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NEUTRON DIFFUSION APPROXIMATION SOLUTION FOR THE THREE LAYER BOREHOLE CYLINDRICAL GEOMETRY

PART I: THEORETICAL DESCRIPTION

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ABSTRACT

A solution of the neutron diffusion equation is given for a three layer cylindrical coaxial geometry. The calculation is performed in two neutron-energy groups which distinguish the thermal and epithermal neutron fluxes in the media irradiated by the fast point neutron source. The aim of the calculation is to define the neutron slowing down and migration lengths which are observed at a given point of the system.

Generally, the slowing down and migration lengths are defined for an infinite homogeneous medium (irradiated by the point neutron source) as a quotient of the neutron flux moment of the $(2n+2)$-order to the moment of the $2n$-order. CZUBEK (1992) introduced in the same manner the apparent neutron slowing down length and the apparent migration length for a given multi-region cylindrical geometry.

The solutions in the present paper are applied to the method of semi-empirical calibration of neutron well-logging tools. The three-region cylindrical geometry corresponds to the borehole of radius R_1 surrounded by the intermediate region (*e.g.* mud cake) of thickness (R_2-R_1) and finally surrounded by the geological formation which spreads from R_2 up to infinity. The cylinders of an infinite lenght are considered.

The paper gives detailed solutions for the 0-th, 2-nd and 4-th neutron moments of the neutron fluxes for each neutron energy group and in each cylindrical layer. A comprehensive list of the solutions for integrals containing Bessel functions or their derivatives, which are absent in common tables of integrals, is also included.

1. INTRODUCTION

The semi-empirical method of neutron tool calibration, elaborated by Czubek (CZUBEK, 1992, 1993, 1994, CZUBEK *et al.*, 1995) was used to generate the general and standard calibration curves as well as all porosity correction charts for different borehole and formation conditions. The borehole geometry was approximated by a two-layer cylindrical region *i.e.* a borehole and a geological formation, not taking into account the case where a mud cake plasters the borehole wall. This is because the theoretical solution for the neutron flux moments along the borehole axis obtained in the two group diffusion approximation is comparatively simple for two cylindrical regions. The mud cake requires a solution of the three region problem as it is shown in Fig. 1.

The signal of a neutron porosity tool depends on the formation, borehole and tool properties. The most important neutron property of the complex borehole-formation system is the neutron relaxation length: it is the thermal neutron diffusion length, L_d, for a thermal neutron source, whereas for a fast neutron source it is the migration length, L_m. The migration length is defined as

$$L_m^2 = L_s^2 + L_d^2 \tag{1}$$

where L_s is the slowing down length.

The thermal neutron flux $\varphi_{th}(r)$ observed in an infinite medium can be approximated (CZUBEK, 1992) by a function

$$\varphi_{th}(r) \cong \frac{P}{\Sigma_a} F(L_m) \tag{2}$$

where P is the probability of escaping resonance absorption during the slowing down process and Σ_a is the absorption cross section of the medium of interest. When the spatial distribution of epithermal neutrons is considered, the function F in Eq.(2) is dependent on the slowing down length, L_s.

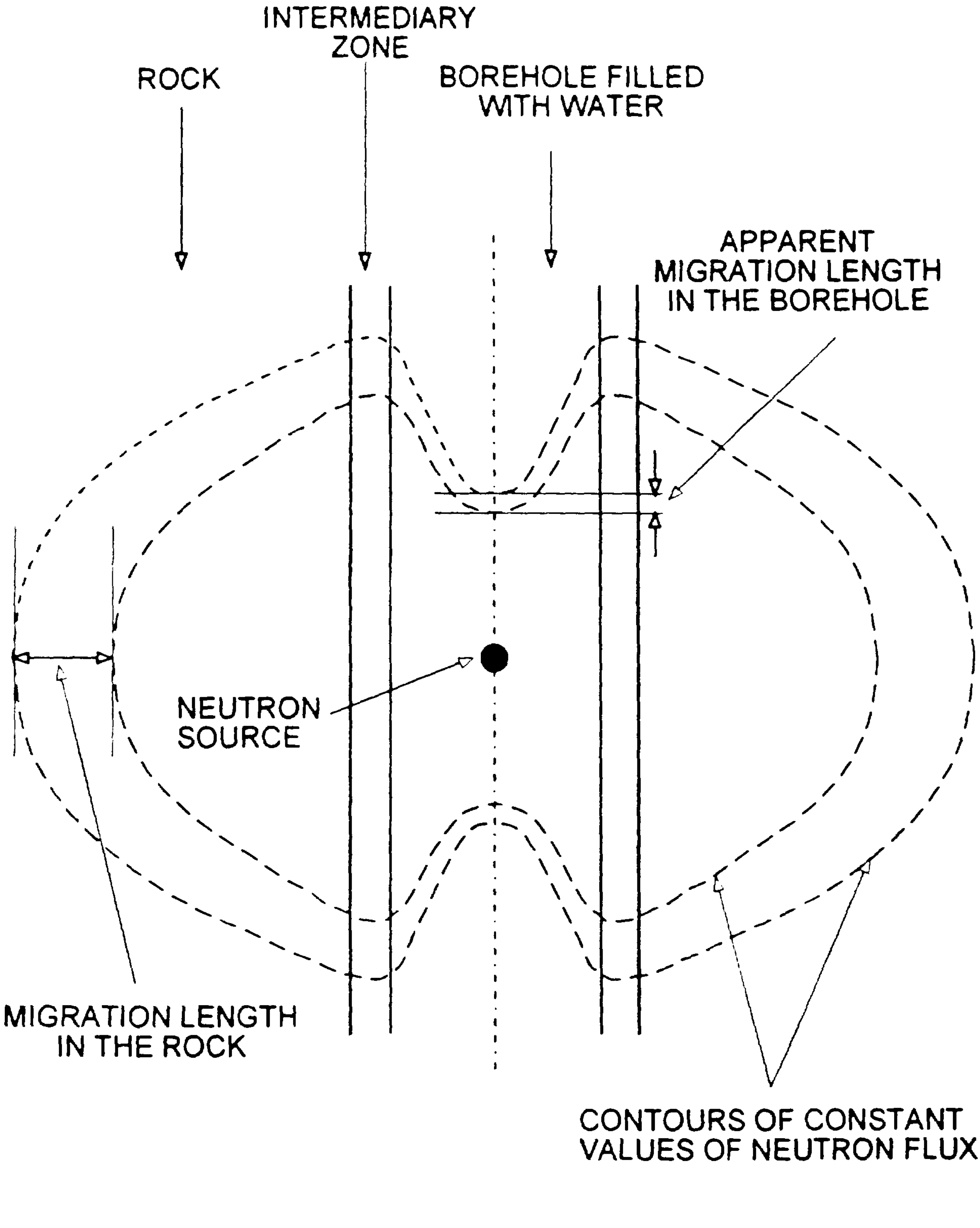

Fig.1. Contours of constant values of neutron flux around the point isotropic neutron source in the three layer cylindrical coaxial medium.

The simple introduction presented above shows the necessity of the slowing down and migration length knowledge for neutron well logging calculations. The real borehole - geological formation system requires more complicated neutron field calculations than for the simple infinite medium as shown above. The neutron tool which reads thermal and/or epithermal neutron flux is placed in the borehole, frequently in the asymmetric position to the borehole axis. The neutron parameters determining the neutron flux observed by the tool are complex parameters of the formation, the borehole and the tool materials. It is the reason that Czubek (CZUBEK, 1992) introduced the so-called „apparent" parameters: the apparent slowing down, L_{sap}, and migration, L_{map}, lengths, the apparent absorption cross section for thermal neutrons, Σ_{ap}, and the apparent resonance escape probability, P_{ap}, which describe the heterogeneous borehole - formation system.

The method of calculation of the apparent slowing down and migration lengths requires solving of the diffusion equations for a given cylindrical geometry. Up to now it was done by Czubek in the two-region cylindrical geometry as is mentioned at the beginning of the paper. These calculations gave a possibility to elaborate the method of the semi-empirical calibration for porosity neutron logging tools. The method has been used to the ODSN-102 neutron porosity tool, 70 mm diameter equipped with an Am-Be neutron source at the calibration facility, Zielona Góra, Poland (CZUBEK *et al.*, 1995).

A method of calculation of the apparent neutron slowing down length, L_{sap}, and the apparent neutron migration length, L_{map}, in the three-region cylindrical geometry by the two-group diffusion approximation is presented in the paper.

2. WHY A NEW APPROACH IS NECESSARY

In the previous papers listed above the apparent slowing down and migration lengths were found by solving the two-group diffusion approximation for the neutron fluxes $\varphi_{ij}^{\alpha}(r,z) \equiv \varphi(r,z)$ (i = b, r – borehole or rock, j = 1, 2 – number of neutron diffusion group, α = ^{o}, * or ** – indicator of original neutrons from the real sources or neutrons reflected from the boundaries, the so-called apparent neutron sources). The equations are:

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}-\lambda_{ij}^2\right)\varphi(r,z)=-4\pi\rho(r,z) \quad , \tag{3}$$

where $\rho(r,z)=\rho_{ij}^{\alpha}(r,z)$ is the neutron source density distribution for each neutron diffusion group and inside each geometric region.

The solution is found by applying the Fourier transform:

$$\varphi_{ij}^{\alpha}(r,z)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\overline{\varphi}_{ij}^{\alpha}(r,\eta)\,e^{-i\eta z}\mathrm{d}\eta \quad . \tag{4}$$

For fixed values of indices i and j the summation of partial neutron fluxes $\varphi_{ij}^{\alpha}(r,z)$ over the index α gives the physical flux φ "really" existing inside a borehole or formation. The existence of the apparent sources on the boundaries is realised by 4 equations resulting from the boundary conditions. The Fourier transforms $\overline{\varphi}_{ij}^{\alpha}(r,\eta)$ are combinations of the modified Bessel functions $I_0(x)$, $I_1(x)$, $I_0(X)$, $I_1(X)$, $K_0(x)$, $K_1(x)$, $K_0(X)$, and $K_1(X)$, where $x=r\sqrt{\lambda_{ij}^2+\eta^2}$ and $X=R\sqrt{\lambda_{ij}^2+\eta^2}$. In these expressions the quantity $\lambda_{ij}=1/L_{ij}$ is the reciprocal of the neutron diffusion length in each energy group and in each geometric region. R is the borehole radius.

The semi-empirical method of calibration is based on the calculation of the moments $m_{2n}(\varphi)$ of the $2n$-order from the flux $\varphi_{b2}(r,z)$ along the z axis for $r=0$ *i.e.* of the expression

$$m_{2n}(\varphi)=\int_{-\infty}^{\infty}z^{2n}\varphi(0,z)\mathrm{d}z \quad . \tag{5}$$

Mathematically, this operation is equivalent to the expression:

$$m_{2n}(\varphi)=(-1)^n\frac{\mathrm{d}^{2n}}{\mathrm{d}\eta^{2n}}\overline{\varphi}(r,\eta)_{|\eta=0,\,r=0} \quad . \tag{6}$$

Thus, the entire procedure is reduced to the calculation of derivatives of order $2n$ of the functions $\bar{\varphi}_{ij}^{\alpha}(r,\eta)$ with respect to the η variable. Which is needed in these calculations are the derivatives of the order $2n = 0, 2, 4$. In spite of the fact that the derivatives are always possible to calculate, the calculation of the derivative of the fourth order is, in this case, a very complicated procedure giving many technical problems. This can lead to many errors in the final formulae. For this reason a similar approach to the more complicated three region problem is hopeless. Some other approach to the solution of the three region problem is needed if one wants to treat the mud cake and/or the steel lining problem. This paper presents such a new approach.

3. EQUATIONS FOR AXIAL MOMENTS OF THE NEUTRON FLUX

The idea, how to overcome the laborious calculations of higher order derivatives for the z-axial moments of the neutron fluxes, is to establish the differential equations directly for the flux moments. Let us multiply every diffusion equation for the partial neutron flux by z^{2n} and integrate over z :

$$\int_{-\infty}^{\infty} z^{2n}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}-\lambda_{ij}^2\right)\varphi_{ij}^{\alpha}(r,z)\,\mathrm{d}z = -4\pi\int_{-\infty}^{\infty}\rho_{ij}^{\alpha}(r,z)\,z^{2n}\mathrm{d}z \tag{7}$$

Using the definition of the neutron flux z-axial moments $m_{2n}(\varphi;r)$ we have

$$\int_{-\infty}^{\infty} z^{2n}\frac{\partial^2}{\partial r^2}\varphi(r,z)\,\mathrm{d}z = \frac{\partial^2}{\partial r^2}\big(m_{2n}(\varphi;r)\big) \tag{8}$$

and similarly

$$\int_{-\infty}^{\infty} z^{2n}\frac{1}{r}\frac{\partial}{\partial r}\varphi(r,z)\,\mathrm{d}z = \frac{1}{r}\frac{\partial}{\partial r}\big(m_{2n}(\varphi;r)\big) \tag{9}$$

Integrating Eq. (7) by parts, and bearing in mind that for $z=\pm\infty$ the neutron fluxes as well as their derivatives vanish, one obtains:

$$\int_{-\infty}^{\infty} z^{2n} \frac{\partial^2}{\partial z^2} \varphi(r,z)\, dz = -2n \int_{-\infty}^{\infty} z^{2n-1} \frac{\partial}{\partial z} \varphi(r,z)\, dz =$$
$$= 2n(2n-1) \int_{-\infty}^{\infty} z^{2n-2} \varphi(r,z)\, dz = 2n(2n-1) m_{2n-2}(\varphi;r) \quad . \tag{10}$$

The source term in Eq. (7) can be rewritten as:

$$\int_{-\infty}^{\infty} z^{2n} \rho_{ij}^{\alpha}(r,z)\, dz = m_{2n}(\rho_{ij}^{\alpha};r) \quad . \tag{11}$$

When the moments and $m_{2n}(\varphi;r)$ and $m_{2n}(\rho;r)$ are presented by Hankel transforms:

$$m_{2n}(\varphi;r) = \int_0^{\infty} \tilde{m}_{2n}(\varphi;k)\, k\, J_0(kr)\, dk \quad ,$$
$$m_{2n}(\rho;r) = \int_0^{\infty} \tilde{m}_{2n}(\rho;k)\, k\, J_0(kr)\, dk \quad , \tag{12}$$

where $J_0(kr)$ is the Bessel function of the first kind, one obtains algebraic equations for the Hankel transforms of the moments $\tilde{m}_{2n}(\varphi;k)$ and $\tilde{m}_{2n}(\rho;k)$:

$$-\left(k^2 + \lambda_{ij}^2\right) \tilde{m}_{2n}(\varphi;k) + 2n(2n-1)\, \tilde{m}_{2n-2}(\varphi;k) = -4\pi\, \tilde{m}_{2n}(\rho;k) \quad , \tag{13}$$

where $\tilde{m}_{2n}(\rho;k)$ is the Hankel transform of the moment $2n$ from the neutron source distribution $\rho_{ij}(r,z)$. Now the problem to obtain the desired moments is reduced to the solution of the Hankel integral:

$$m_{2n}(\varphi;r) = \int_0^{\infty} k\, J_0(kr) \left(4\pi \frac{\tilde{m}_{2n}(\rho;k)}{k^2 + \lambda_{ij}^2} + 2n(2n-1) \frac{\tilde{m}_{2n-2}(\varphi;k)}{k^2 + \lambda_{ij}^2} \right) dk \tag{14}$$

consecutively for $n = 0, 1, 2$ and respecting the proper boundary conditions. The integrals in Eq. (14) are relatively simple to calculate. In the general case they are:

$$I_{n_1,n_2}(r,R,\lambda_{ij},\lambda_{lp}) = \int_0^\infty \frac{k\,J_0(kr)\,J_0(kR)}{\left(k^2+\lambda_{ij}^2\right)^{n_1}\left(k^2+\lambda_{lp}^2\right)^{n_2}}\,dk \tag{15}$$

which is usually not included in the tables of integrals, but they are easy to calculate for integer values of n_1 and n_2. For $R = 0$ we are going to use the notation:

$$J_{n_1,n_2}(r,\lambda_{ij},\lambda_{lp}) = \int_0^\infty \frac{k\,J_0(kr)}{\left(k^2+\lambda_{ij}^2\right)^{n_1}\left(k^2+\lambda_{lp}^2\right)^{n_2}}\,dk\ . \tag{16}$$

When the above mathematical approach is applied to the calculation of moments for the two-region problem, one should calculate every component of the moment (*i.e.* from the real and apparent sources) which gives 24 equations plus 12 algebraic equations derived from the boundary conditions for moments. At every boundary both the moment and its first derivative should be continuos. As a result there are many more equations to solve than there were in the previous method. On the other hand, the problem of calculating of the higher order derivatives has disappeared completely. For two boundaries (*i.e.* three coaxial regions) it is necessary to solve, in the two-group diffusion approximation, the set of 42 equations for the moments and the set of 24 algebraic equations resulting from the boundary conditions.

4. EQUATIONS FOR THE PARTIAL NEUTRON FLUXES AND FOR THE PARTIAL MOMENTS

Let us take the following geometric situation depicted in Fig 2: the borehole has a radius R_1, and parameters indicated by i = b, the intermediate region representing the mud cake is situated in the space between R_1 and R_2. All neutron properties (from the mud cake zone) are marked by the index i = c or i = d, depending upon from which boundary they originated. The properties of the geological formation outside the radius R_2 are marked with the index i = r. All neutron partial fluxes and source densities have the upper index $\alpha = {}^{o}$, $*$ or $**$ depending upon the origin of neutrons being treated. The index $\alpha = {}^{o}$ belongs to neutrons coming from the real sources distributed in the space. The index $\alpha = *$ characterises all neutrons coming from the apparent neutron sources distributed on the boundary surfaces. The upper index $\alpha = **$ belongs to fluxes and neutron source densities of the second group (*i.e.* j = 2) originating from the fluxes created by apparent sources with the index $\alpha = *$. Each geometric region has its own neutron transport properties: the diffusion coefficient D_{ij} and the reciprocal of the diffusion length $\lambda_{ij} = 1/L_{ij}$. The total neutron fluxes in each region are the sums of the partial fluxes $\varphi_{ij}^{\alpha}(r,z)$ (i = b, c, d, r; j = 1, 2; $\alpha = {}^{o}, *, **$). Every partial flux satisfies the diffusion Eq. (3):

For the first neutron group there are

$$
\begin{aligned}
\varphi_{b1}(r,z) &= \overset{o}{\varphi}_{b1}(r,z) + \overset{*}{\varphi}_{b1}(r,z) \\
\varphi_{c1}(r,z) &= \overset{*}{\varphi}_{c1}(r,z) + \overset{*}{\varphi}_{d1}(r,z) \\
\varphi_{r1}(r,z) &= \overset{*}{\varphi}_{r1}(r,z)
\end{aligned}
\tag{17}
$$

whereas for the second neutron group

$$
\begin{aligned}
\varphi_{b2}(r,z) &= \overset{o}{\varphi}_{b2}(r,z) + \overset{*}{\varphi}_{b2}(r,z) + \overset{**}{\varphi}_{b2}(r,z) \\
\varphi_{c2}(r,z) &= \overset{*}{\varphi}_{c2}(r,z) + \overset{**}{\varphi}_{c2}(r,z) + \overset{*}{\varphi}_{d2}(r,z) + \overset{**}{\varphi}_{d2}(r,z) \\
\varphi_{r2}(r,z) &= \overset{*}{\varphi}_{r2}(r,z) + \overset{**}{\varphi}_{r2}(r,z)
\end{aligned}
\tag{18}
$$

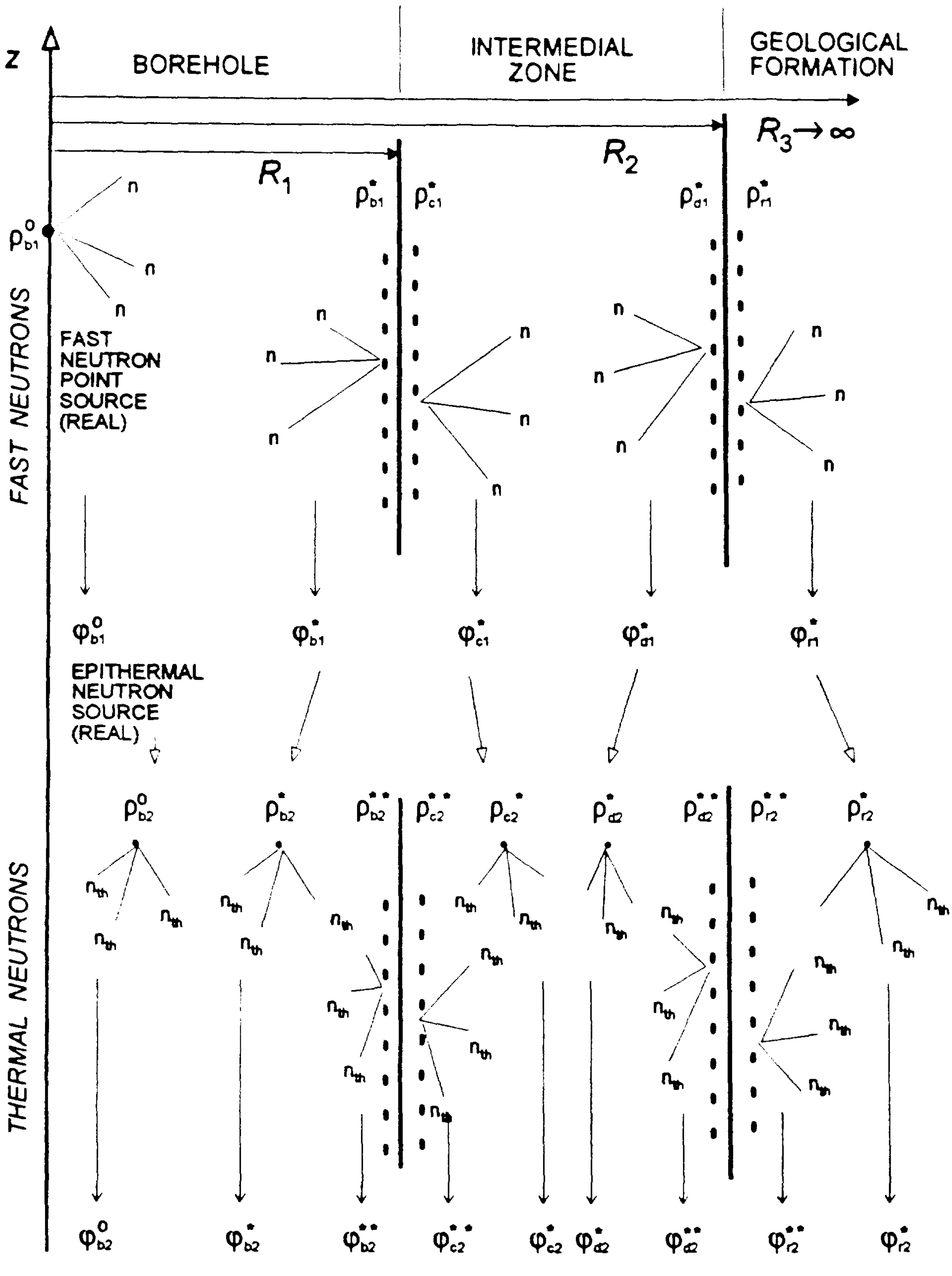

Fig.2. Real ρ_i^o and apparent ρ_i^*, ρ_i^{**} neutron sources and counterpart partial neutron fluxes φ_i^o, φ_i^*, φ_i^{**} in three region cylindrical system.

All total fluxes (j = 1, 2) have to satisfy the boundary conditions of the continuity of both the flux and the current:

$$\begin{aligned}
&\varphi_{bj}(R_1,z)=\varphi_{cj}(R_1,z)\\
&D_{bj}\frac{\partial}{\partial r}\varphi_{bj}(r,z)\big|_{r=R_1}=D_{cj}\frac{\partial}{\partial r}\varphi_{cj}(r,z)\big|_{r=R_1}\\
&\varphi_{cj}(R_2,z)=\varphi_{rj}(R_2,z)\\
&D_{cj}\frac{\partial}{\partial r}\varphi_{cj}(r,z)\big|_{r=R_2}=D_{rj}\frac{\partial}{\partial r}\varphi_{rj}(r,z)\big|_{r=R_2}
\end{aligned}\tag{19}$$

From Eqs (17, 18 and 19), in agreement with the definition of moments given by Eq. (5), one has the equations for the total moments:

1. For the first neutron group (j = 1):

$$\begin{aligned}
&m_{2n}(\varphi_{b1};r)=m_{2n}(\overset{\circ}{\varphi}_{b1};r)+m_{2n}(\dot{\varphi}_{b1};r)\\
&m_{2n}(\varphi_{c1};r)=m_{2n}(\dot{\varphi}_{c1};r)+m_{2n}(\dot{\varphi}_{d1};r)\\
&m_{2n}(\varphi_{r1};r)=m_{2n}(\dot{\varphi}_{r1};r)
\end{aligned}\tag{20}$$

2. For the second neutron group (j = 2):

$$\begin{aligned}
&m_{2n}(\varphi_{b2};r)=m_{2n}(\overset{\circ}{\varphi}_{b2};r)+m_{2n}(\dot{\varphi}_{b2};r)+m_{2n}(\ddot{\varphi}_{b2};r)\\
&m_{2n}(\varphi_{c2};r)=m_{2n}(\dot{\varphi}_{c2};r)+m_{2n}(\ddot{\varphi}_{c2};r)+m_{2n}(\dot{\varphi}_{d2};r)+m_{2n}(\ddot{\varphi}_{d2};r)\\
&m_{2n}(\varphi_{r2};r)=m_{2n}(\dot{\varphi}_{r2};r)+m_{2n}(\ddot{\varphi}_{r2};r)
\end{aligned}\tag{21}$$

which for n = 0, 1, 2 gives 42 equations for the partial moments $m_{2n}(\varphi_{ij}^{a};r)$. The boundary conditions for the total moments, according to Eq.(19), are

$$\begin{aligned}
&m_{2n}(\varphi_{bj};R_1)=m_{2n}(\varphi_{cj};R_1)\\
&D_{bj}\frac{\partial}{\partial r}m_{2n}(\varphi_{bj};r)\big|_{r=R_1}=D_{cj}\frac{\partial}{\partial r}m_{2n}(\varphi_{cj};r)\big|_{r=R_1}\\
&m_{2n}(\varphi_{cj};R_2)=m_{2n}(\varphi_{rj};R_2)\\
&D_{cj}\frac{\partial}{\partial r}m_{2n}(\varphi_{cj};r)\big|_{r=R_2}=D_{rj}\frac{\partial}{\partial r}m_{2n}(\varphi_{rj};r)\big|_{r=R_2}
\end{aligned}\tag{22}$$

which for $n = 0, 1, 2;\ j = 1, 2$ for all partial moments give 24 equations.

To be able to solve the three-region coaxial problem for the moments of neutron fluxes one needs to define first the proper source distribution densities $\rho_{ij}^{\alpha}(r,z)$. This is presented in Appendix A. Then, the Hankel transforms $\tilde{m}_{2n}(\rho_{ij}^{\alpha};k)$ of the axial moments of the source distributions have to be established. This is done in Appendix B. When these moments are already known, from Eq. (13) rewritten in the form:

$$\tilde{m}_{2n}(\varphi_{ij}^{\alpha};k) = \frac{4\pi}{k^2+\lambda_{ij}^2}\tilde{m}_{2n}(\rho_{ij}^{\alpha};k) + 2n(2n-1)\frac{1}{k^2+\lambda_{ij}^2}\tilde{m}_{2n-2}(\varphi_{ij}^{\alpha};k) \quad , \tag{23}$$

it is possible to calculate the Hankel transforms $\tilde{m}_{2n}(\varphi_{ij}^{\alpha};k)$ of the neutron flux moments. This is shown in Appendix C. To obtain the partial moments of the fluxes one takes the inverse Hankel transform, according to Eq. (12). Some definite integrals containing Bessel functions, which are usually not included in the common tables, are needed to solve Eq. (12). These integrals are given in Appendix D, whereas the equations for the moments $m_{2n}(\varphi_{ij}^{\alpha};r)$ are presented in Appendix E. Once the equations for the moments $m_{2n}(\varphi_{ij}^{\alpha};r)$ are known one has to find the 24 numbers $B_{2n,ij}$, $C_{2n,ij}$, $D_{2n,ij}$, and $E_{2n,ij}$ for $n = 0, 1, 2$; i = b, c, d, r and $j = 1, 2$ appearing in Eqs (E.1 to E.42). They are obtained by introducing the moments into Eq. (22) describing the boundary conditions. These equations are given in Appendix F.

When the numbers $B_{2n,ij}$, $C_{2n,ij}$, $D_{2n,ij}$ and $E_{2n,ij}$ are known, it is possible to calculate the apparent slowing down length and the apparent migration length along the z-axis (CZUBEK, 1992) according to the formulae:

$$L_{\mathrm{sap}}^2 = \frac{1}{6}\frac{m_4(\varphi_{b1};0)}{m_2(\varphi_{b1};0)} \tag{24}$$

$$L_{\mathrm{map}}^2 = \frac{1}{6}\frac{m_4(\varphi_{b2};0)}{m_2(\varphi_{b2};0)} \tag{25}$$

5. CONCLUSIONS

The paper presents the theoretical calculation which finally gives a possibility to calculate the apparent neutron slowing down and migration lengths in the three-layer cylindrical system which represents the borehole, the intermediate zone (*e.g.* mud cake at the borehole walls), and the geological formation. The semi-empirical method of neutron tool calibration correlates the tool reading observed experimentally with the general neutron parameter, GNP. The GNP is defined as

$$\mathrm{GNP} = L_{map}\Sigma_{ap}^{n}P_{ap}^{m} \tag{26}$$

where the exponents n and m are obtained experimentally during the calibration procedure.

The apparent migration length can be calculated using the calculation procedure described above. The neutron absorption cross section, the elemental compositions and the densities of the borehole fluid, of the mud cake and of the geological formation as well as geometrical dimensions of the system have to be known. Knowledge of the mud cake parameters is the worst among them. A research in this field is going on.

There is no well-defined method to calculate the apparent absorption cross section Σ_{ap} and the apparent resonance probability P_{ap}. The method proposed by Morstin (MORSTIN AND KREFT, 1984; CZUBEK, 1992) has been used in the semi-empirical neutron tool calibration up to now. The mud cake allowance in the calculation make the problem more difficult and must be solved.

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APPENDIX A

Neutron source densities for three coaxial regions

According to the notation used in Fig. 2 there are following source densities related with the diffusion Eq. (3). In all source distributions the $\beta_{ij}(z)$ and $\gamma_{ij}(z)$ are functions of the source distributions on the boundary surfaces. They are unknown functions which have to be found from appropriate boundary conditions. The $\delta(r)$ is the Dirac delta.

A. First neutron group (slowing down neutrons), $j=1$

1. Borehole region, $i=$b

$\varphi^{o}_{b1}(r,z)$ - epithermal neutron flux originating from the real source of fast neutrons situated at the origin of the cylindrical co-ordinate system (1 neutron/s) having the space density

$$\rho^{o}_{b1}(r,z) = \frac{1}{4\pi D_{b1}} \frac{\delta(r)}{2\pi r} \delta(z) \tag{A.1}$$

$\varphi^{\bullet}_{b1}(r,z)$ - epithermal neutron flux from the apparent fast neutron sources distributed on the lateral boundary between the borehole of radius R_1 (on the side of the borehole) and the mud cake. The space density of these sources is given by the expression

$$\rho^{\bullet}_{b1}(r,z) = \frac{1}{4\pi D_{b1}} \beta_{b1}(z) \frac{\delta(r-R_1)}{2\pi R_1} \tag{A.2}$$

2. Intermediary region (the mud cake), $i=$c, d

$\varphi^{\bullet}_{c1}(r,z)$ - epithermal neutron flux from the apparent fast neutron sources distributed on the lateral boundary between the borehole of radius R_1 (on the side of the mud cake) and the mud cake. The space density of these sources is given by the expression

$$\rho^{\bullet}_{c1}(r,z) = \frac{1}{4\pi D_{c1}} \beta_{c1}(z) \frac{\delta(r-R_1)}{2\pi R_1} \tag{A.3}$$

$\varphi^{\bullet}_{d1}(r,z)$ - epithermal neutron flux from the apparent fast neutron sources distributed on the lateral boundary between the mud cake of radius R_2 (on the side of the mud cake) and the formation. The space density of these sources is given by the expression

$$\rho^{\bullet}_{d1}(r,z) = \frac{1}{4\pi D_{c1}} \gamma_{c1}(z) \frac{\delta(r - R_2)}{2\pi R_2} \qquad (A.4)$$

3. Geological formation, i=r

$\varphi^{\bullet}_{r1}(r,z)$ - epithermal neutron flux from the apparent fast neutron sources distributed on the lateral boundary between the mud cake of radius R_2 (on the side of the formation) and the formation. The space density of these sources is given by the expression

$$\rho^{\bullet}_{r1}(r,z) = \frac{1}{4\pi D_{r1}} \gamma_{r1}(z) \frac{\delta(r - R_2)}{2\pi R_2} \qquad (A.5)$$

B. Second neutron group (thermal neutrons), j=2

1. Borehole region, i=b

$\varphi^{o}_{b2}(r,z)$ - thermal neutron flux originating from the real source of epithermal neutrons having the space density

$$\rho^{o}_{b2}(r,z) = \frac{1}{4\pi D_{b2}} D_{b1} \lambda^2_{b1} \varphi^{o}_{b1}(r,z) \qquad (A.6)$$

$\varphi^{\bullet}_{b2}(r,z)$ - thermal neutron flux originating from the apparent epithermal neutron flux $\varphi^{\bullet}_{b1}(r,z)$ which has the spatial source distribution given by

$$\rho^{\bullet}_{b2}(r,z) = \frac{1}{4\pi D_{b2}} D_{b1} \lambda^2_{b1} \varphi^{\bullet}_{b1}(r,z) \qquad (A.7)$$

$\varphi^{\bullet\bullet}_{b2}(r,z)$ - thermal neutron flux from apparent epithermal neutron sources distributed on the wall of a borehole of radius R_1 at the boundary between the borehole and the intermediary region (on the side of the borehole) with the space source density given by

$$\rho^{\bullet\bullet}_{b2}(r,z) = \frac{1}{4\pi D_{b2}} \beta_{b2}(z) \frac{\delta(r - R_1)}{2\pi R_1} \qquad (A.8)$$

2. Intermediary region (the mud cake), i=c, d

$\varphi^{\bullet}_{c2}(r,z)$ - thermal neutron flux from apparent epithermal neutron sources given by the flux $\varphi^{\bullet}_{c1}(r,z)$ with the space source density given by

$$\rho^{\bullet}_{c2}(r,z) = \frac{1}{4\pi D_{c2}} D_{c1} \lambda^2_{c1} \varphi^{\bullet}_{c1}(r,z) \qquad (A.9)$$

$\overset{\bullet}{\varphi}_{d2}(r,z)$ - thermal neutron flux from apparent epithermal neutron sources given by the flux $\overset{\bullet}{\varphi}_{d1}(r,z)$ with the space source density given by

$$\overset{\bullet}{\rho}_{d2}(r,z) = \frac{1}{4\pi D_{c2}} D_{c1}\lambda_{c1}^{2}\overset{\bullet}{\varphi}_{d1}(r,z) \qquad \text{(A.10)}$$

$\overset{\bullet\bullet}{\varphi}_{c2}(r,z)$ - thermal neutron flux from the apparent epithermal neutron sources distributed on the lateral boundary between the borehole of radius R_1 (on the side of the mud cake) and the mud cake. The space density of these sources is given by the expression

$$\overset{\bullet\bullet}{\rho}_{c2}(r,z) = \frac{1}{4\pi D_{c2}}\beta_{c2}(z)\frac{\delta(r-R_1)}{2\pi R_1} \qquad \text{(A.11)}$$

$\overset{\bullet\bullet}{\varphi}_{d2}(r,z)$ - thermal neutron flux from the apparent epithermal neutron sources distributed on the lateral boundary between the mud cake of radius R_2 (on the side of the mud cake) and the formation. The space density of these sources is given by the expression

$$\overset{\bullet\bullet}{\rho}_{d2}(r,z) = \frac{1}{4\pi D_{c2}}\gamma_{c2}(z)\frac{\delta(r-R_2)}{2\pi R_2} \qquad \text{(A.12)}$$

3. Geological formation, i=r

$\overset{\bullet}{\varphi}_{r2}(r,z)$ - thermal neutron flux from the apparent epithermal neutron flux $\overset{\bullet}{\varphi}_{r1}(r,z)$ with sources given by the expression

$$\overset{\bullet}{\rho}_{r2}(r,z) = \frac{1}{4\pi D_{r2}} D_{r1}\lambda_{r1}^{2}\overset{\bullet}{\varphi}_{r1}(r,z) \qquad \text{(A.13)}$$

$\overset{\bullet\bullet}{\varphi}_{r2}(r,z)$ - thermal neutron flux from the apparent epithermal neutron sources distributed on the lateral boundary between the mud cake of radius R_2 (on the side of the formation) and the formation. The space density of these sources is given by the expression

$$\overset{\bullet\bullet}{\rho}_{r2}(r,z) = \frac{1}{4\pi D_{r2}}\gamma_{r2}(z)\frac{\delta(r-R_2)}{2\pi R_2} \qquad \text{(A.14)}$$

APPENDIX B

Definition of the moments of source densities

When Eqs (19) are multiplied on both sides by z^{2n} and integrated over z, the boundary conditions for the flux moments are found. At first one has to calculate the moments of the source distributions given by Eqs (A.1 to A.14). The following 24 numbers are introduced:

$$B_{2n,bj} = \int_{-\infty}^{\infty} z^{2n} \beta_{bj}(z) \mathrm{d}z \qquad \text{(B.1a)}$$

$$C_{2n,cj} = \int_{-\infty}^{\infty} z^{2n} \beta_{cj}(z) \mathrm{d}z \qquad \text{(B.1b)}$$

$$D_{2n,cj} = \int_{-\infty}^{\infty} z^{2n} \gamma_{cj}(z) \mathrm{d}z \qquad \text{(B.1c)}$$

$$E_{2n,tj} = \int_{-\infty}^{\infty} z^{2n} \gamma_{tj}(z) \mathrm{d}z \qquad \text{(B.1d)}$$

$$j = 1, 2; \; n = 0, 1, 2$$

When one uses the Hankel transforms ,

$$\delta(r) = r \int_0^{\infty} k J_0(kr) \, \mathrm{d}k$$

$$\delta(r - R) = R \int_0^{\infty} k J_0(kr) J_0(kR) \, \mathrm{d}k \qquad \text{(B.2)}$$

the following Hankel transforms, according to Eq.(12), are obtained for the moments of the source distributions:

$$\tilde{m}_0(\rho^{\circ}_{b1}; k) = \frac{1}{4\pi D_{b1}} \frac{1}{2\pi} \qquad \text{(B.3)}$$

$$\tilde{m}_2(\rho^{\circ}_{b1}; k) = \tilde{m}_4(\rho^{\circ}_{b1}; k) = 0$$

$$\tilde{m}_{2n}(\dot{\rho}_{b1}; k) = \frac{1}{4\pi D_{b1}} B_{2n,b1} \frac{1}{2\pi} J_0(kR_1) \qquad \text{(B.4)}$$

$$\tilde{m}_{2n}(\dot{\rho}_{c1}; k) = \frac{1}{4\pi D_{c1}} C_{2n,c1} \frac{1}{2\pi} J_0(kR_1) \qquad \text{(B.5)}$$

$$\tilde{m}_{2n}(\dot{\rho}_{d1};k) = \frac{1}{4\pi D_{c1}} \mathrm{D}_{2n,c1} \frac{1}{2\pi} J_0(kR_2) \qquad (B.6)$$

$$\tilde{m}_{2n}(\dot{\rho}_{r1};k) = \frac{1}{4\pi D_{r1}} \mathrm{E}_{2n,r1} \frac{1}{2\pi} J_0(kR_2) \qquad (B.7)$$

$$\tilde{m}_{2n}(\overset{\circ}{\rho}_{b2};k) = \frac{1}{4\pi D_{b2}} D_{b1}\, \lambda^2_{b1}\, \tilde{m}_{2n}(\overset{\circ}{\varphi}_{b1};k) \qquad (B.8)$$

$$\tilde{m}_{2n}(\dot{\rho}_{b2};k) = \frac{1}{4\pi D_{b2}} D_{b1}\, \lambda^2_{b1}\, \tilde{m}_{2n}(\dot{\varphi}_{b1};k) \qquad (B.9)$$

$$\tilde{m}_{2n}(\ddot{\rho}_{b2};k) = \frac{1}{4\pi D_{b2}} \mathrm{B}_{2n,b2} \frac{1}{2\pi} J_0(kR_1) \qquad (B.10)$$

$$\tilde{m}_{2n}(\dot{\rho}_{c2};k) = \frac{1}{4\pi D_{c2}} D_{c1}\, \lambda^2_{c1}\, \tilde{m}_{2n}(\dot{\varphi}_{c1};k) \qquad (B.11)$$

$$\tilde{m}_{2n}(\dot{\rho}_{d2};k) = \frac{1}{4\pi D_{c2}} D_{c1}\, \lambda^2_{c1}\, \tilde{m}_{2n}(\dot{\varphi}_{d1};k) \qquad (B.12)$$

$$\tilde{m}_{2n}(\ddot{\rho}_{c2};k) = \frac{1}{4\pi D_{c2}} \mathrm{C}_{2n,c2} \frac{1}{2\pi} J_0(kR_1) \qquad (B.13)$$

$$\tilde{m}_{2n}(\ddot{\rho}_{d2};k) = \frac{1}{4\pi D_{c2}} \mathrm{D}_{2n,c2} \frac{1}{2\pi} J_0(kR_2) \qquad (B.14)$$

$$\tilde{m}_{2n}(\dot{\rho}_{r2};k) = \frac{1}{4\pi D_{r2}} D_{r1}\, \lambda^2_{r1}\, \tilde{m}_{2n}(\dot{\varphi}_{r1};k) \qquad (B.15)$$

$$\tilde{m}_{2n}(\ddot{\rho}_{r2};k) = \frac{1}{4\pi D_{r2}} \mathrm{E}_{2n,r2} \frac{1}{2\pi} J_0(kR_2) \qquad (B.16)$$

Note that the Hankel transform for the moments given by Eqs (B.3 to B.16) have to be calculated consecutively for $n = 0$, 1, 2.

APPENDIX C

Hankel transforms of the moments

The Hankel transform of z-axial moments of neutron fluxes has been defined in Eq. (23). Here the transforms are given explicitly for all n, i and j needed in the calculations. When Eqs (B.3 to B.16) are inserted consecutively into Eq. (23), the following transforms of the moments are obtained:

A. For n=0 there are 14 moments:

$$\tilde{m}_0(\varphi^{\circ}_{b1};k) = \frac{1}{2\pi D_{b1}} \frac{1}{k^2+\lambda^2_{b1}} \tag{C.1}$$

$$\tilde{m}_0(\varphi^{\bullet}_{b1};k) = \frac{1}{2\pi D_{b1}} B_{0,b1} \frac{J_0(kR_1)}{k^2+\lambda^2_{b1}} \tag{C.2}$$

$$\tilde{m}_0(\varphi^{\bullet}_{c1};k) = \frac{1}{2\pi D_{c1}} C_{0,c1} \frac{J_0(kR_1)}{k^2+\lambda^2_{c1}} \tag{C.3}$$

$$\tilde{m}_0(\varphi^{\bullet}_{d1};k) = \frac{1}{2\pi D_{c1}} D_{0,c1} \frac{J_0(kR_2)}{k^2+\lambda^2_{c1}} \tag{C.4}$$

$$\tilde{m}_0(\varphi^{\bullet}_{r1};k) = \frac{1}{2\pi D_{r1}} E_{0,r1} \frac{J_0(kR_2)}{k^2+\lambda^2_{r1}} \tag{C.5}$$

$$\tilde{m}_0(\varphi^{\circ}_{b2};k) = \frac{\lambda^2_{b1}}{2\pi D_{b2}} \frac{1}{\left(k^2+\lambda^2_{b1}\right)\left(k^2+\lambda^2_{b2}\right)} = \frac{\lambda^2_{b1}}{2\pi D_{b2}} \frac{1}{\lambda^2_{b2}-\lambda^2_{b1}} \left(\frac{1}{k^2+\lambda^2_{b1}} - \frac{1}{k^2+\lambda^2_{b2}}\right) \tag{C.6}$$

$$\tilde{m}_0(\varphi^{\bullet}_{b2};k) = \frac{\lambda^2_{b1}}{2\pi D_{b2}} B_{0,b1} \frac{J_0(kR_1)}{\left(k^2+\lambda^2_{b1}\right)\left(k^2+\lambda^2_{b2}\right)} = \frac{\lambda^2_{b1}}{2\pi D_{b2}} B_{0,b1} \frac{1}{\lambda^2_{b2}-\lambda^2_{b1}} \left[\frac{J_0(kR_1)}{k^2+\lambda^2_{b1}} - \frac{J_0(kR_1)}{k^2+\lambda^2_{b2}}\right] \tag{C.7}$$

$$\tilde{m}_0(\varphi_{b2}^{\bullet\bullet};k) = \frac{1}{2\pi D_{b2}} B_{0,b2} \frac{J_0(kR_1)}{k^2+\lambda_{b2}^2} \tag{C.8}$$

$$\tilde{m}_0(\varphi_{c2}^{\bullet};k) = \frac{\lambda_{c1}^2}{2\pi D_{c2}} C_{0,c1} \frac{J_0(kR_1)}{\left(k^2+\lambda_{c1}^2\right)\left(k^2+\lambda_{c2}^2\right)} = \frac{\lambda_{c1}^2}{2\pi D_{c2}} C_{0,c1} \frac{1}{\lambda_{c2}^2-\lambda_{c1}^2}\left[\frac{J_0(kR_1)}{k^2+\lambda_{c1}^2} - \frac{J_0(kR_1)}{k^2+\lambda_{c2}^2}\right] \tag{C.9}$$

$$\tilde{m}_0(\varphi_{d2}^{\bullet};k) = \frac{\lambda_{c1}^2}{2\pi D_{c2}} D_{0,c1} \frac{J_0(kR_2)}{\left(k^2+\lambda_{c1}^2\right)\left(k^2+\lambda_{c2}^2\right)} = \frac{\lambda_{c1}^2}{2\pi D_{c2}} D_{0,c1} \frac{1}{\lambda_{c2}^2-\lambda_{c1}^2}\left[\frac{J_0(kR_2)}{k^2+\lambda_{c1}^2} - \frac{J_0(kR_2)}{k^2+\lambda_{c2}^2}\right] \tag{C.10}$$

$$\tilde{m}_0(\varphi_{c2}^{\bullet\bullet};k) = \frac{1}{2\pi D_{c2}} C_{0,c2} \frac{J_0(kR_1)}{k^2+\lambda_{c2}^2} \tag{C.11}$$

$$\tilde{m}_0(\varphi_{d2}^{\bullet\bullet};k) = \frac{1}{2\pi D_{c2}} D_{0,c2} \frac{J_0(kR_2)}{k^2+\lambda_{c2}^2} \tag{C.12}$$

$$\tilde{m}_0(\varphi_{r2}^{\bullet};k) = \frac{\lambda_{r1}^2}{2\pi D_{r2}} E_{0,r1} \frac{J_0(kR_2)}{\left(k^2+\lambda_{r1}^2\right)\left(k^2+\lambda_{r2}^2\right)} = \frac{\lambda_{r1}^2}{2\pi D_{r2}} E_{0,r1} \frac{1}{\lambda_{r2}^2-\lambda_{r1}^2}\left[\frac{J_0(kR_2)}{k^2+\lambda_{r1}^2} - \frac{J_0(kR_2)}{k^2+\lambda_{r2}^2}\right] \tag{C.13}$$

$$\tilde{m}_0(\varphi_{r2}^{\bullet\bullet};k) = \frac{1}{2\pi D_{r2}} E_{0,r2} \frac{J_0(kR_2)}{k^2+\lambda_{r2}^2} \tag{C.14}$$

B. For $n=1$ there are also 14 transforms of moments:

$$\tilde{m}_2(\varphi_{b1}^{o};k) = \frac{2}{2\pi D_{b1}} \frac{1}{\left(k^2+\lambda_{b1}^2\right)^2} \tag{C.15}$$

$$\tilde{m}_2(\overset{\bullet}{\varphi}_{b1},k) = \frac{1}{2\pi D_{b1}} B_{2,b1} \frac{J_0(kR_1)}{k^2+\lambda_{b1}^2} + \frac{2}{2\pi D_{b1}} B_{0,b1} \frac{J_0(kR_1)}{\left(k^2+\lambda_{b1}^2\right)^2} \tag{C.16}$$

$$\tilde{m}_2(\overset{\bullet}{\varphi}_{c1},k) = \frac{1}{2\pi D_{c1}} C_{2,c1} \frac{J_0(kR_1)}{k^2+\lambda_{c1}^2} + \frac{2}{2\pi D_{c1}} C_{0,c1} \frac{J_0(kR_1)}{\left(k^2+\lambda_{c1}^2\right)^2} \tag{C.17}$$

$$\tilde{m}_2(\overset{\bullet}{\varphi}_{d1},k) = \frac{1}{2\pi D_{c1}} D_{2,c1} \frac{J_0(kR_2)}{k^2+\lambda_{c1}^2} + \frac{2}{2\pi D_{c1}} D_{0,c1} \frac{J_0(kR_2)}{\left(k^2+\lambda_{c1}^2\right)^2} \tag{C.18}$$

$$\tilde{m}_2(\overset{\bullet}{\varphi}_{r1},k) = \frac{1}{2\pi D_{r1}} E_{2,r1} \frac{J_0(kR_2)}{k^2+\lambda_{r1}^2} + \frac{2}{2\pi D_{r1}} E_{0,r1} \frac{J_0(kR_2)}{\left(k^2+\lambda_{r1}^2\right)^2} \tag{C.19}$$

$$\tilde{m}_2(\overset{\circ}{\varphi}_{b2};k) = \frac{2\lambda_{b1}^2}{2\pi D_{b2}}\left[\frac{1}{\left(k^2+\lambda_{b1}^2\right)^2\left(k^2+\lambda_{b2}^2\right)} + \frac{1}{\left(k^2+\lambda_{b1}^2\right)\left(k^2+\lambda_{b2}^2\right)^2}\right] = \frac{2\lambda_{b1}^2}{2\pi D_{b2}} \frac{1}{\lambda_{b2}^2-\lambda_{b1}^2}\left[\frac{1}{\left(k^2+\lambda_{b1}^2\right)^2} - \frac{1}{\left(k^2+\lambda_{b2}^2\right)^2}\right] \tag{C.20}$$

$$\tilde{m}_2(\overset{\bullet}{\varphi}_{b2};k) = \frac{\lambda_{b1}^2}{2\pi D_{b2}} B_{2,b1} \frac{J_0(kR_1)}{\left(k^2+\lambda_{b1}^2\right)\left(k^2+\lambda_{b2}^2\right)} + \frac{2\lambda_{b1}^2}{2\pi D_{b2}} B_{0,b1} \frac{J_0(kR_1)}{\left(k^2+\lambda_{b1}^2\right)^2\left(k^2+\lambda_{b2}^2\right)} + \frac{2\lambda_{b1}^2}{2\pi D_{b2}} B_{0,b1} \frac{J_0(kR_1)}{\left(k^2+\lambda_{b1}^2\right)\left(k^2+\lambda_{b2}^2\right)^2} =$$

$$= \frac{\lambda_{b1}^2}{2\pi D_{b2}} \frac{1}{\lambda_{b2}^2-\lambda_{b1}^2}\left\{B_{2,b1}\left[\frac{J_0(kR_1)}{k^2+\lambda_{b1}^2} - \frac{J_0(kR_1)}{k^2+\lambda_{b2}^2}\right] + 2B_{0,b1}\left[\frac{J_0(kR_1)}{\left(k^2+\lambda_{b1}^2\right)^2} - \frac{J_0(kR_1)}{\left(k^2+\lambda_{b2}^2\right)^2}\right]\right\} \tag{C.21}$$

$$\tilde{m}_2(\ddot{\varphi}_{b2},k) = \frac{1}{2\pi D_{b2}} \mathrm{B}_{2,b2} \frac{J_0(kR_1)}{k^2+\lambda_{b2}^2} + \frac{2}{2\pi D_{b2}} \mathrm{B}_{0,b2} \frac{J_0(kR_1)}{\left(k^2+\lambda_{b2}^2\right)^2} \tag{C.22}$$

$$\tilde{m}_2(\dot{\varphi}_{c2},k) = \frac{\lambda_{c1}^2}{2\pi D_{c2}} \frac{1}{\lambda_{c2}^2-\lambda_{c1}^2} \left\{ \mathrm{C}_{2,c1}\left[\frac{J_0(kR_1)}{k^2+\lambda_{c1}^2} - \frac{J_0(kR_1)}{k^2+\lambda_{c2}^2}\right] + 2\mathrm{C}_{0,c1}\left[\frac{J_0(kR_1)}{\left(k^2+\lambda_{c1}^2\right)^2} - \frac{J_0(kR_1)}{\left(k^2+\lambda_{c2}^2\right)^2}\right]\right\} \tag{C.23}$$

$$\tilde{m}_2(\dot{\varphi}_{d2},k) = \frac{\lambda_{c1}^2}{2\pi D_{c2}} \frac{1}{\lambda_{c2}^2-\lambda_{c1}^2} \left\{ \mathrm{D}_{2,c1}\left[\frac{J_0(kR_2)}{k^2+\lambda_{c1}^2} - \frac{J_0(kR_2)}{k^2+\lambda_{c2}^2}\right] + 2\mathrm{D}_{0,c1}\left[\frac{J_0(kR_2)}{\left(k^2+\lambda_{c1}^2\right)^2} - \frac{J_0(kR_2)}{\left(k^2+\lambda_{c2}^2\right)^2}\right]\right\} \tag{C.24}$$

$$\tilde{m}_2(\ddot{\varphi}_{c2},k) = \frac{1}{2\pi D_{c2}}\left[\mathrm{C}_{2,c2}\frac{J_0(kR_1)}{k^2+\lambda_{c2}^2} + 2\mathrm{C}_{0,c2}\frac{J_0(kR_1)}{\left(k^2+\lambda_{c2}^2\right)^2}\right] \tag{C.25}$$

$$\tilde{m}_2(\ddot{\varphi}_{d2},k) = \frac{1}{2\pi D_{c2}}\left[\mathrm{D}_{2,c2}\frac{J_0(kR_2)}{k^2+\lambda_{c2}^2} + 2\mathrm{D}_{0,c2}\frac{J_0(kR_2)}{\left(k^2+\lambda_{c2}^2\right)^2}\right] \tag{C.26}$$

$$\tilde{m}_2(\dot{\varphi}_{r2},k) = \frac{\lambda_{r1}^2}{2\pi D_{r2}} \frac{1}{\lambda_{r2}^2-\lambda_{r1}^2} \left\{ \mathrm{E}_{2,r1}\left[\frac{J_0(kR_2)}{k^2+\lambda_{r1}^2} - \frac{J_0(kR_2)}{k^2+\lambda_{r2}^2}\right] + 2\mathrm{E}_{0,r1}\left[\frac{J_0(kR_2)}{\left(k^2+\lambda_{r1}^2\right)^2} - \frac{J_0(kR_2)}{\left(k^2+\lambda_{r2}^2\right)^2}\right]\right\} \tag{C.27}$$

$$\tilde{m}_2(\varphi_{r2}^{\bullet\bullet};k) = \frac{1}{2\pi D_{r2}}\left[E_{2,r2}\frac{J_0(kR_2)}{k^2+\lambda_{r2}^2} + 2E_{0,r2}\frac{J_0(kR_2)}{\left(k^2+\lambda_{r2}^2\right)^2}\right] \quad (C.28)$$

C. For n=2 there are 14 following transforms of moments:

$$\tilde{m}_4(\varphi_{b1}^{o};k) = \frac{24}{2\pi D_{b1}}\frac{1}{\left(k^2+\lambda_{b1}^2\right)^3} \quad (C.29)$$

$$\tilde{m}_4(\varphi_{b1}^{\bullet};k) = \frac{1}{2\pi D_{b1}}\left[B_{4,b1}\frac{J_0(kR_1)}{k^2+\lambda_{b1}^2} + 12B_{2,b1}\frac{J_0(kR_1)}{\left(k^2+\lambda_{b1}^2\right)^2} + 24B_{0,b1}\frac{J_0(kR_1)}{\left(k^2+\lambda_{b1}^2\right)^3}\right] \quad (C.30)$$

$$\tilde{m}_4(\varphi_{c1}^{\bullet};k) = \frac{1}{2\pi D_{c1}}\left[C_{4,c1}\frac{J_0(kR_1)}{k^2+\lambda_{c1}^2} + 12C_{2,c1}\frac{J_0(kR_1)}{\left(k^2+\lambda_{c1}^2\right)^2} + 24C_{0,c1}\frac{J_0(kR_1)}{\left(k^2+\lambda_{c1}^2\right)^3}\right] \quad (C.31)$$

$$\tilde{m}_4(\varphi_{d1}^{\bullet},k) = \frac{1}{2\pi D_{c1}}\left[D_{4,c1}\frac{J_0(kR_2)}{k^2+\lambda_{c1}^2} + 12D_{2,c1}\frac{J_0(kR_2)}{\left(k^2+\lambda_{c1}^2\right)^2} + 24D_{0,c1}\frac{J_0(kR_2)}{\left(k^2+\lambda_{c1}^2\right)^3}\right] \quad (C.32)$$

$$\tilde{m}_4(\overset{\bullet}{\varphi}_{r1},k) = \frac{1}{2\pi D_{r1}}\left[\mathrm{E}_{4,r1}\frac{J_0(kR_2)}{k^2+\lambda_{r1}^2} + 12\mathrm{E}_{2,r1}\frac{J_0(kR_2)}{\left(k^2+\lambda_{r1}^2\right)^2} + 24\mathrm{E}_{0,r1}\frac{J_0(kR_2)}{\left(k^2+\lambda_{r1}^2\right)^3}\right] \tag{C.33}$$

$$\tilde{m}_4(\overset{\circ}{\varphi}_{b2},k) = \frac{24\lambda_{b1}^2}{2\pi D_{b2}}\,\frac{1}{\lambda_{b2}^2-\lambda_{b1}^2}\left[\frac{1}{\left(k^2+\lambda_{b1}^2\right)^3} - \frac{1}{\left(k^2+\lambda_{b2}^2\right)^3}\right] \tag{C.34}$$

$$\tilde{m}_4(\overset{\bullet}{\varphi}_{b2},k) = \frac{\lambda_{b1}^2}{2\pi D_{b2}}\,\frac{1}{\lambda_{b2}^2-\lambda_{b1}^2}\left\{\mathrm{B}_{4,b1}\left[\frac{J_0(kR_1)}{k^2+\lambda_{b1}^2} - \frac{J_0(kR_1)}{k^2+\lambda_{b2}^2}\right] + 12\mathrm{B}_{2,b1}\left[\frac{J_0(kR_1)}{\left(k^2+\lambda_{b1}^2\right)^2} - \frac{J_0(kR_1)}{\left(k^2+\lambda_{b2}^2\right)^2}\right] + 24\mathrm{B}_{0,b1}\left[\frac{J_0(kR_1)}{\left(k^2+\lambda_{b1}^2\right)^3} - \frac{J_0(kR_1)}{\left(k^2+\lambda_{b2}^2\right)^3}\right]\right\} \tag{C.35}$$

$$\tilde{m}_4(\overset{\bullet\bullet}{\varphi}_{b2},k) = \frac{1}{2\pi D_{b2}}\left[\mathrm{B}_{4,b2}\frac{J_0(kR_1)}{k^2+\lambda_{b2}^2} + 12\mathrm{B}_{2,b2}\frac{J_0(kR_1)}{\left(k^2+\lambda_{b2}^2\right)^2} + 24\mathrm{B}_{0,b2}\frac{J_0(kR_1)}{\left(k^2+\lambda_{b2}^2\right)^3}\right] \tag{C.36}$$

$$\tilde{m}_4(\overset{\bullet}{\varphi}_{c2},k) = \frac{\lambda_{c1}^2}{2\pi D_{c2}}\,\frac{1}{\lambda_{c2}^2-\lambda_{c1}^2}\left\{\mathrm{C}_{4,c1}\left[\frac{J_0(kR_1)}{k^2+\lambda_{c1}^2} - \frac{J_0(kR_1)}{k^2+\lambda_{c2}^2}\right] + 12\mathrm{C}_{2,c1}\left[\frac{J_0(kR_1)}{\left(k^2+\lambda_{c1}^2\right)^2} - \frac{J_0(kR_1)}{\left(k^2+\lambda_{c2}^2\right)^2}\right] + 24\mathrm{C}_{0,c1}\left[\frac{J_0(kR_1)}{\left(k^2+\lambda_{c1}^2\right)^3} - \frac{J_0(kR_1)}{\left(k^2+\lambda_{c2}^2\right)^3}\right]\right\} \tag{C.37}$$

$$\tilde{m}_4(\dot{\varphi}_{d2};k)=\frac{\lambda_{c1}^2}{2\pi D_{c2}}\frac{1}{\lambda_{c2}^2-\lambda_{c1}^2}\left\{D_{4,c1}\left[\frac{J_0(kR_2)}{k^2+\lambda_{c1}^2}-\frac{J_0(kR_2)}{k^2+\lambda_{c2}^2}\right]+12D_{2,c1}\left[\frac{J_0(kR_2)}{\left(k^2+\lambda_{c1}^2\right)^2}-\frac{J_0(kR_2)}{\left(k^2+\lambda_{c2}^2\right)^2}\right]+24D_{0,c1}\left[\frac{J_0(kR_2)}{\left(k^2+\lambda_{c1}^2\right)^3}-\frac{J_0(kR_2)}{\left(k^2+\lambda_{c2}^2\right)^3}\right]\right\} \tag{C.38}$$

$$\tilde{m}_4(\ddot{\varphi}_{c2};k)=\frac{1}{2\pi D_{c2}}\left[C_{4,c2}\frac{J_0(kR_1)}{k^2+\lambda_{c2}^2}+12C_{2,c2}\frac{J_0(kR_1)}{\left(k^2+\lambda_{c2}^2\right)^2}+24C_{0,c2}\frac{J_0(kR_1)}{\left(k^2+\lambda_{c2}^2\right)^3}\right] \tag{C.39}$$

$$\tilde{m}_4(\ddot{\varphi}_{d2};k)=\frac{1}{2\pi D_{c2}}\left[D_{4,c2}\frac{J_0(kR_2)}{k^2+\lambda_{c2}^2}+12D_{2,c2}\frac{J_0(kR_2)}{\left(k^2+\lambda_{c2}^2\right)^2}+24D_{0,c2}\frac{J_0(kR_2)}{\left(k^2+\lambda_{c2}^2\right)^3}\right] \tag{C.40}$$

$$\tilde{m}_4(\dot{\varphi}_{r2};k)=\frac{\lambda_{r1}^2}{2\pi D_{r2}}\frac{1}{\lambda_{r2}^2-\lambda_{r1}^2}\left\{E_{4,r1}\left[\frac{J_0(kR_2)}{k^2+\lambda_{r1}^2}-\frac{J_0(kR_2)}{k^2+\lambda_{r2}^2}\right]+12E_{2,r1}\left[\frac{J_0(kR_2)}{\left(k^2+\lambda_{r1}^2\right)^2}-\frac{J_0(kR_2)}{\left(k^2+\lambda_{r2}^2\right)^2}\right]+24E_{0,r1}\left[\frac{J_0(kR_2)}{\left(k^2+\lambda_{r1}^2\right)^3}-\frac{J_0(kR_2)}{\left(k^2+\lambda_{r2}^2\right)^3}\right]\right\} \tag{C.41}$$

$$\tilde{m}_4(\ddot{\varphi}_{r2};k)=\frac{1}{2\pi D_{r2}}\left[E_{4,r2}\frac{J_0(kR_2)}{k^2+\lambda_{r2}^2}+12E_{2,r2}\frac{J_0(kR_2)}{\left(k^2+\lambda_{r2}^2\right)^2}+24E_{0,c2}\frac{J_0(kR_2)}{\left(k^2+\lambda_{r2}^2\right)^3}\right] \tag{C.42}$$

APPENDIX D

Some useful definite integrals containing Bessel functions

Here some definite integrals containing Bessel functions are given together with their first derivatives with respect to the radial co-ordinate r. For these integrals some special notation is used.

Integrals containing one Bessel function $J_0(kr)$:

$$J_1(r,\lambda) = \int_0^\infty \frac{k\,J_0(kr)}{k^2+\lambda^2}\,dk = K_0(\lambda r) \tag{D.1}$$

$$J_2(r,\lambda) = \int_0^\infty \frac{k\,J_0(kr)}{\left(k^2+\lambda^2\right)^2}\,dk = \frac{1}{2}\frac{r}{\lambda}K_1(\lambda r) \tag{D.2}$$

$$J_3(r,\lambda) = \int_0^\infty \frac{k\,J_0(kr)}{\left(k^2+\lambda^2\right)^3}\,dk = \frac{1}{8}\left(\frac{r}{\lambda}\right)^2 K_2(\lambda r) \tag{D.3}$$

and in general (BATEMAN, 1953, p. 96, Eq. 59):

$$J_{n+1}(r,\lambda) = \int_0^\infty \frac{k\,J_0(kr)}{\left(k^2+\lambda^2\right)^{n+1}}\,dk = \frac{1}{2^n n!}\left(\frac{r}{\lambda}\right)^n K_n(r\lambda) \tag{D.4}$$

Definite integrals containing Bessel functions $J_0(kr)$ and $J_0(kR)$:

Integrals of the type

$$I_n(r,R,\lambda) = \int_0^\infty \frac{k\,J_0(kr)\,J_0(kR)}{\left(k^2+\lambda^2\right)^n}\,dk \tag{D.5}$$

have to be calculated depending on the condition $r<R$ or $r>R$ and they are marked in the text as $I_n(r<R,\lambda)$ or $I_n(r>R,\lambda)$ according to the case. From BATEMAN (1953, p. 96, Eq. 57) there is:

$$I_1(r,R,\lambda) = \int_0^\infty \frac{k\,J_0(kr)\,J_0(kR)}{k^2+\lambda^2}\,dk = \begin{cases} I_0(\lambda r)K_0(\lambda R), & r<R \\ I_0(\lambda R)K_0(\lambda r), & r>R \end{cases} \tag{D.6}$$

The other integrals (*i.e.* for $n>1$ in Eq.D.5) are not tabulated. They have to be calculated by consecutive derivation of the integral $I_1(r,R,\lambda)$ with respect to the parameter λ:

$$I_{n+1}(r,R,\lambda) = -\frac{1}{2\lambda}\frac{\partial}{\partial\lambda} I_n(r,R,\lambda) \tag{D.7}$$

which gives

$$I_2(r,R,\lambda) = \int_0^\infty \frac{k\,J_0(kr)\,J_0(kR)}{\left(k^2+\lambda^2\right)^2}\,\mathrm{d}k = \begin{cases} \frac{1}{2\lambda^2}\left[\lambda R\ I_0(\lambda r)\,K_1(\lambda R) - \lambda r\ I_1(\lambda r)\,K_0(\lambda R)\right], & r < R \\ \frac{1}{2\lambda^2}\left[\lambda r\ I_0(\lambda R)\,K_1(\lambda r) - \lambda R\ I_1(\lambda R)\,K_0(\lambda r)\right], & r > R \end{cases} \tag{D.8}$$

$$I_3(r,R,\lambda) = \int_0^\infty \frac{k\,J_0(kr)\,J_0(kR)}{\left(k^2+\lambda^2\right)^3}\,\mathrm{d}k =$$

$$= \begin{cases} \frac{1}{4\lambda^4}\left[\ \lambda R\ I_0(\lambda r)\,K_1(\lambda R) - \lambda r\ I_1(\lambda r)\,K_0(\lambda R) - \lambda r\ \lambda R\ I_1(\lambda r)\,K_1(\lambda R) + I_0(\lambda r)\,K_0(\lambda R)\frac{(\lambda r)^2+(\lambda R)^2}{2}\ \right], & r < R \\ \frac{1}{4\lambda^4}\left[\ \lambda r\ I_0(\lambda R)\,K_1(\lambda r) - \lambda R\ I_1(\lambda R)\,K_0(\lambda r) - \lambda r\ \lambda R\ I_1(\lambda R)\,K_1(\lambda r) + I_0(\lambda R)\,K_0(\lambda r)\frac{(\lambda r)^2+(\lambda R)^2}{2}\ \right], & r > R \end{cases} \tag{D.9}$$

First derivatives of the integrals $J_n(r,\lambda)$ with respect to r:

$$J_1'(r,\lambda) = -\lambda\, K_1(\lambda r) \tag{D.10}$$

$$J_2'(r,\lambda) = -\frac{1}{2}\, r\, K_0(\lambda r) \tag{D.11}$$

$$J_3'(r,\lambda) = -\frac{1}{8}\frac{1}{\lambda}\, r^2 K_1(\lambda r) \tag{D.12}$$

First derivatives of the integrals $I_n(r,R,\lambda)$ with respect to r:

$$I_1'(r < R,\lambda) = \lambda\, I_1(\lambda r)\, K_0(\lambda R) \tag{D.13}$$

$$I_1'(r > R,\lambda) = -\lambda\, I_0(\lambda R)\, K_1(\lambda r) \tag{D.14}$$

$$I_2'(r < R,\lambda) = \frac{1}{2}\left[R\, I_1(\lambda r)\, K_1(\lambda R) - r\, I_0(\lambda r)\, K_0(\lambda R)\right] \tag{D.15}$$

$$I_2'(r > R,\lambda) = \frac{1}{2}\left[R\, I_1(\lambda R)\, K_1(\lambda r) - r\, I_0(\lambda R)\, K_0(\lambda r)\right] \tag{D.16}$$

$$I_3'(r < R,\lambda) = \frac{1}{4\lambda^3}\left[\lambda R\, I_1(\lambda r)\, K_1(\lambda R) - \lambda R\, \lambda r\, I_0(\lambda r)\, K_1(\lambda R) + \frac{(\lambda r)^2 + (\lambda R)^2}{2} I_1(\lambda r)\, K_0(\lambda R)\right] \tag{D.17}$$

$$I_3'(r > R,\lambda) = \frac{1}{4\lambda^3}\left[\lambda R\, I_1(\lambda R)\, K_1(\lambda r) + \lambda R\, \lambda r\, I_1(\lambda R)\, K_0(\lambda r) - \frac{(\lambda r)^2 + (\lambda R)^2}{2} I_0(\lambda R)\, K_1(\lambda r)\right] \tag{D.18}$$

Special values of integrals $J_n(r,\lambda)$ for $r = 0$:

Here the general formula is used:

$$\lim_{x\to 0} x^n K_n(x) = (n-1)!\; 2^{n-1}$$

which gives

$$J_2(0,\lambda) = \frac{1}{2\lambda^2} \tag{D.19}$$

$$J_3(0,\lambda) = \frac{1}{4\lambda^4} \tag{D.20}$$

Special values of integrals $I_n(r,R,\lambda)$ for $r = 0$:

$$I_1(0,R,\lambda) = K_0(\lambda R) \tag{D.21}$$

$$I_2(0,R,\lambda) = \frac{1}{2\lambda^2}\lambda R\, K_1(\lambda R) \tag{D.22}$$

$$I_3(0,R,\lambda) = \frac{1}{8\lambda^4}(\lambda R)^2 K_2(\lambda R) \qquad \text{(D.23)}$$

Special values of integrals $I_n(r,R,\lambda)$ for $r=R$:

$$I_1(R,R,\lambda) = I_0(\lambda R)\, K_0(\lambda R) \qquad \text{(D.24)}$$

$$I_2(R,R,\lambda) = \frac{\lambda R}{2\lambda^2}\left[I_0(\lambda R)\, K_1(\lambda R) - I_1(\lambda R)\, K_0(\lambda R)\right] \qquad \text{(D.25)}$$

$$I_3(R,R,\lambda) = \frac{1}{4\lambda^4}\left\{2\lambda^2\, I_2(R,R,\lambda) - (\lambda R)^2\left[I_1(\lambda R)\, K_1(\lambda R) - I_0(\lambda R)\, K_0(\lambda R)\right]\right\} \qquad \text{(D.26)}$$

Special values of the first derivatives of the integrals $I_n'(r,R,\lambda)$ for $r = R$:

$$I_1'(R,R,\lambda) = \begin{cases} \lambda\, I_1(\lambda R)\, K_0(\lambda R), & r < R \\ -\lambda\, I_0(\lambda R)\, K_1(\lambda R), & r > R \end{cases} \qquad \text{(D.27)}$$

$$I_2'(R,R,\lambda) = \frac{R}{2}\left[I_1(\lambda R)\, K_1(\lambda R) - I_0(\lambda R)\, K_0(\lambda R)\right] \qquad \text{(D.28)}$$

$$I_3'(R,R,\lambda) = \begin{cases} \frac{1}{4\lambda^2}\left\{2\, I_2^{(1)}(R,R,\lambda) - R\left[1 - 2\lambda R\, I_1(\lambda R)\, K_0(\lambda R) - K_0(\lambda R)\, I_0(\lambda R)\right]\right\}, & r < R \\ \frac{1}{4\lambda^2}\left\{2\, I_2^{(1)}(R,R,\lambda) - R\left[1 - K_0(\lambda R)\, I_0(\lambda R)\right]\right\}, & r > R \end{cases} \qquad \text{(D.29)}$$

$$I_3^{(1)}(R,R,\lambda)\big|_{r>R} = \frac{1}{4\lambda^2}\left\{2\, I_2^{(1)}(R,R,\lambda) - R\left[1 - K_0(\lambda R)\, I_0(\lambda R)\right]\right\} \qquad \text{(D.30)}$$

APPENDIX E

Inverse Hankel transforms of the z-axial moments of partial fluxes

The Hankel transforms of moments given in Appendix C have to be inverted to the form depending on the radial co-ordinate r, *i.e.* one has to perform the inverse transform given by Eq. (12). Keeping the notation of integrals given in Appendix D the following partial moments are obtained:

A. For $n = 0$ *the following inverse transforms one obtains from Eq.* (12) *for the transforms given by Eqs* (C.1 *to* C.14):

$$m_0(\varphi^{o}_{b1};r) = \frac{1}{2\pi D_{b1}} J_1(r,\lambda_{b1}) \quad \text{(E.1)}$$

$$m_0(\varphi^{\bullet}_{b1};r) = \frac{1}{2\pi D_{b1}} \mathrm{B}_{0,b1} I_1(r < R_1,\lambda_{b1}) \quad \text{(E.2)}$$

$$m_0(\varphi^{\bullet}_{c1};r) = \frac{1}{2\pi D_{c1}} \mathrm{C}_{0,c1} I_1(r > R_1,\lambda_{c1}) \quad \text{(E.3)}$$

$$m_0(\varphi^{\bullet}_{d1};r) = \frac{1}{2\pi D_{c1}} \mathrm{D}_{0,c1} I_1(r < R_2,\lambda_{c1}) \quad \text{(E.4)}$$

$$m_0(\varphi^{\bullet}_{r1};r) = \frac{1}{2\pi D_{r1}} \mathrm{E}_{0,r1} I_1(r > R_2,\lambda_{r1}) \quad \text{(E.5)}$$

$$m_0(\varphi^{o}_{b2};r) = \frac{\lambda^2_{b1}}{2\pi D_{b2}} \frac{1}{\lambda^2_{b2} - \lambda^2_{b1}} \left[J_1(r,\lambda_{b1}) - J_1(r,\lambda_{b2})\right] \quad \text{(E..6)}$$

$$m_0(\varphi^{\bullet}_{b2};r) = \frac{\lambda^2_{b1}}{2\pi D_{b2}} \mathrm{B}_{0,b1} \frac{1}{\lambda^2_{b2} - \lambda^2_{b1}} \left[I_1(r < R_1,\lambda_{b1}) - I_1(r < R_1,\lambda_{b2})\right]$$

(E.7)

$$m_0(\varphi^{\bullet\bullet}_{b2};r) = \frac{1}{2\pi D_{b2}} \mathrm{B}_{0,b2} I_1(r < R_1,\lambda_{b2}) \quad \text{(E.8)}$$

$$m_0(\varphi^{\bullet}_{c2};r) = \frac{\lambda^2_{c1}}{2\pi D_{c2}} \mathrm{C}_{0,c1} \frac{1}{\lambda^2_{c2} - \lambda^2_{c1}} \left[I_1(r > R_1,\lambda_{c1}) - I_1(r > R_1,\lambda_{c2})\right]$$

(E.9)

$$m_0(\varphi^{\bullet}_{d2};r) = \frac{\lambda^2_{c1}}{2\pi D_{c2}} \mathrm{D}_{0,c1} \frac{1}{\lambda^2_{c2} - \lambda^2_{c1}} \left[I_1(r < R_2,\lambda_{c1}) - I_1(r < R_2,\lambda_{c2})\right]$$

(E.10)

$$m_0(\ddot{\varphi}_{c2};r) = \frac{1}{2\pi D_{c2}} C_{0,c2}\, I_1(r > R_1, \lambda_{c2}) \tag{E.11}$$

$$m_0(\ddot{\varphi}_{d2};r) = \frac{1}{2\pi D_{c2}} D_{0,c2}\, I_1(r < R_2, \lambda_{c2}) \tag{E.12}$$

$$m_0(\dot{\varphi}_{r2};r) = \frac{\lambda_{r1}^2}{2\pi D_{r2}} E_{0,r1} \frac{1}{\lambda_{r2}^2 - \lambda_{r1}^2}\left[I_1(r > R_2, \lambda_{r1}) - I_1(r > R_2, \lambda_{r2})\right] \tag{E.13}$$

$$m_0(\ddot{\varphi}_{r2};r) = \frac{1}{2\pi D_{r2}} E_{0,r2}\, I_1(r > R_2, \lambda_{r2}) \tag{E.14}$$

B. For $n = 1$ the following inverse transforms one obtains from Eq. (12) *for the transforms given by Eqs* (C.15 *to* C.28):

$$m_2(\mathring{\varphi}_{b1};r) = \frac{2}{2\pi D_{b1}} J_2(r, \lambda_{b1}) \tag{E.15}$$

$$m_2(\dot{\varphi}_{b1};r) = \frac{1}{2\pi D_{b1}}\left[B_{2,b1}\, I_1(r < R_1, \lambda_{b1}) + 2B_{0,b1}\, I_2(r < R_1, \lambda_{b1})\right] \tag{E.16}$$

$$m_2(\dot{\varphi}_{c1};r) = \frac{1}{2\pi D_{c1}}\left[C_{2,c1}\, I_1(r > R_1, \lambda_{c1}) + 2C_{0,c1}\, I_2(r > R_1, \lambda_{c1})\right] \tag{E.17}$$

$$m_2(\dot{\varphi}_{d1};r) = \frac{1}{2\pi D_{c1}}\left[D_{2,c1}\, I_1(r < R_2, \lambda_{c1}) + 2D_{0,c1}\, I_2(r < R_2, \lambda_{c1})\right] \tag{E.18}$$

$$m_2(\dot{\varphi}_{r1};r) = \frac{1}{2\pi D_{r1}}\left[E_{2,r1}\, I_1(r > R_2, \lambda_{r1}) + 2E_{0,r1}\, I_2(r > R_2, \lambda_{r1})\right] \tag{E.19}$$

$$m_2(\mathring{\varphi}_{b2};r) = \frac{2\lambda_{b1}^2}{2\pi D_{b2}} \frac{1}{\lambda_{b2}^2 - \lambda_{b1}^2}\left[J_2(r, \lambda_{b1}) - J_2(r, \lambda_{b2})\right] \tag{E.20}$$

$$m_2(\dot{\varphi}_{b2};r) = \frac{\lambda_{b1}^2}{2\pi D_{b2}} \frac{1}{\lambda_{b2}^2 - \lambda_{b1}^2}\left\{B_{2,b1}\left[I_1(r < R_1, \lambda_{b1}) - I_1(r < R_1, \lambda_{b2})\right] + 2B_{0,b1}\left[I_2(r < R_1, \lambda_{b1}) - I_2(r < R_1, \lambda_{b2})\right]\right\} \tag{E.21}$$

$$m_2(\ddot{\varphi}_{b2};r) = \frac{1}{2\pi D_{b2}}\left[B_{2,b2}\, I_1(r < R_1, \lambda_{b2}) + 2B_{0,b2}\, I_2(r < R_1, \lambda_{b2})\right] \tag{E.22}$$

$$m_2(\varphi^{\bullet}_{c2};r) = \frac{\lambda^2_{c1}}{2\pi D_{c2}}\frac{1}{\lambda^2_{c2}-\lambda^2_{c1}}\left\{C_{2,c1}\left[I_1(r>R_1,\lambda_{c1})-I_1(r>R_1,\lambda_{c2})\right] + 2C_{0,c1}\left[I_2(r>R_1,\lambda_{c1})-I_2(r>R_1,\lambda_{c2})\right]\right\} \quad \text{(E.23)}$$

$$m_2(\varphi^{\bullet}_{d2};r) = \frac{\lambda^2_{c1}}{2\pi D_{c2}}\frac{1}{\lambda^2_{c2}-\lambda^2_{c1}}\left\{D_{2,c1}\left[I_1(r<R_2,\lambda_{c1})-I_1(r<R_2,\lambda_{c2})\right] + 2D_{0,c1}\left[I_2(r<R_2,\lambda_{c1})-I_2(r<R_2,\lambda_{c2})\right]\right\} \quad \text{(E.24)}$$

$$m_2(\varphi^{\bullet\bullet}_{c2};r) = \frac{1}{2\pi D_{c2}}\left[C_{2,c2}\,I_1(r>R_1,\lambda_{c2})+2C_{0,c2}\,I_2(r>R_1,\lambda_{c2})\right] \quad \text{(E.25)}$$

$$m_2(\varphi^{\bullet\bullet}_{d2};r) = \frac{1}{2\pi D_{c2}}\left[D_{2,c2}\,I_1(r<R_2,\lambda_{c2})+2D_{0,c2}\,I_2(r<R_2,\lambda_{c2})\right] \quad \text{(E.26)}$$

$$m_2(\varphi^{\bullet}_{r2};r) = \frac{\lambda^2_{r1}}{2\pi D_{r2}}\frac{1}{\lambda^2_{r2}-\lambda^2_{r1}}\left\{E_{2,r1}\left[I_1(r>R_2,\lambda_{r1})-I_1(r>R_2,\lambda_{r2})\right] + 2E_{0,r1}\left[I_2(r>R_2,\lambda_{r1})-I_2(r>R_2,\lambda_{r2})\right]\right\} \quad \text{(E.27)}$$

$$m_2(\varphi^{\bullet\bullet}_{r2};r) = \frac{1}{2\pi D_{r2}}\left[E_{2,r2}\,I_1(r>R_2,\lambda_{r2})+2E_{0,r2}\,I_2(r>R_2,\lambda_{r2})\right] \quad \text{(E.28)}$$

C. For $n = 2$ *the following inverse transforms one obtains from Eq.* (12) *for the transforms given by Eqs* (C.29 *to* C.42):

$$m_4(\varphi^{o}_{b1};r) = \frac{24}{2\pi D_{b1}}J_3(r,\lambda_{b1}) \quad \text{(E.29)}$$

$$m_4(\varphi^{\bullet}_{b1};r) = \frac{1}{2\pi D_{b1}}\left[B_{4,b1}\,I_1(r<R_1,\lambda_{b1})+12B_{2,b1}\,I_2(r<R_1,\lambda_{b1}) + 24B_{0,b1}\,I_3(r<R_1,\lambda_{b1})\right] \quad \text{(E.30)}$$

$$m_4(\varphi^{\bullet}_{c1};r) = \frac{1}{2\pi D_{c1}}\left[C_{4,c1}\,I_1(r>R_1,\lambda_{c1})+12C_{2,c1}\,I_2(r>R_1,\lambda_{c1}) + 24C_{0,c1}\,I_3(r>R_1,\lambda_{c1})\right] \quad \text{(E.31)}$$

$$m_4(\overset{\bullet}{\varphi}_{d1};r) = \frac{1}{2\pi D_{c1}}\left[\mathrm{D}_{4,c1}\, I_1(r < R_2, \lambda_{c1}) + 12\mathrm{D}_{2,c1}\, I_2(r < R_2, \lambda_{c1}) + 24\mathrm{D}_{0,c1}\, I_3(r < R_2, \lambda_{c1})\right] \tag{E.32}$$

$$m_4(\overset{\bullet}{\varphi}_{r1};r) = \frac{1}{2\pi D_{r1}}\left[\mathrm{E}_{4,r1}\, I_1(r > R_2, \lambda_{r1}) + 12\mathrm{E}_{2,r1}\, I_2(r > R_2, \lambda_{r1}) + 24\mathrm{E}_{0,r1}\, I_3(r > R_2, \lambda_{r1})\right] \tag{E.33}$$

$$m_4(\overset{\circ}{\varphi}_{b2};r) = \frac{24\lambda_{b1}^2}{2\pi D_{b2}}\ \frac{1}{\lambda_{b2}^2 - \lambda_{b1}^2}\left[J_3(r, \lambda_{b1}) - J_3(r, \lambda_{b2})\right] \tag{E.34}$$

$$m_4(\overset{\bullet}{\varphi}_{b2};r) = \frac{\lambda_{b1}^2}{2\pi D_{b2}}\ \frac{1}{\lambda_{b2}^2 - \lambda_{b1}^2}\left\{\mathrm{B}_{4,b1}\left[I_1(r < R_1, \lambda_{b1}) - I_1(r < R_1, \lambda_{b2})\right] + \right.$$

$$\left. + 12\mathrm{B}_{2,b1}\left[I_2(r < R_1, \lambda_{b1}) - I_2(r < R_1, \lambda_{b2})\right] + 24\mathrm{B}_{0,b1}\left[I_3(r < R_1, \lambda_{b1}) - I_3(r < R_1, \lambda_{b2})\right]\right\} \tag{E.35}$$

$$m_4(\overset{\bullet\bullet}{\varphi}_{b2};r) = \frac{1}{2\pi D_{b2}}\left[\mathrm{B}_{4,b2}\, I_1(r < R_1, \lambda_{b2}) + 12\mathrm{B}_{2,b2}\, I_2(r < R_1, \lambda_{b2}) + 24\mathrm{B}_{0,b2}\, I_3(r < R_1, \lambda_{b2})\right] \tag{E.36}$$

$$m_4(\overset{\bullet}{\varphi}_{c2};r) = \frac{\lambda_{c1}^2}{2\pi D_{c2}}\ \frac{1}{\lambda_{c2}^2 - \lambda_{c1}^2}\left\{\mathrm{C}_{4,c1}\left[I_1(r > R_1, \lambda_{c1}) - I_1(r > R_1, \lambda_{c2})\right] + \right.$$

$$\left. + 12\mathrm{C}_{2,c1}\left[I_2(r > R_1, \lambda_{c1}) - I_2(r > R_1, \lambda_{c2})\right] + 24\mathrm{C}_{0,c1}\left[I_3(r > R_1, \lambda_{c1}) - I_3(r > R_1, \lambda_{c2})\right]\right\} \tag{E.37}$$

$$m_4(\dot{\varphi}_{d2};r) = \frac{\lambda_{c1}^2}{2\pi D_{c2}} \frac{1}{\lambda_{c2}^2 - \lambda_{c1}^2} \Big\{ \mathrm{D}_{4,c1}\big[I_1(r < R_2, \lambda_{c1}) - I_1(r < R_2, \lambda_{c2}) \big] +$$

$$+ 12\mathrm{D}_{2,c1}\big[I_2(r < R_2, \lambda_{c1}) - I_2(r < R_2, \lambda_{c2}) \big] + 24\mathrm{D}_{0,c1}\big[I_3(r < R_2, \lambda_{c1}) - I_3(r < R_2, \lambda_{c2}) \big] \Big\} \tag{E.38}$$

$$m_4(\ddot{\varphi}_{c2};r) = \frac{1}{2\pi D_{c2}} \big[\mathrm{C}_{4,c2}\, I_1(r > R_1, \lambda_{c2}) + 12\mathrm{C}_{2,c2}\, I_2(r > R_1, \lambda_{c2}) + 24\mathrm{C}_{0,c2}\, I_3(r > R_1, \lambda_{c2}) \big] \tag{E.39}$$

$$m_4(\ddot{\varphi}_{d2};r) = \frac{1}{2\pi D_{c2}} \big[\mathrm{D}_{4,c2}\, I_1(r < R_2, \lambda_{c2}) + 12\mathrm{D}_{2,c2}\, I_2(r < R_2, \lambda_{c2}) + 24\mathrm{D}_{0,c2}\, I_3(r < R_2, \lambda_{c2}) \big] \tag{E.40}$$

$$m_4(\dot{\varphi}_{r2};r) = \frac{\lambda_{r1}^2}{2\pi D_{r2}} \frac{1}{\lambda_{r2}^2 - \lambda_{r1}^2} \Big\{ \mathrm{E}_{4,r1}\big[I_1(r > R_2, \lambda_{r1}) - I_1(r > R_2, \lambda_{r2}) \big] +$$

$$+ 12\mathrm{E}_{2,r1}\big[I_2(r > R_2, \lambda_{r1}) - I_2(r > R_2, \lambda_{r2}) \big] + 24\mathrm{E}_{0,r1}\big[I_3(r > R_2, \lambda_{r1}) - I_3(r > R_2, \lambda_{r2}) \big] \Big\} \tag{E.41}$$

$$m_4(\ddot{\varphi}_{r2};r) = \frac{1}{2\pi D_{r2}} \big[\mathrm{E}_{4,r2}\, I_1(r > R_2, \lambda_{r2}) + 12\mathrm{E}_{2,r2}\, I_2(r > R_2, \lambda_{r2}) + 24\mathrm{E}_{0,r2}\, I_3(r > R_2, \lambda_{r2}) \big] \tag{E.42}$$

APPENDIX F

Calculation of constants for the boundary conditions.

For the calculation of the z-axial moments after the formulae given in Appendix E one has to know first the constant numbers, $B_{2n,bj}$, $C_{2n,cj}$, $D_{2n,cj}$ and $E_{2n,rj}$ appearing there (cf. Eq.(B.1) for the meaning of these constants). First, using the boundary conditions given by Eqs (22), one has to find for $n = 0$ the constants, $B_{0,b1}$, $C_{0,c1}$, $D_{0,c1}$, $E_{0,r1}$, for the first neutron group, and then the constants, $B_{0,b2}$, $C_{0,c2}$, $D_{0,c2}$, $E_{0,r2}$, for the second neutron group.

Knowing these constants one can calculate from the same Eqs (22) for $n = 1$ the constants $B_{2,b1}$, $C_{2,c1}$, $D_{2,c1}$, $E_{2,r1}$ and $B_{2,b2}$, $C_{2,c2}$, $D_{2,c2}$, $E_{2,r2}$, and after that it is possible to calculate the constants $B_{4,b1}$, $C_{4,c1}$, $D_{4,c1}$, $E_{4,r1}$ and $B_{4,b2}$, $C_{4,c2}$, $D_{4,c2}$, $E_{4,r2}$. This is because for a given n in the linear Eqs(22) the constants obtained for $(n-1)$ enter as the already known parameters. When Eqs (22) are established for a given n the derivatives with respect to the variable r of the functions $J_n(r,\lambda)$ and $I_n(r,R,\lambda)$ are needed at $r = R_1$ and $r = R_2$. The reader can find all these functions and their derivatives in Appendix D. All equations which one has to solve are the set of linear equations with four unknowns. As an example the boundary Eqs (22) are written below for $n = 0$ and $\dot{j} = 1$:

$$m_0(\varphi^{\circ}_{b1};R_1) + m_0(\dot{\varphi}_{b1};R_1) = m_0(\dot{\varphi}_{c1};R_1) + m_0(\dot{\varphi}_{d1};R_1) \tag{F.1a}$$

$$D_{b1}\frac{\partial}{\partial r}\left[m_0(\varphi^{\circ}_{b1};r) + m_0(\dot{\varphi}_{b1};r)\right]\Big|_{r=R_1} = D_{c1}\frac{\partial}{\partial r}\left[m_0(\dot{\varphi}_{c1};r) + m_0(\dot{\varphi}_{d1};r)\right]\Big|_{r=R_1} \tag{F.1b}$$

$$m_0(\dot{\varphi}_{c1};R_2) + m_0(\dot{\varphi}_{d1};R_2) = m_0(\dot{\varphi}_{r1};R_2) \tag{F.1c}$$

$$D_{c1}\frac{\partial}{\partial r}\left[m_0(\dot{\varphi}_{c1};r) + m_0(\dot{\varphi}_{d1};r)\right]\Big|_{r=R_2} = D_{r1}\frac{\partial}{\partial r}\left[m_0(\dot{\varphi}_{r1};r)\right]\Big|_{r=R_2} \tag{F.1d}$$

Equation (F.1) has to be rewritten using the formulae (E.1) to (E.14) which gives:

$$\frac{1}{D_{b1}}\left[J_1(R_1,\lambda_{b1}) + B_{0,b1}\, I_1(R_1,R_1,\lambda_{b1})\right] = \frac{1}{D_{c1}}\left[C_{0,c1}\, I_1(R_1,R_1,\lambda_{c1}) + D_{0,c1}\, I_1(R_1,R_2,\lambda_{c1})\right] \tag{F.2a}$$

$$J_1'(R_1,\lambda_{b1}) + B_{0,b1}\, I_1'(R_1,R_1,\lambda_{b1})_{r<R_1} = C_{0,c1}\, I_1'(R_1,R_1,\lambda_{c1})_{r>R_1} + D_{0,c1}\, I_1'(R_1,R_2,\lambda_{c1})_{r<R_2} \tag{F.2b}$$

$$\frac{1}{D_{c1}}\left[C_{0,c1}\, I_1(R_2,R_1,\lambda_{c1}) + D_{0,c1}\, I_1(R_2,R_2,\lambda_{c1})\right] = \frac{1}{D_{r1}} E_{0,r1}\, I_1(R_2,R_2,\lambda_{r1}) \tag{F.2c}$$

$$C_{0,c1}\, I_1'(R_2,R_1,\lambda_{c1})_{r>R_1} + D_{0,c1}\, I_1'(R_2,R_2,\lambda_{c1})_{r<R_2} = E_{0,r1}\, I_1'(R_2,R_2,\lambda_{r1})_{r>R_2} \tag{F.2d}$$

where the derivation with respect to r is always going for the first variable of the function $I_1(r,R,\lambda)$. The set of Eqs (F.2) is reordered according to the sequence of the unknowns $B_{0,b1}$, $C_{0,c1}$, $D_{0,c1}$, $E_{0,r1}$, which gives

$$\left.\begin{array}{lllll}
\frac{I_1(R_1,R_1,\lambda_{b1})}{D_{b1}} B_{0,b1} & -\frac{I_1(R_1,R_1,\lambda_{c1})}{D_{c1}} C_{0,c1} & -\frac{I_1(R_1,R_2,\lambda_{c1})}{D_{c1}} D_{0,c1} & & = -\frac{J_1(R_1,\lambda_{b1})}{D_{b1}} \\
I_1'(R_1,R_1,\lambda_{b1})_{r<R_1} B_{0,b1} & -I_1'(R_1,R_1,\lambda_{c1})_{r>R_1} C_{0,c1} & -I_1'(R_1,R_2,\lambda_{c1})_{r<R_2} D_{0,c1} & & = -J_1'(R_1,\lambda_{b1}) \\
 & \frac{I_1(R_2,R_1,\lambda_{c1})}{D_{c1}} C_{0,c1} & +\frac{I_1(R_2,R_2,\lambda_{c1})}{D_{c1}} D_{0,c1} & -\frac{I_1(R_2,R_2,\lambda_{r1})}{D_{r1}} E_{0,r1} & = 0 \\
 & I_1'(R_2,R_1,\lambda_{c1})_{r>R_1} C_{0,c1} & +I_1'(R_2,R_2,\lambda_{c1})_{r<R_2} D_{0,c1} & -I_1'(R_2,R_2,\lambda_{r1})_{r>R_2} E_{0,r1} & = 0
\end{array}\right\} \tag{F.3}$$

$$
\begin{aligned}
{}^{2}b_{0,2} &= 2\pi\left\{-D_{b2}\frac{\partial}{\partial r}\Big[m_0(\varphi^{\circ}_{b2};r)+m_0(\varphi^{\bullet}_{b2};r)\Big]_{r=R_1} + D_{c2}\frac{\partial}{\partial r}\Big[m_0(\varphi^{\bullet}_{c2};r)+m_0(\varphi^{\bullet}_{d2};r)\Big]_{r=R_1}\right\} = \\
&= -\Lambda_b^2\Big\{J_1'(R_1,\lambda_{b1}) - J_1'(R_1,\lambda_{b2}) + B_{0,b1}\big[I_1'(R_1,R_1,\lambda_{b1}) - I_1'(R_1,R_1,\lambda_{b2})\big]_{r<R_1}\Big\} + \\
&\quad + \Lambda_c^2\Big\{C_{0,c1}\big[I_1'(R_1,R_1,\lambda_{c1}) - I_1'(R_1,R_1,\lambda_{c2})\big]_{r>R_1} + D_{0,c1}\big[I_1'(R_1,R_2,\lambda_{c1}) - I_1'(R_1,R_2,\lambda_{c2})\big]_{r<R_2}\Big\}
\end{aligned}
\tag{F.10b}
$$

$$
\begin{aligned}
{}^{3}b_{0,2} &= 2\pi\Big[-m_0(\varphi^{\bullet}_{c2};R_2) - m_0(\varphi^{\bullet}_{d2};R_2) + m_0(\varphi^{\bullet}_{r2};R_2)\Big] = \\
&= -\frac{1}{D_{c2}}\Lambda_c^2\Big\{C_{0,c1}\big[I_1(R_2,R_1,\lambda_{c1}) - I_1(R_2,R_1,\lambda_{c2})\big] + D_{0,c1}\big[I_1(R_2,R_2,\lambda_{c1}) - I_1(R_2,R_2,\lambda_{c2})\big]\Big\} + \\
&\quad + \frac{1}{D_{r2}}\Lambda_r^2 E_{0,r1}\big[I_1(R_2,R_2,\lambda_{r1}) - I_1(R_2,R_2,\lambda_{r2})\big]
\end{aligned}
\tag{F.10c}
$$

$$
\begin{aligned}
{}^{4}b_{0,2} &= 2\pi\left\{-D_{c2}\frac{\partial}{\partial r}\Big[m_0(\varphi^{\circ}_{c2};r)+m_0(\varphi^{\bullet}_{d2};r)\Big]_{r=R_2} + D_{r2}\frac{\partial}{\partial r}\Big[m_0(\varphi^{\bullet}_{r2};r)\Big]_{r=R_2}\right\} = \\
&= -\Lambda_c^2\Big\{C_{0,c1}\big[I_1'(R_2,R_1,\lambda_{c1}) - I_1'(R_2,R_1,\lambda_{c2})\big]_{r>R_1} + D_{0,c1}\big[I_1'(R_2,R_2,\lambda_{c1}) - I_1'(R_2,R_2,\lambda_{c2})\big]_{r<R_2}\Big\} + \\
&\quad + \Lambda_r^2 E_{0,r1}\big[I_1'(R_2,R_2,\lambda_{r1}) - I_1'(R_2,R_2,\lambda_{r2})\big]_{r>R_2}
\end{aligned}
\tag{F.10d}
$$

and the free terms are defined as:

for $n = 0$ and $j = 1$:

$$ {}^{1}b_{0,1} = -\frac{J_1(R_1,\lambda_{b1})}{D_{b1}} \qquad \text{(F.6a)} $$

$$ {}^{2}b_{0,1} = -J_1'(R_1,\lambda_{b1}) \qquad \text{(F.6b)} $$

$$ {}^{3}b_{0,1} = 0 \qquad \text{(F.6c)} $$

$$ {}^{4}b_{0,1} = 0 \qquad \text{(F.6d)} $$

for $2n = 2$ and $j = 1$:

$$ {}^{1}b_{2,1} = -\frac{2}{D_{b1}} J_2(R_1\lambda_{b1}) - \frac{2\mathrm{B}_{0,b1}}{D_{b1}} I_2(R_1,R_1,\lambda_{b1}) + \frac{2\mathrm{C}_{0,c1}}{D_{c1}} I_2(R_1,R_1,\lambda_{c1}) + \frac{2\mathrm{D}_{0,c1}}{D_{c1}} I_2(R_1,R_2,\lambda_{c1}) \qquad \text{(F.7a)} $$

$$ {}^{2}b_{2,1} = -2J_2'(R_1,\lambda_{b1}) - 2\mathrm{B}_{0,b1} I_2'(R_1,R_1,\lambda_{b1}) + 2\mathrm{C}_{0,c1} I_2'(R_1,R_1,\lambda_{c1}) + 2\mathrm{D}_{0,c1} I_2'(R_1,R_2,\lambda_{c1}) \qquad \text{(F.7b)} $$

$$ {}^{3}b_{2,1} = -\frac{2\mathrm{C}_{0,c1}}{D_{c1}} I_2(R_2,R_1,\lambda_{c1}) - \frac{2\mathrm{D}_{0,c1}}{D_{c1}} I_2(R_2,R_2,\lambda_{c1}) + \frac{2\mathrm{E}_{0,r1}}{D_{r1}} I_2(R_2,R_2,\lambda_{r1}) \qquad \text{(F.7c)} $$

$$ {}^{4}b_{2,1} = -2\mathrm{C}_{0,c1} I_2'(R_2,R_1,\lambda_{c1})_{r>R_1} - 2\mathrm{D}_{0,c1} I_2'(R_2,R_2,\lambda_{c1})_{r<R_2} + 2\mathrm{E}_{0,r1} I_2'(R_2,R_2,\lambda_{r1})_{r>R_2} \qquad \text{(F.7d)} $$

for $2n = 4$ and $j = 1$:

$$
{}^{1}b_{4,1} = -\frac{24}{D_{b1}} J_3(R_1,\lambda_{b1}) - \frac{12B_{2,b1}}{D_{b1}} I_2(R_1,R_1,\lambda_{b1}) - \frac{24B_{0,b1}}{D_{b1}} I_3(R_1,R_1,\lambda_{b1}) + \frac{12C_{2,c1}}{D_{c1}} I_2(R_1,R_1,\lambda_{c1}) + \frac{24C_{0,c1}}{D_{c1}} I_3(R_1,R_1,\lambda_{c1}) + \frac{12D_{2,c1}}{D_{c1}} I_2(R_1,R_2,\lambda_{c1}) + \frac{24D_{0,c1}}{D_{c1}} I_3(R_1,R_2,\lambda_{c1}) \tag{F.8a}
$$

$$
{}^{2}b_{4,1} = -24J_3'(R_1,\lambda_{b1}) - 12B_{2,b1}\, I_2'(R_1,R_1,\lambda_{b1})_{r<R_1} - 24B_{0,b1}\, I_3'(R_1,R_1,\lambda_{b1}) + 12C_{2,c1}\, I_2'(R_1,R_1,\lambda_{c1})_{r>R_1} + 24C_{0,c1}\, I_3'(R_1,R_1,\lambda_{c1})_{r>R_1} + 12D_{2,c1}\, I_2'(R_1,R_2,\lambda_{c1})_{r<R_2} + 24D_{0,c1}\, I_3'(R_1,R_2,\lambda_{c1})_{r<R_2} \tag{F.8b}
$$

$$
{}^{3}b_{4,1} = -\frac{12C_{2,c1}}{D_{c1}} I_2(R_2,R_1,\lambda_{c1}) - \frac{24C_{0,c1}}{D_{c1}} I_3(R_2,R_1,\lambda_{c1}) - \frac{12D_{2,c1}}{D_{c1}} I_2(R_2,R_2,\lambda_{c1}) - \frac{24D_{0,c1}}{D_{c1}} I_3(R_2,R_2,\lambda_{c1}) + \frac{12E_{2,r1}}{D_{r1}} I_2(R_2,R_2,\lambda_{r1}) + \frac{24E_{0,r1}}{D_{r1}} I_3(R_2,R_2,\lambda_{r1}) \tag{F.8c}
$$

$$
{}^{4}b_{4,1} = -12C_{2,c1}\, I_2'(R_2,R_1,\lambda_{c1})_{r>R_1} - 24C_{0,c1}\, I_3'(R_2,R_1,\lambda_{c1})_{r>R_1} - 12D_{2,c1}\, I_2'(R_2,R_2,\lambda_{c1}) - 24D_{0,c1}\, I_3'(R_2,R_2,\lambda_{c1})_{r<R_2} + 12E_{2,r1} I_2'(R_2,R_2,\lambda_{r1})_{r>R_2} + 24E_{0,r1}\, I_3'(R_2,R_2,\lambda_{r1})_{r>R_2} \tag{F.8d}
$$

For n = 0, 1, 2 and j = 2 the coefficients a_{kl} are of the form:

$$
\left[a_{kl}\right] = \begin{bmatrix}
\frac{1}{D_{b2}} I(R_1, R_1, \lambda_{b2}) & -\frac{1}{D_{c2}} I(R_1, R_1, \lambda_{c2}) & -\frac{1}{D_{c2}} I(R_1, R_2, \lambda_{c2}) & 0 \\
I_1'(R_1, R_1, \lambda_{b2})_{r<R_1} & -I_1'(R_1, R_1, \lambda_{c2})_{r>R_1} & -I_1'(R_1, R_2, \lambda_{c2})_{r<R_2} & 0 \\
0 & \frac{1}{D_{c2}} I(R_2, R_1, \lambda_{c2}) & \frac{1}{D_{c2}} I(R_2, R_2, \lambda_{c2}) & -\frac{1}{D_{r2}} I(R_2, R_2, \lambda_{r2}) \\
0 & I_1'(R_2, R_1, \lambda_{c2})_{r>R_1} & I_1'(R_2, R_2, \lambda_{c2})_{r<R_2} & -I_1'(R_2, R_2, \lambda_{r2})_{r>R_2}
\end{bmatrix} \tag{F.9}
$$

using the substitution $\Lambda_i^2 = \dfrac{\lambda_{i1}^2}{\lambda_{i2}^2 - \lambda_{i1}^2}$ $(i$ = b, c, r), the free terms are defined as:

for n = 0 and j = 2:

$$
\begin{aligned}
{}^1b_{0,2} &= 2\pi\left[-m_0(\varphi_{b2}^o, R_1) - m_0(\varphi_{b2}^\bullet; R_1) + m_0(\varphi_{c2}^\bullet; R_1) + m_0(\varphi_{d2}^\bullet; R_1)\right] = \\
&= -\frac{1}{D_{b2}} \Lambda_b^2 \left\{J_1(R_1, \lambda_{b1}) - J_1(R_1, \lambda_{b2}) + \mathrm{B}_{0,b1}\left[I_1(R_1, R_1, \lambda_{b1}) - I_1(R_1, R_1, \lambda_{b2})\right]\right\} + \\
&\quad + \frac{1}{D_{c2}} \Lambda_c^2 \left\{\mathrm{C}_{0,c1}\left[I_1(R_1, R_1, \lambda_{c1}) - I_1(R_1, R_1, \lambda_{c2})\right] + \mathrm{D}_{0,c1}\left[I_1(R_1, R_2, \lambda_{c1}) - I_1(R_1, R_2, \lambda_{c2})\right]\right\}
\end{aligned} \tag{F.10a}
$$

$$
\begin{aligned}
{}^{2}b_{0,2} &= 2\pi\left\{-D_{b2}\frac{\partial}{\partial r}\left[m_0(\varphi^{o}_{b2};r)+m_0(\varphi^{\bullet}_{b2};r)\right]_{r=R_1} + D_{c2}\frac{\partial}{\partial r}\left[m_0(\varphi^{\bullet}_{c2};r)+m_0(\varphi^{\bullet}_{d2};r)\right]_{r=R_1}\right\} = \\
&= -\Lambda_b^2\left\{J_1'(R_1,\lambda_{b1}) - J_1'(R_1,\lambda_{b2}) + B_{0,b1}\left[I_1'(R_1,R_1,\lambda_{b1}) - I_1'(R_1,R_1,\lambda_{b2})\right]_{r<R_1}\right\} + \\
&+ \Lambda_c^2\left\{C_{0,c1}\left[I_1'(R_1,R_1,\lambda_{c1}) - I_1'(R_1,R_1,\lambda_{c2})\right]_{r>R_1} + D_{0,c1}\left[I_1'(R_1,R_2,\lambda_{c1}) - I_1'(R_1,R_2,\lambda_{c2})\right]_{r<R_2}\right\}
\end{aligned}
\tag{F.10b}
$$

$$
\begin{aligned}
{}^{3}b_{0,2} &= 2\pi\left[-m_0(\varphi^{\bullet}_{c2};R_2) - m_0(\varphi^{\bullet}_{d2};R_2) + m_0(\varphi^{\bullet}_{r2};R_2)\right] = \\
&= -\frac{1}{D_{c2}}\Lambda_c^2\left\{C_{0,c1}\left[I_1(R_2,R_1,\lambda_{c1}) - I_1(R_2,R_1,\lambda_{c2})\right] + D_{0,c1}\left[I_1(R_2,R_2,\lambda_{c1}) - I_1(R_2,R_2,\lambda_{c2})\right]\right\} + \\
&+ \frac{1}{D_{r2}}\Lambda_r^2 E_{0,r1}\left[I_1(R_2,R_2,\lambda_{r1}) - I_1(R_2,R_2,\lambda_{r2})\right]
\end{aligned}
\tag{F.10c}
$$

$$
\begin{aligned}
{}^{4}b_{0,2} &= 2\pi\left\{-D_{c2}\frac{\partial}{\partial r}\left[m_0(\varphi^{o}_{c2};r)+m_0(\varphi^{\bullet}_{d2};r)\right]_{r=R_2} + D_{r2}\frac{\partial}{\partial r}\left[m_0(\varphi^{\bullet}_{r2};r)\right]_{r=R_2}\right\} = \\
&= -\Lambda_c^2\left\{C_{0,c1}\left[I_1'(R_2,R_1,\lambda_{c1}) - I_1'(R_2,R_1,\lambda_{c2})\right]_{r>R_1} + D_{0,c1}\left[I_1'(R_2,R_2,\lambda_{c1}) - I_1'(R_2,R_2,\lambda_{c2})\right]_{r<R_2}\right\} + \\
&+ \Lambda_r^2 E_{0,r1}\left[I_1'(R_2,R_2,\lambda_{r1}) - I_1'(R_2,R_2,\lambda_{r2})\right]_{r>R_2}
\end{aligned}
\tag{F.10d}
$$

for $2n = 2$ and $j = 2$:

(F.11a)

$$^{1}b_{2,2} = -2\pi\left[m_2(\varphi^{o}_{b2};R_1) + m_2(\varphi^{\bullet}_{b2};R_1) - m_2(\varphi^{\bullet}_{c2};R_1) - m_2(\varphi^{\bullet}_{d2};R_1)\right] - \frac{2B_{0,b2}}{D_{b2}} I_2(R_1,R_1,\lambda_{b2}) + \frac{2C_{0,c2}}{D_{c2}} I_2(R_1,R_1,\lambda_{c2}) + \frac{2D_{0,c2}}{D_{c2}} I_2(R_1,R_2,\lambda_{c2}) =$$

$$= -\frac{1}{D_{b2}} \Lambda_b^2 \left\{2\left[J_2(R_1,\lambda_{b1}) - J_2(R_1,\lambda_{b2})\right] + B_{2,b1}\left[I_1(R_1,R_1,\lambda_{b1}) - I_1(R_1,R_1,\lambda_{b2})\right] + 2B_{0,b1}\left[I_2(R_1,R_1,\lambda_{b1}) - I_2(R_1,R_1,\lambda_{b2})\right]\right\} +$$

$$+ \frac{1}{D_{c2}} \Lambda_c^2 \left\{C_{2,c1}\left[I_1(R_1,R_1,\lambda_{c1}) - I_1(R_1,R_1,\lambda_{c2})\right] + 2C_{0,c1}\left[I_2(R_1,R_1,\lambda_{c1}) - I_2(R_1,R_1,\lambda_{c2})\right] + D_{2,c1}\left[I_1(R_1,R_2,\lambda_{c1}) - I_1(R_1,R_2,\lambda_{c2})\right] +\right.$$

$$\left. + 2D_{0,c1}\left[I_2(R_1,R_2,\lambda_{c1}) - I_2(R_1,R_2,\lambda_{c2})\right]\right\} - \frac{2B_{0,b2}}{D_{b2}} I_2(R_1,R_1,\lambda_{b2}) + \frac{2C_{0,c2}}{D_{c2}} I_2(R_1,R_1,\lambda_{c2}) + \frac{2D_{0,c2}}{D_{c2}} I_2(R_1,R_2,\lambda_{c2})$$

(F.11b)

$$^{2}b_{2,2} = -2\pi D_{b2} \frac{\partial}{\partial r}\left[m_2(\varphi^{o}_{b2};r) + m_2(\varphi^{\bullet}_{b2};r)\right]_{r=R_1} + 2\pi D_{c2} \frac{\partial}{\partial r}\left[m_2(\varphi^{\bullet}_{c2};r) + m_2(\varphi^{\bullet}_{d2};r)\right]_{r=R_1} -$$

$$- 2B_{0,b2} I_2'(R_1,R_1,\lambda_{b2}) + 2C_{0,c2} I_2'(R_1,R_1,\lambda_{c2}) + 2D_{0,c2} I_2'(R_1,R_2,\lambda_{c2}) =$$

$$= -\Lambda_b^2 \left\{2\left[J_2'(R_1,\lambda_{b1}) - J_2'(R_1,\lambda_{b2})\right] + B_{2,b1}\left[I_1'(R_1,R_1,\lambda_{b1}) - I_1'(R_1,R_1,\lambda_{b2})\right]_{r<R_1} + 2B_{0,b1}\left[I_2'(R_1,R_1,\lambda_{b1}) - I_2'(R_1,R_1,\lambda_{b2})\right]_{r<R_1}\right\} +$$

$$+ \Lambda_c^2 \left\{C_{2,c1}\left[I_1'(R_1,R_1,\lambda_{c1}) - I_1'(R_1,R_1,\lambda_{c2})\right]_{r>R_1} + 2C_{0,c1}\left[I_2'(R_1,R_1,\lambda_{c1}) - I_2'(R_1,R_1,\lambda_{c2})\right]_{r>R_1} + D_{2,c1}\left[I_1'(R_1,R_2,\lambda_{c1}) - I_1'(R_1,R_2,\lambda_{c2})\right]_{r<R_2} +\right.$$

$$\left. + 2D_{0,c1}\left[I_2'(R_1,R_2,\lambda_{c1}) - I_2'(R_1,R_2,\lambda_{c2})\right]_{r<R_2}\right\} - 2B_{0,b2} I_2'(R_1,R_1,\lambda_{b2})_{r<R_1} + 2C_{0,c2} I_2'(R_1,R_1,\lambda_{c2})_{r>R_1} + 2D_{0,c2} I_2'(R_1,R_2,\lambda_{c2})_{r<R_2}$$

$$ {}^{3}b_{2,2} = -2\pi\left[m_2(\overset{\bullet}{\varphi}_{c2};R_2) + m_2(\overset{\bullet}{\varphi}_{d2};R_2) - m_2(\overset{\bullet}{\varphi}_{r2};R_2)\right] - \frac{2\mathrm{C}_{0,c2}}{D_{c2}} I_2(R_2,R_1,\lambda_{c2}) - \frac{2\mathrm{D}_{0,c2}}{D_{c2}} I_2(R_2,R_2,\lambda_{c2}) + \frac{2\mathrm{E}_{0,r2}}{D_{r2}} I_2(R_2,R_2,\lambda_{r2}) = \tag{F.11c} $$

$$ = -\frac{1}{D_{c2}}\Lambda_c^2\Big\{\mathrm{C}_{2,c1}\left[I_1(R_2,R_1,\lambda_{c1}) - I_1(R_2,R_1,\lambda_{c2})\right] + 2\mathrm{C}_{0,c1}\left[I_2(R_2,R_1,\lambda_{c1}) - I_2(R_2,R_1,\lambda_{c2})\right] + \mathrm{D}_{2,c1}\left[I_1(R_2,R_2,\lambda_{c1}) - I_1(R_2,R_2,\lambda_{c2})\right] + $$

$$ + 2\mathrm{D}_{0,c1}\left[I_2(R_2,R_2,\lambda_{c1}) - I_2(R_2,R_2,\lambda_{c2})\right]\Big\} + \frac{1}{D_{r2}}\Lambda_r^2\Big\{\mathrm{E}_{2,r1}\left[I_1(R_2,R_2,\lambda_{r1}) - I_1(R_2,R_2,\lambda_{r2})\right] + $$

$$ + 2\mathrm{E}_{0,r1}\left[I_2(R_2,R_2,\lambda_{r1}) - I_2(R_2,R_2,\lambda_{r2})\right]\Big\} - \frac{2\mathrm{C}_{0,c2}}{D_{c2}} I_2(R_2,R_1,\lambda_{c2}) - \frac{2\mathrm{D}_{0,c2}}{D_{c2}} I_2(R_2,R_2,\lambda_{c2}) + \frac{2\mathrm{E}_{0,r2}}{D_{r2}} I_2(R_2,R_2,\lambda_{r2}) $$

$$ {}^{4}b_{2,2} = -2\pi D_{c2}\frac{\partial}{\partial r}\left[m_2(\overset{\bullet}{\varphi}_{c2};r) + m_2(\overset{\bullet}{\varphi}_{d2};r)\right]_{r=R_2} + 2\pi D_{r2}\frac{\partial}{\partial r}\left[m_2(\overset{\bullet}{\varphi}_{r2};r)\right]_{r=R_2} - \tag{F.11d} $$

$$ - 2\mathrm{C}_{0,c2} I_2'(R_2,R_1,\lambda_{c2}) - 2\mathrm{D}_{0,c2} I_2'(R_2,R_2,\lambda_{c2}) + 2\mathrm{E}_{0,r2} I_2'(R_2,R_2,\lambda_{r2}) = $$

$$ = -\Lambda_c^2\Big\{\mathrm{C}_{2,c1}\left[I_1'(R_2,R_1,\lambda_{c1}) - I_1'(R_2,R_1,\lambda_{c2})\right]_{r>R_1} + 2\mathrm{C}_{0,c1}\left[I_2'(R_2,R_1,\lambda_{c1}) - I_2'(R_2,R_1,\lambda_{c2})\right]_{r>R_1} + $$

$$ + \mathrm{D}_{2,c1}\left[I_1'(R_2,R_2,\lambda_{c1}) - I_1'(R_2,R_2,\lambda_{c2})\right]_{r<R_2} + 2\mathrm{D}_{0,c1}\left[I_2'(R_2,R_2,\lambda_{c1}) - I_2'(R_2,R_2,\lambda_{c2})\right]_{r<R_2}\Big\} + $$

$$ + \Lambda_r^2\Big\{\mathrm{E}_{2,r1}\left[I_1'(R_2,R_2,\lambda_{r1}) - I_1'(R_2,R_2,\lambda_{r2})\right]_{r>R_2} + 2\mathrm{E}_{0,r1}\left[I_2'(R_2,R_2,\lambda_{r1}) - I_2'(R_2,R_2,\lambda_{r2})\right]_{r>R_2}\Big\} - $$

$$ - 2\mathrm{C}_{0,c2} I_2'(R_2,R_1,\lambda_{c2})_{r>R_1} - 2\mathrm{D}_{0,c2} I_2'(R_2,R_2,\lambda_{c2})_{r<R_2} + 2\mathrm{E}_{0,r2} I_2'(R_2,R_2,\lambda_{r2})_{r>R_2} $$

for $2n = 4$ and $j = 2$:

$$
\begin{aligned}
{}^1b_{4,2} &= -2\pi\left[m_4(\varphi^{o}_{b2};R_1) + m_4(\dot\varphi_{b2};R_1) - m_4(\dot\varphi_{c2};R_1) - m_4(\dot\varphi_{d2};R_1)\right] - \frac{12B_{2,b2}}{D_{b2}} I_2(R_1,R_1,\lambda_{b2}) - \frac{24B_{0,b2}}{D_{b2}} I_3(R_1,R_1,\lambda_{b2}) + \\
&+ \frac{12C_{2,c2}}{D_{c2}} I_2(R_1,R_1,\lambda_{c2}) + \frac{24C_{0,c2}}{D_{c2}} I_3(R_1,R_1,\lambda_{c2}) + \frac{12D_{2,c2}}{D_{c2}} I_2(R_1,R_2,\lambda_{c2}) + \frac{24D_{0,c2}}{D_{c2}} I_3(R_1,R_2,\lambda_{c2}) = \\
&= -\frac{1}{D_{b2}}\Lambda_b^2\Big\{24\left[J_3(R_1,\lambda_{b1}) - J_3(R_1,\lambda_{b2})\right] + B_{4,b1}\left[I_1(R_1,R_1,\lambda_{b1}) - I_1(R_1,R_1,\lambda_{b2})\right] + 12B_{2,b1}\left[I_2(R_1,R_1,\lambda_{b1}) - I_2(R_1,R_1,\lambda_{b2})\right] + \\
&+ 24B_{0,b1}\left[I_3(R_1,R_1,\lambda_{b1}) - I_3(R_1,R_1,\lambda_{b2})\right]\Big\}_{r<R_1} + \frac{1}{D_{c2}}\Lambda_c^2\Big\{C_{4,c1}\left[I_1(R_1,R_1,\lambda_{c1}) - I_1(R_1,R_1,\lambda_{c2})\right] + 12C_{2,c1}\left[I_2(R_1,R_1,\lambda_{c1}) - I_2(R_1,R_1,\lambda_{c2})\right] + \\
&+ 24C_{0,c1}\left[I_3(R_1,R_1,\lambda_{c1}) - I_3(R_1,R_1,\lambda_{c2})\right]_{r>R_1} + D_{4,c1}\left[I_1(R_1,R_2,\lambda_{c1}) - I_1(R_1,R_2,\lambda_{c2})\right] + 12D_{2,c1}\left[I_2(R_1,R_2,\lambda_{c1}) - I_2(R_1,R_2,\lambda_{c2})\right] + \\
&+ 24D_{0,c1}\left[I_3(R_1,R_2,\lambda_{c1}) - I_3(R_1,R_2,\lambda_{c2})\right]_{r<R_2}\Big\} - \frac{12B_{2,b2}}{D_{b2}} I_2(R_1,R_1,\lambda_{b2}) - \frac{24B_{0,b2}}{D_{b2}} I_3(R_1,R_1,\lambda_{b2}) + \frac{12C_{2,c2}}{D_{c2}} I_2(R_1,R_1,\lambda_{c2}) + \\
&+ \frac{24C_{0,c2}}{D_{c2}} I_3(R_1,R_1,\lambda_{c2}) + \frac{12D_{2,c2}}{D_{c2}} I_2(R_1,R_2,\lambda_{c2}) + \frac{24D_{0,c2}}{D_{c2}} I_3(R_1,R_2,\lambda_{c2})
\end{aligned}
\tag{F.12a}
$$

(F.12b)

$$^2b_{4,2} = -2\pi D_{b2}\frac{\partial}{\partial r}\Big[m_4(\varphi^{o}_{b2};R_1) + m_4(\varphi^{\bullet}_{b2};R_1)\Big]_{r=R_1} + 2\pi D_{c2}\frac{\partial}{\partial r}\Big[m_2(\varphi^{\bullet}_{c2};R_1) + m_2(\varphi^{\bullet}_{d2};R_1)\Big]_{r=R_1} - 12B_{2,b2}\, I_2'(R_1,R_1,\lambda_{b2}) -$$

$$-24B_{0,b2}\, I_3'(R_1,R_1,\lambda_{b2})_{r<R_1} + 12C_{2,c2}\, I_2'(R_1,R_1,\lambda_{c2}) + 24C_{0,c2}\, I_3'(R_1,R_1,\lambda_{c2})_{r>R_1} + 12D_{2,c2}\, I_2'(R_1,R_2,\lambda_{c2}) + 24D_{0,c2}\, I_3'(R_1,R_2,\lambda_{c2})_{r<R_2} =$$

$$= -\Lambda_b^2\Big\{24\big[J_3'(R_1,\lambda_{b1}) - J_3'(R_1,\lambda_{b2})\big] + B_{4,b1}\big[I_1'(R_1,R_1,\lambda_{b1}) - I_1'(R_1,R_1,\lambda_{b2})\big]_{r<R_1} + 12B_{2,b1}\big[I_2'(R_1,R_1,\lambda_{b1}) - I_2'(R_1,R_1,\lambda_{b2})\big]_{r<R_1} +$$

$$+24B_{0,b1}\big[I_3'(R_1,R_1,\lambda_{b1}) - I_3'(R_1,R_1,\lambda_{b2})\big]_{r<R_1}\Big\} + \Lambda_c^2\Big\{C_{4,c1}\big[I_1'(R_1,R_1,\lambda_{c1}) - I_1'(R_1,R_1,\lambda_{c2})\big]_{r>R_1} + 12C_{2,c1}\big[I_2'(R_1,R_1,\lambda_{c1}) - I_2'(R_1,R_1,\lambda_{c2})\big]_{r>R_1} +$$

$$+24C_{0,c1}\big[I_3'(R_1,R_1,\lambda_{c1}) - I_3'(R_1,R_1,\lambda_{c2})\big]_{r>R_1} + D_{4,c1}\big[I_1'(R_1,R_2,\lambda_{c1}) - I_1'(R_1,R_2,\lambda_{c2})\big]_{r<R_2} + 12D_{2,c1}\big[I_2'(R_1,R_2,\lambda_{c1}) - I_2'R_1,R_2,\lambda_{c2})\big]_{r<R_2} +$$

$$+24D_{0,c1}\big[I_3'(R_1,R_2,\lambda_{c1}) - I_3'(R_1,R_2,\lambda_{c2})\big]_{r<R_2}\Big\} - 12B_{2,b2}\, I_2'(R_1,R_1,\lambda_{b2})_{r<R_1} - 24B_{0,b2} I_3'(R_1,R_1,\lambda_{b2})_{r<R_1} + 12C_{2,c2}\, I_2'(R_1,R_1,\lambda_{c2})_{r>R_1} +$$

$$+24C_{0,c2} I_3'(R_1,R_1,\lambda_{c2})_{r>R_1} + 12D_{2,c2}\, I_2'(R_1,R_2,\lambda_{c2})_{r<R_2} + 24D_{0,c2}\, I_3'(R_1,R_2,\lambda_{c2})_{r<R_2}$$

(F.12c)

$$
\begin{aligned}
{}^{3}b_{4,2} &= -2\pi\left[m_4(\varphi^{*}_{c2};R_2) + m_4(\varphi^{*}_{d2};R_2) - m_4(\varphi^{*}_{r2};R_2)\right] - \frac{12C_{2,c2}}{D_{c2}} I_2(R_2,R_1,\lambda_{c2}) - \frac{24C_{0,c2}}{D_{c2}} I_3(R_2,R_1,\lambda_{c2}) - \\
&- \frac{12D_{2,c2}}{D_{c2}} I_2(R_2,R_2,\lambda_{c2}) - \frac{24D_{0,c2}}{D_{c2}} I_3(R_2,R_2,\lambda_{c2}) + \frac{12E_{2,r2}}{D_{r2}} I_2(R_2,R_2,\lambda_{r2}) + \frac{24E_{0,r2}}{D_{r2}} I_3(R_2,R_2,\lambda_{r2}) = \\
&= -\frac{1}{D_{c2}} \Lambda_c^2 \Big\{ C_{4,c1}\left[I_1(R_2,R_1,\lambda_{c1}) - I_1(R_2,R_1,\lambda_{c2})\right] + 12C_{2,c1}\left[I_2(R_2,R_1,\lambda_{c1}) - I_2(R_2,R_1,\lambda_{c2})\right] + \\
&+ 24C_{0,c1}\left[I_3(R_2,R_1,\lambda_{c1}) - I_3(R_2,R_1,\lambda_{c2})\right]_{r>R_1} + D_{4,c1}\left[I_1(R_2,R_2,\lambda_{c1}) - I_1(R_2,R_2,\lambda_{c2})\right] + \\
&+ 12D_{2,c1}\left[I_2(R_2,R_2,\lambda_{c1}) - I_2(R_2,R_2,\lambda_{c2})\right] + 24D_{0,c1}\left[I_3(R_2,R_2,\lambda_{c1}) - I_3(R_2,R_2,\lambda_{c2})\right]_{r<R_2} \Big\} + \\
&+ \frac{1}{D_{r2}} \Lambda_r^2 \Big\{ E_{4,r1}\left[I_1(R_2,R_2,\lambda_{r1}) - I_1(R_2,R_2,\lambda_{r2})\right] + 12E_{2,r1}\left[I_2(R_2,R_2,\lambda_{r1}) - I_2(R_2,R_2,\lambda_{r2})\right] + \\
&+ 24E_{0,r1}\left[I_3(R_2,R_2,\lambda_{r1}) - I_3(R_2,R_2,\lambda_{r2})\right]_{r>R_2} \Big\} - \frac{12C_{2,c2}}{D_{c2}} I_2(R_2,R_1,\lambda_{c2}) - \frac{24C_{0,c2}}{D_{c2}} I_3(R_2,R_1,\lambda_{c2}) - \\
&- \frac{12D_{2,c2}}{D_{c2}} I_2(R_2,R_2,\lambda_{c2}) - \frac{24D_{0,c2}}{D_{c2}} I_3(R_2,R_2,\lambda_{c2}) + \frac{12E_{2,r2}}{D_{r2}} I_2(R_2,R_2,\lambda_{r2}) + \frac{24E_{0,r2}}{D_{r2}} I_3(R_2,R_2,\lambda_{r2})
\end{aligned}
$$

(F.12d)

$$
\begin{aligned}
{}^{4}b_{4,2} = & -2\pi D_{c2} \frac{\partial}{\partial r}\left[m_4(\overset{\bullet}{\varphi}_{c2};r) + m_4(\overset{\bullet}{\varphi}_{d2};r)\right]_{r=R_2} + 2\pi D_{r2} \frac{\partial}{\partial r}\left[m_4(\overset{\bullet}{\varphi}_{r2};r)\right]_{r=R_2} - 12C_{2,c2}\, I_2'(R_2,R_1,\lambda_{c2}) - 24C_{0,c2}\, I_3'(R_2,R_1,\lambda_{c2})_{r>R_1} - \\
& - 12D_{2,c2}\, I_2'(R_2,R_2,\lambda_{c2}) - 24D_{0,c2}\, I_3'(R_2,R_2,\lambda_{c2})_{r<R_2} + 12E_{2,r2}\, I_2'(R_2,R_2,\lambda_{r2}) + 24E_{0,r2}\, I_3'(R_2,R_2,\lambda_{r2})_{r>R_2} = \\
& = -\Lambda_c^2\Big\{C_{4,c1}\left[I_1'(R_2,R_1,\lambda_{c1}) - I_1'(R_2,R_1,\lambda_{c2})\right]_{r>R_1} + 12C_{2,c1}\left[I_2'(R_2,R_1,\lambda_{c1}) - I_2'(R_2,R_1,1_{c2})\right]_{r>R_1} + \\
& + 24C_{0,c1}\left[I_3'(R_2,R_1,\lambda_{c1}) - I_3'(R_2,R_1,\lambda_{c2})\right]_{r>R_1} + D_{4,c1}\left[I_1'(R_2,R_2,\lambda_{c1}) - I_1'(R_2,R_2,\lambda_{c2})\right]_{r<R_2} + \\
& + 12D_{2,c1}\left[I_2'(R_2,R_2,\lambda_{c1}) - I_2'(R_2,R_2,\lambda_{c2})\right]_{r<R_2} + 24D_{0,c1}\left[I_3'(R_2,R_2,\lambda_{c1}) - I_3'(R_2,R_2,\lambda_{c2})\right]_{r<R_2}\Big\} + \\
& + \Lambda_r^2\Big\{E_{4,r1}\left[I_1'(R_2,R_2,\lambda_{r1}) - I_1'(R_2,R_2,\lambda_{r2})\right]_{r>R_2} + 12E_{2,r1}\left[I_2'(R_2,R_2,\lambda_{r1}) - I_2'(R_2,R_2,\lambda_{r2})\right]_{r>R_2} + \\
& + 24E_{0,r1}\left[I_3'(R_2,R_2,\lambda_{r1}) - I_3'(R_2,R_2,\lambda_{r2})\right]_{r>R_2}\Big\} - 12C_{2,c2}\, I_2'(R_2,R_1,\lambda_{c2}) - 24C_{0,c2}\, I_3'(R_2,R_1,\lambda_{c2})_{r>R_1} - \\
& - 12D_{2,c2} I_2'(R_2,R_2,\lambda_{c2}) - 24D_{0,c2}\, I_3'(R_2,R_2,\lambda_{c2})_{r<R_2} + 12E_{2,r2} I_2'(R_2,R_2,\lambda_{r2}) + 24E_{0,r2}\, I_3'(R_2,R_2,\lambda_{r2})_{r>R_2}
\end{aligned}
$$

In that way all equations and constants needed to calculate the second and fourth z-axial moments of all neutron fluxes for the three coaxial layer configuration have been established.

APPENDIX G

Solution of the set of linear equations related to the boundary conditions.

From the boundary conditions given by Eq.(22) one always has four linear equations with four unknowns x_1, x_2, x_3, x_4 of the form

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + 0 \cdot x_4 &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + 0 \cdot x_4 &= b_2 \\
0 \cdot x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4 &= b_3 \\
0 \cdot x_1 + a_{42}x_2 + a_{43}x_3 + a_{44}x_4 &= b_4
\end{aligned} \qquad \text{(G.1)}$$

The notation is used:

$$\begin{aligned}
c_{11} &= a_{12}a_{21} - a_{22}a_{11} \\
c_{12} &= a_{13}a_{21} - a_{23}a_{11} \\
c_{21} &= a_{22}a_{44} - a_{42}a_{34} \\
c_{22} &= a_{33}a_{44} - a_{43}a_{34} \\
d_1 &= b_1a_{21} - b_2a_{11} \\
d_2 &= b_3a_{44} - b_4a_{34}
\end{aligned} \qquad \text{(G.2)}$$

By the elimination of the two unknowns x_1, x_4 one obtains a set of two equations

$$\begin{aligned}
c_{11}x_2 + c_{12}x_3 &= d_1 \\
c_{21}x_2 + c_{22}x_3 &= d_2
\end{aligned} \qquad \text{(G.3)}$$

By the calculation of the determinants for Eq.(G.3) one obtains

$$\begin{aligned}
D_{23} &= c_{11}c_{22} - c_{21}c_{12} \\
D_2 &= d_1c_{22} - d_2c_{12} \\
D_3 &= d_2c_{11} - d_1c_{21}
\end{aligned} \qquad \text{(G.4)}$$

which gives the solution for x_2 and x_3:

$$x_2 = \frac{\Delta_2}{\Delta_{23}} \qquad x_3 = \frac{\Delta_3}{\Delta_{23}} \qquad \text{(G.5)}$$

After the substitution of the solutions given by Eqs (G.5) into Eq.(G.1) one obtains for x_1 and x_4:

$$\begin{aligned}
x_1 &= \frac{1}{a_{11}}\left(b_1 - a_{12}x_2 - a_{13}x_3\right) \\
x_4 &= \frac{1}{a_{34}}\left(b_3 - a_{32}x_2 - a_{33}x_3\right)
\end{aligned} \qquad \text{(G.6)}$$

When $b_3 = b_4 = 0$ in Eq.(G.1), then $d_2 = 0$, and:

$$\begin{aligned}
&\Delta_2 = d_1c_{22} \qquad \Delta_3 = -d_1c_{21} \\
&x_4 = -\frac{1}{a_{34}}\left(a_{32}x_2 + a_{33}x_3\right)
\end{aligned} \qquad \text{(G.7)}$$

which makes the calculations easier.

APPENDIX H

Apparent neutron parameters

The apparent neutron slowing down and migration lengths are defined by Eqs (24 and 25). The 2-nd and 4-th neutron flux moments on the borehole axis, *i.e.* for $r = 0$, are calculated from Eqs (20 and 21) using the solutions (E.15, E.16, E.20, E.21, E.22, E.29, E.30, E.35 and E.36). The particular values of integrals $J_n(r,1)$ and $I_n(r,R,1)$ for $r = 0$ are obtained from formulae (D.19 to D.23).

$$m_2(\varphi_{b1};0) = \frac{1}{2\pi D_{b1}}\left[2J_2(0,\lambda_{b1}) + B_{2,b1}\, I_1(0,R_1,\lambda_{b1}) + 2B_{0,b1}\, I_2(0,R_1,\lambda_{b1})\right] = \frac{1}{2\pi D_{b1}}\left[\frac{1}{\lambda_{b1}^2} + B_{0,b1}\frac{R_1}{\lambda_{b1}}K_1(R_1\lambda_{b1}) + B_{2,b1}K_0(R_1\lambda_{b1})\right] \tag{H.1}$$

$$\begin{aligned} m_2(\varphi_{b2};0) = \frac{1}{2\pi D_{b2}}\Bigg\{&\frac{2\lambda_{b1}^2}{\lambda_{b2}^2-\lambda_{b1}^2}\Big\{J_2(0,\lambda_{b1}) - J_2(0,\lambda_{b2}) + B_{2,b1}\left[I_1(0,R_1,\lambda_{b1}) - I_1(0,R_1,\lambda_{b2})\right] + 2B_{0,b1}\left[I_2(0,R_1,\lambda_{b1}) - I_2(0,R_1,\lambda_{b2})\right]\Big\} + \\ &+ B_{2,b2}\, I_1(0,R_1,\lambda_{b2}) + 2B_{0,b2}\, I_2(0,R_1,\lambda_{b2})\Bigg\} = \frac{1}{2\pi D_{b2}}\left\{\frac{\lambda_{b1}^2}{\lambda_{b2}^2-\lambda_{b1}^2}\left\{\frac{1}{\lambda_{b1}} - \frac{1}{\lambda_{b2}} + B_{2,b1}\left[K_0(R_1\lambda_{b1}) - K_0(R_1\lambda_{b2})\right] + \right.\right. \\ &\left.\left. + R_1 B_{0,b1}\left[\frac{1}{\lambda_{b1}}K_1(R_1\lambda_{b1}) - \frac{1}{\lambda_{b2}}K_1(R_1\lambda_{b2})\right]\right\} + B_{2,b2}K_0(R_1\lambda_{b2}) + B_{0,b2}\frac{1}{\lambda_{b2}}R_1K_1(R_1\lambda_{b2})\right\} \end{aligned} \tag{H.2}$$

$$m_4(\varphi_{b1};0) = \frac{1}{2\pi D_{b1}}\left[24 J_3(0,\lambda_{b1}) + B_{4,b1}\, I_1(0,R_1,\lambda_{b1}) + 12 B_{2,b1}\, I_2(0,R_1,\lambda_{b1}) + 24 B_{0,b1}\, I_3(0,R_1,\lambda_{b1})\right] =$$

$$= \frac{1}{2\pi D_{b1}}\left[\frac{6}{\lambda_{b1}^4} + B_{0,b1}\frac{3}{\lambda_{b1}^2} R_1^2 K_2(R_1\lambda_{b1}) + B_{2,b1}\frac{6}{\lambda_{b1}} R_1 K_1(R_1\lambda_{b1}) + B_{4,b1} K_0(R_1\lambda_{b1})\right] \tag{H.3}$$

$$m_4(\varphi_{b2};0) = \frac{1}{2\pi D_{b2}}\left\{\frac{\lambda_{b1}^2}{\lambda_{b2}^2-\lambda_{b1}^2}\left\{24\left[J_3(0,\lambda_{b1}) - J_3(0,\lambda_{b2})\right] + B_{4,b1}\left[I_1(0,R_1,\lambda_{b1}) - I_1(0,R_1,\lambda_{b2})\right] + 12 B_{2,b1}\left[I_2(0,R_1,\lambda_{b1}) - I_2(0,R_1,\lambda_{b2})\right] +\right.\right.$$

$$\left.\left. + 24 B_{0,b1}\left[I_3(0,R_1,\lambda_{b1}) - I_3(0,R_1,\lambda_{b2})\right]\right\} + B_{4,b2}\, I_1(0,R_1,\lambda_{b2}) + 12 B_{2,b2}\, I_2(0,R_1,\lambda_{b2}) + 24 B_{0,b2}\, I_3(0,R_1,\lambda_{b2})\right\} =$$

$$= \frac{1}{2\pi D_{b2}}\left\{\frac{\lambda_{b1}^2}{\lambda_{b2}^2-\lambda_{b1}^2}\left\{6\left(\frac{1}{\lambda_{b1}^4} - \frac{1}{\lambda_{b2}^4}\right) + B_{4,b1}\left[K_0(R_1\lambda_{b1}) - K_0(R_1\lambda_{b2})\right] + 6 R_1 B_{2,b1}\left[\frac{1}{\lambda_{b1}} K_1(R_1\lambda_{b1}) - \frac{1}{\lambda_{b2}} K_1(R_1\lambda_{b2})\right] +\right.\right.$$

$$\left. + 3R_1^2 B_{0,b1}\left[\frac{1}{\lambda_{b1}^2} K_2(R_1\lambda_{b1}) - \frac{1}{\lambda_{b2}^2} K_2(R_1\lambda_{b2})\right]\right\} + B_{4,b2} K_0(R_1\lambda_{b2}) + 6 B_{2,b2}\frac{1}{\lambda_{b2}} I_2(R_1\lambda_{b2}) + 3 B_{0,b2}\frac{1}{\lambda_{b2}^2} R_1 K_2(R_1\lambda_{b2})\Bigg\} \tag{H.4}$$