right) \left(1 - \left(1 - \beta_1^{\frac{1}{r}}\right) \Phi_1 \right) \leq \beta_G$$

or,

$$\frac{n_2}{N} \geq 1 - \left(\left(\frac{\beta_G}{1 - (1 - \beta_1^{\frac{1}{\gamma}})\Phi_1} - (1 - \Phi_2) \right) / \Phi_2 \right)^{\frac{x}{G}} \quad (12)$$

For any given defect fraction, γ , the lower bound of the sample sizes required is given by this formula. For the inspection strategy to cover all diversion strategies, the sample size used must be the maximum (over all possible defect fractions) of the lower bound of these calculated, sizes.

To simplify the following discussions, we assume that a defect will not be missed by the second method of measurement, so that $F_2(1)$ is negligible. Then

$$\frac{n_2}{N} \geq 1 - \left(\frac{\beta_G}{1 - (1 - \beta_1^{\frac{1}{\gamma}})\Phi_1} \right)^{\frac{x}{G}} \quad (13)$$

Considered as a function of γ , the sample size n_2 may be determined by calculating the value of γ at which n_2 is maximized. An analytical expression of the critical γ can be obtained, but it involves quite complicated expressions and cannot be readily solved manually. Using a PC which has an approximation for the cumulative normal distribution function, the sample size can be computed readily. A few examples of the level 2 sample sizes are given below for two values of G/x . For simplicity β_1 is taken equal to β_G .

$G/x=1$	$1-\beta$	σ	γ	n/N
	0.2	0.125	0.245	0.04114
	0.2	0.0625	0.1	0.01668
	0.2	0.01	0.015	0.002279
	0.5	0.125	0.275	0.14288
	0.5	0.0625	0.125	0.06746
	0.5	0.01	0.02	0.01036
	0.9	0.125	0.33	0.47515
	0.9	0.0625	0.17	0.26901
	0.9	0.01	0.025	0.04786
$G/x=10$	$1-\beta$	σ	γ	n/N
	0.2	0.125	0.245	0.004193
	0.2	0.0625	0.1	0.00168
	0.2	0.01	0.015	0.000228
	0.5	0.125	0.275	0.01530
	0.5	0.0625	0.125	0.006961
	0.5	0.01	0.02	0.00105
	0.9	0.125	0.33	0.0624
	0.9	0.0625	0.17	0.03085
	0.9	0.01	0.025	0.004892

We note that n/N varies with σ and x/G more strongly than linearly. The defect fraction at the needed sample size can be conservatively estimated by 2σ for 50% (or below) detection probability, and 2.25σ for 90% detection probability. With the estimated defect fraction, the corresponding sample size needed can be estimated conservatively from equation (3). Such a rule of thumb is useful for getting a first-order estimate of the resources required for the sampling inspection. Using the analytical expression derived above reduces such requirements.

To show the effects of the accuracy of measurement on the sample sizes, we plotted the sampling fraction needed as a function of defect fraction (Figure 1). Four curves are shown, corresponding to measurement standard deviations of 12.5%, 10%, 8%, and 6.25%; $G/x=16$ and $\beta=0.1$ were used. Figure 2 shows the effect of the detection probability on the requirements for the sample size. $G/x=10$ and $\sigma_1=6.25\%$ were used.

If $F_2(1)$ is not negligible, Figure 3 shows the lower bound of the sample sizes as a function the defect fraction when the second-level measurements' standard deviations are 6.25%, 3%, 1%, and 0.01%. The first-level measurement's standard deviation is assumed to be 12.5%, the detection probability is 90%, and $G/x=16$. It is interesting to note that when the combination of the uncertainty in the second-level measurement and the defect fraction satisfies the relation

$$\Phi_2 \approx 1 - \frac{\beta G}{1 - \left(1 - \beta \frac{1}{G}\right) \Phi_1} = 1 - \beta_c$$

the sample size diverges very rapidly. In fact, it diverges so rapidly that a reasonable strategy is to use the local maximum determined (in Fig. 3, near $\gamma=0.34$, $n/N=0.04$); thus, diversions with a diversion fraction

$$\gamma < \gamma'$$

where γ' is the limiting value defined by the above relation, will need to be dealt with by other inspection techniques, e.g., a variable test. Using sample sizes determined

assuming a perfect second-level measurement ($F_2 = 0$) is slightly non-conservative, as seen from Fig. 4.

If expression (4) is used to approximate the sampling probability instead of expression (2), the required sample size is reduced to

$$\frac{n_2}{N} \geq \left(1 - \left(\frac{\beta_G}{1 - (1 - \beta_G)^{\frac{1}{\gamma}}} \Phi_1 \right)^{\frac{1}{G}} \right) \left(1 - \frac{G}{2\gamma x N} + \frac{1}{2N} \right) \quad (14)$$

The reduction factor becomes important when the number of items making a goal quantity approaches the total population (i.e., for small fractional diversions). Fig. 4 shows the effect of this reduction factor, f , for $G/x=10$, $1-\beta = 90\%$, $N = 100$ and 1000 . The curves representing n_1 assume a perfect second-level measurement, while those representing n_2 assume $\sigma_1 = 12.5\%$ and $\sigma_2 = 6.25\%$. For $N=100$, the sample fraction (at local maximum) is reduced from 6.25% to 5.35%.

The procedure may then be summarized as follows:

1: Calculate the total sample size from

$$\frac{n}{N} = 1 - \beta^{\frac{1}{G}}$$

where $\gamma=1$ is used (i.e. the standard approach).

2: For any given σ_1 and σ_2 , calculate the local maximum of Q as a function of γ , (locate the local peak as shown in Fig. 3):

$$Q = \left(1 - \left(\left(\frac{\beta_G}{1 - (1 - \beta_G^{\frac{1}{G}})\Phi_1} - (1 - \Phi_2) \right) / \Phi_2 \right)^{\frac{K}{G}} \right) \left(1 - \frac{G}{2\gamma \times N} + \frac{1}{2N} \right)$$

This can be accomplished with a calculator or a PC which contains an approximation to the cumulative normal distribution function. When a local maximum does not exist, set

$$\Phi_2 = 1$$

in the expression above, and then find the local maximum.

3: The level-2 sample size is $n_2 = NQ$ while the level-1 sample size is $n - n_2$.

4: Caution must be exercised in situations when small defects are not detected. For example, as shown in Fig. 3, defects with a defect fraction less than 11% will not be detected when measurements with standard deviations of $\sigma_1 = 12.5\%$ and $\sigma_2 = 3\%$ are used. A more accurate instrument, if available, should be used to detect such defects.

5: The level 3 sample size can be determined using an analogous formalism, with one more step of calculation.

§4. Random and Systematic Errors

4.1 Background

It is important to distinguish between situations when errors are random versus those when they are systematic. Consider the problem of sampling a defected item, and then detecting it (with probability Φ). When there are N items *in toto*, m of which have defects, and assuming random errors, the non-detection probability β is

$$\beta = \sum_{j=0}^n \frac{\binom{m}{j} \binom{N-m}{n-j}}{\binom{N}{n}} (1-\Phi)^j . \quad (15)$$

By the properties of the combination symbols, this is also equal to

$$\beta = \sum_{j=0}^n \frac{\binom{n}{j} \binom{N-n}{m-j}}{\binom{N}{m}} (1-\Phi)^j . \quad (16)$$

Since

$$\binom{n}{k} = 0 \text{ for } k > n$$

the sum becomes

$$\beta = \sum_{j=0}^{\min(n,m)} \frac{\binom{n}{j} \binom{N-n}{m-j}}{\binom{N}{m}} (1-\Phi)^j , \quad (17)$$

or

$$\beta = \sum_{j=0}^{\min(n,m)} \frac{\binom{m}{j} \binom{N-m}{n-j}}{\binom{N}{n}} (1-\Phi)^j \quad (18)$$

Applying the usual method of binomial approximation (for $m, n \ll N$), so that

$$\frac{\binom{m}{j} \binom{N-m}{n-j}}{\binom{N}{n}} \approx \left(\frac{n}{N}\right)^j \left(\frac{N-m}{N}\right)^{m-j} ,$$

eventually we find that

$$\beta \approx \left(1 - \frac{n\Phi}{N}\right)^m \quad (19)$$

or

$$\beta \approx \left(1 - \frac{m\Phi}{N}\right)^n \quad (20)$$

The first expression is a better approximation when $m < n$, while the second is better when $n < m$.

These formulae clearly display the underlying symmetry between n and m in this model, and their derivation also underlies the approach used in the following discussions.

4.2 Application

The model described in §3 above uses $F(1)$ to solve for a lower bound for the sample sizes. However, there the error is assumed to be systematic so that the probability of detecting j defects, $j \geq 1$, is the same as the probability of detecting one defect. When random errors dominate, the sample size so obtained is too conservative. The following section is devoted to this situation.

As indicated in sub-section 4.1 above, for $n \leq m$ the expression is

$$\beta \leq \left(1 - \frac{G\Phi}{N_{\gamma X}}\right)^n \quad (21)$$

while for $n \geq m$, a better expression is

$$\beta \leq \left(1 - \frac{n\Phi}{N}\right)^{\frac{G}{n}} \quad (22)$$

where (as before) Φ denotes

$$\Phi\left(\frac{\gamma - r\sigma}{(1-\gamma)\sigma}\right).$$

The IAEA has been using the first-level sample size

$$\frac{n_1}{N} = \left(1 - \beta \frac{x}{G}\right)$$

Using expression (22), the non-detection probability of level-1 sampling and measurement satisfies the relation

$$\beta_1(\gamma) \leq \left(1 - \left(1 - \beta \frac{x}{G}\right) \Phi_1\right)^{\frac{G}{n}}$$

for any defect fraction γ . Thus, $\beta_1(\gamma) \leq \beta_G$ can not be guaranteed for $\gamma \leq \gamma^*$ where γ^* satisfies

$$\Phi\left(\frac{\gamma^* - r\sigma_1}{(1-\gamma^*)\sigma_1}\right) = \frac{1 - \beta \frac{x}{G}}{1 - \beta \frac{x}{G}}$$

The table below shows the value of γ^* for a few values the standard deviations of measurements, given $G/x=16$, $\beta_G=0.1$, and the IAEA first-level sample size. γ

γ^* (random) was calculated from the above equation, while γ^* (systematic) was determined using the expression for systematic errors developed earlier.

σ_1 (%)	γ^* (random)	γ^* (systematic)
25	0.795	0.825
12.5	0.350	0.465
10	0.255	0.380
8	0.185	0.310
6.25	0.13	0.25
3	0.04	0.125
1	0.001	0.045

Using expression (21) instead of (22) yields very similar answers. In that case, γ^* is the solution of

$$\gamma^* = \Phi \left(\frac{\gamma^* - r\sigma_1}{(1 - \gamma^*)\sigma_1} \right).$$

When the first level measurement is not adequate (for $\gamma \leq \gamma^*$), a more accurate measurement should be employed to ensure the adequacy of the combined detection probability. To attain the desired non-detection probability using expression (22) for the second-level measurement, n_2 items must be sampled and measured with an instrument with a standard deviation σ_2 , such that

$$\beta_2(\gamma) \leq \left(1 - \frac{n\Phi_2}{N} \right)^{\frac{G}{K}}$$

and

$$\beta_1(\gamma) \beta_2(\gamma) \leq \beta_G.$$

Alternatively

$$\frac{n_2}{N} \geq \left(1 - \frac{\beta \frac{x}{G}}{1 - (1 - \beta \frac{x}{G}) \Phi_1} \right) / \Phi_2 \quad (23)$$

The lower bound of the sampling sizes for any γ so obtained must then be maximized over $\gamma \leq \gamma^*$. If the more accurate expression based on equation (4) is used for β , the right-hand side of the above expression is reduced by the factor

$$\left(1 - \frac{G}{2\gamma x N} + \frac{1}{2N} \right).$$

Figs. 5, 6, and 7 compare the sample sizes obtained from (12), (13), and (23).

Thus, the sample size obtained assuming random errors is non-conservative since it predicts a smaller sample size even when the second-level instrument is perfect. As shown in Figs. 6 and 7, the sample size in the random case also increases rapidly when the instrument is not adequate. Although Fig. 7 shows a maximum at a very small gamma, using such a large sampling fraction to cover defect fractions below 0.05 may not be cost effective. A more accurate measurement would be more meaningful.

§5. Summary

Summarizing the above considerations, using the notation described in subsection 2.1, the sampling non-detection probability at any level is given by

$$\beta_s = \prod_{j=1}^m \left(1 - \frac{n}{N-j+1} \right) \leq \left(1 - \frac{n}{N} \right)^m$$

where m is the number of defecting items, and n is the number of samples.

The errors in measurement are defined by a cumulative normal distribution function Φ , with

$$\Phi_i = \Phi\left(\frac{\gamma - r\sigma_i}{(1-\gamma)\sigma_i}\right)$$

where γ denotes the defect fraction at the i -th level, and r the rejection limit defined in terms of the measurement's standard deviation σ_i .

5.1 Random Measurement Errors

For random measurement errors therefore, the overall non-detection probability

$$\beta \leq \left(1 - \frac{n}{N}\Phi\right)^{\frac{G}{\pi}},$$

in the binomial approximation, while for systematic errors

$$\beta \leq 1 - \left(1 - \left(1 - \frac{n}{N}\right)^{\frac{G}{\pi}}\right)\Phi.$$

Using the IAEA prescription

$$\frac{n}{N} = 1 - \beta_G^{\frac{x}{G}},$$

and β_G being the non-detection probability goal, in the case of random measurement errors, the expression is

$$\beta_1 \leq \left(1 - \left(1 - \beta_G^{\frac{x}{G}}\right)\Phi_1\right)^{\frac{G}{\pi}}, \text{ and}$$

$$\beta_2 \leq \left(1 - \frac{n_2}{N}\Phi_2\right)^{\frac{G}{\pi}}.$$

Solving for $\frac{n_2}{N}$, with $\beta_1\beta_2 \leq \beta_G$ leads to

$$\frac{n_2}{N} \geq \left(1 - \frac{\beta_G^{\frac{\pi}{G}}}{1 - (1 - \beta_G^{\frac{x}{G}})\Phi_1} \right) \frac{1}{\Phi_2},$$

and $\frac{n_2}{N}$ must then be maximized over γ .

5.2 Systematic Measurement Errors

For systematic measurement errors, one has correspondingly

$$\beta_1 \leq 1 - \left(1 - \beta_G^{\frac{1}{\gamma}} \right) \Phi_1, \text{ and}$$

$$\beta_2 \leq 1 - \left(1 - \left(1 - \frac{n_2}{N} \right)^{\frac{G}{\pi}} \right) \Phi_2.$$

Again, solving for $\frac{n_2}{N}$ with $\beta_1\beta_2 \leq \beta_G$ now leads to

$$1 - \left(\left(\frac{\beta_G}{1 - (1 - \beta_G^{\frac{1}{\gamma}})\Phi_1} - (1 - \Phi_2) \right) / \Phi_2 \right)^{\frac{\pi}{G}},$$

and, again, $\frac{n_2}{N}$ must be maximized over γ .

In both cases, the right-hand sides are reduced by the factor

$$\left(1 - \frac{G}{2\gamma\pi N} + \frac{1}{2N} \right)$$

when the more accurate approximation for β , formula (4), is used.

Exactly the same procedure can be used to calculate the sample sizes for further levels of inspection.

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Fig. 1

$\sigma = 12.5\%$, 10%, 8%, 6.25%

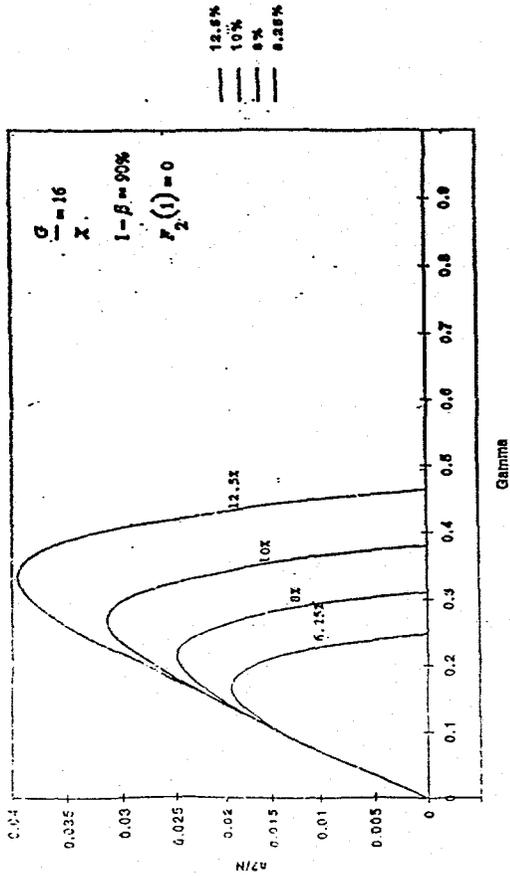


Fig. 2

$1 - \beta = 80\%$, 50%, 20%

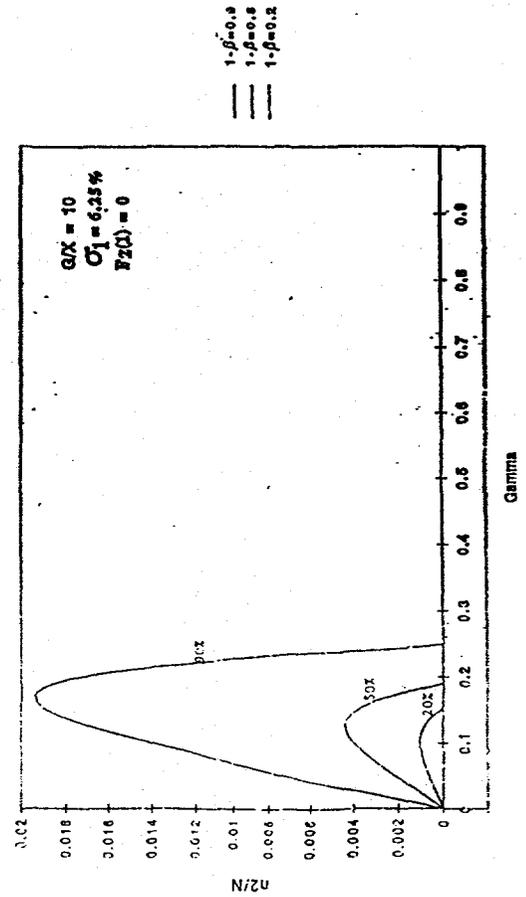


Fig. 3

$\sigma_2 = 6.25\%$, 3%, 1%, .01%; $\sigma_1 = 12.5\%$

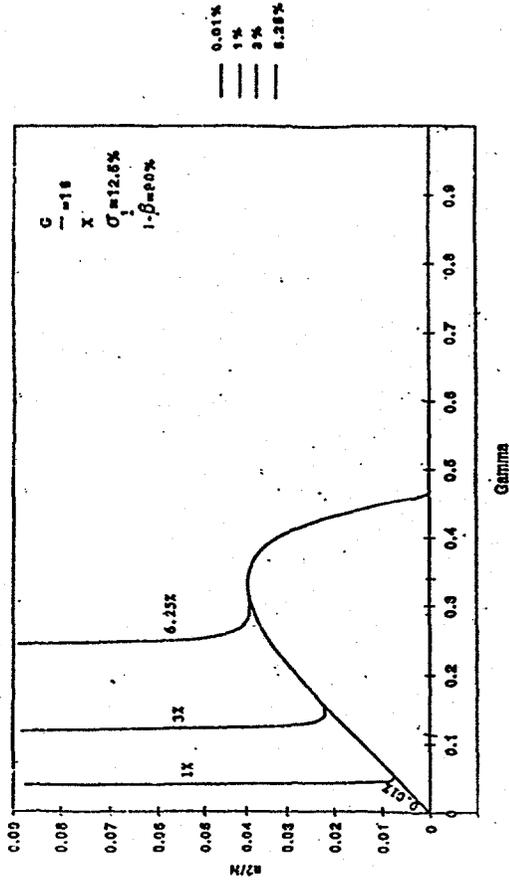


Fig. 4

Sensitivity to Finite Population

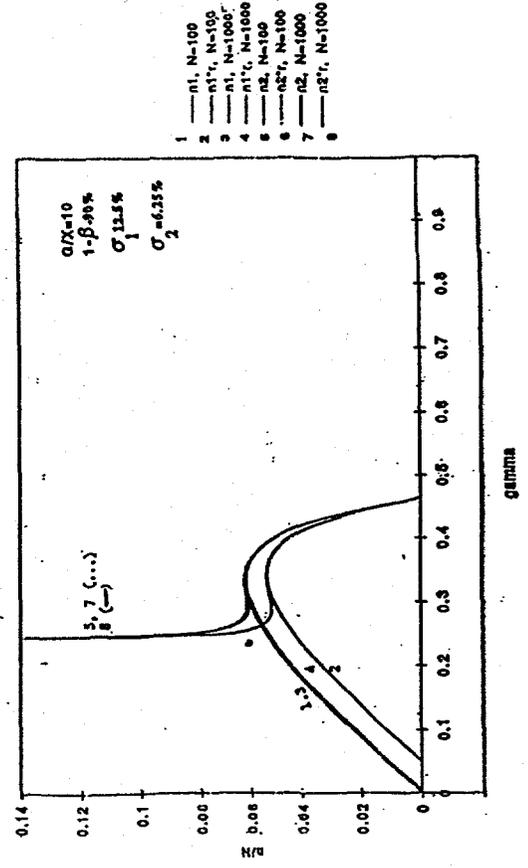


Fig. 5

Error Modeling

$\sigma_1=12.5\%$, $\sigma_2=0.1\%$

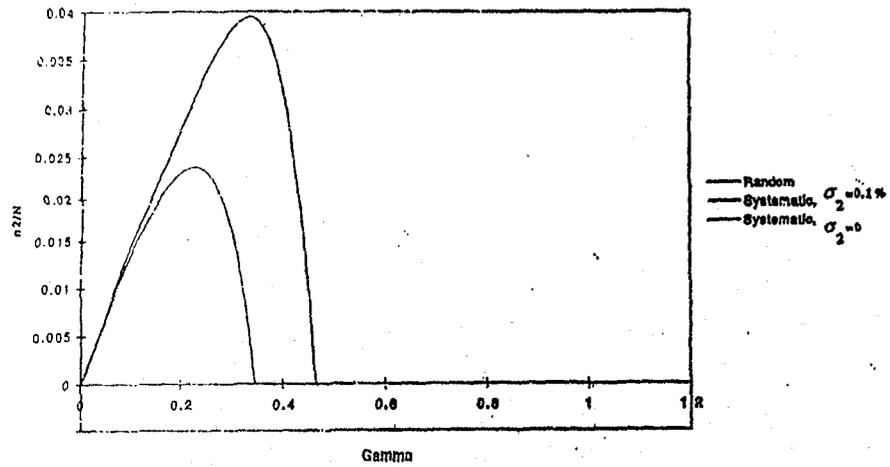


Fig. 7

Error Modeling

$\sigma_1=12.5\%$, $\sigma_2=5\%$

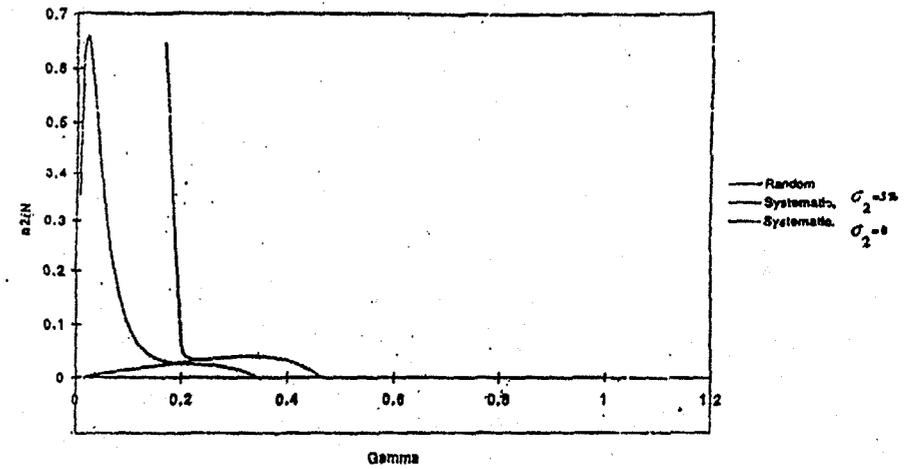


Fig. 8

Error Modeling

$\sigma_1=12.5\%$, $\sigma_2=1\%$

