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A TRANSFORMATION TECHNIQUE TO TREAT STRONG VIBRATING ABSORBERS

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Abstract - Calculation of the neutron noise, induced by small amplitude vibrations of a strong absorbe,r is a difficult task because the traditional linearisation technique cannot be applied. Two methods, based on two different representations of the absorber, were developed earlier to solve the problem. In both methods the rod displacements are described by a Taylor expansion, such that the boundary condition needs only be considered at the surface of a static rod. Only one of the methods is applicable in two dimensions.

In this paper an alternative method is developed and used for the solution of the problem. The essence of the method is a variable transformation by which the moving boundary is transformed into a static one without Taylor expansion. The corresponding equations are solved in a linear manner and the solution is transformed back to the original parameter space. The method is equally applicable in one and two dimensions. The solutions are in complete agreement with those of the previous methods.

I. INTRODUCTION

Because of its diagnostic value, the neutron noise induced by a vibrating absorber has been studied quite extensively in the past. Usually, the absorber is represented by a spatial δ -function, and the vibration is described by the deviation from the equilibrium position. The problem is linear in the vibration amplitude, since the noise vanishes with vanishing vibration amplitude. The linear character of the noise problem is quite apparent if the strength of the absorber is small. In fact, the neutron noise induced by a weak vibrating absorber was already given by Weinberg and Schweinler (1948).

It turned out, however, that the traditional linearization technique breaks down with strong rods. The reasons of this fact were explained by Pázsit (1984), and Pázsit and Karlsson (1997). In these publications, two different solution or linearization techniques were elaborated by which the neutron noise, induced by the small amplitude vibrations of a strong absorber, could be calculated. These two methods are based on two different representations of a localised absorber, both leading to formulae containing the rod in form of δ-functions only. The first one (Pázsit, 1984) is based on a δ-function representation of both the static and the vibrating rod. This method was called the Feinberg-Galanin-Williams (FGW) model, and uses a non-traditional linearisation technique that avoids the pitfall of the traditional method. This method is only applicable in one dimension (1-D). A two-dimensional (2-D) solution was then obtained by extrapolating the 1-D solution to 2-D, using heuristic arguments and a generalisation of the definition of the symmetric derivatives (around the rod position) of both the static flux and the Green's function. The second method, called the ϵ/d-method (Pázsit and Karlsson, 1997) is based on a finite thickness of the static and vibrating rod, uses linearisation in the vibration amplitude, and then contracts the rod into a δ-function. This latter method is applicable in both one and two dimensions. It was shown that the exact 1-D and the heuristic 2-D solutions of the FGW model are identical with the corresponding 1-D and 2-D solutions of the ϵ/d-model, respectively, showing the equivalence of the two methods.

The difficulties in treating the strong vibrating absorber can be formulated as the problem of fulfilling a boundary condition at a moving surface. Indeed, it was shown (Pázsit, 1984) that the FGW solution preserves the correct boundary condition at the moving rod surface, although the solution method itself does not utilize this condition explicitly. The problem of treating a moving boundary is actually wider than, i.e. not restricted to, a vibrating absorber. It is also encountered if the neutron flux needs to be calculated in a system of variable size, such as a fluidized bed core, a molten salt reactor, or a reactor with variable water level such as under accident condition etc.

To handle such cases, a transformation technique was recently suggested by Sahni according to which moving surfaces, both external and internal, correspond to fixed end point values of the new variables (Garis et al, 1996). Boundaries that are originally static are also projected to fixed end points, i.e. remain static. In this way the problem is reduced to one with stationary boundary conditions, which can be treated by standard methods. The problem can also be linearized by standard methods. To obtain the solution of the original problem, the solution needs to be transformed back to the original parameter space.

This technique has already been used for the calculation of the neutron noise induced by a moving boundary (Garis et al, 1996). In the present paper the same technique will be used to calculate the noise induced by the vibrations of a strong absorber. Solutions will be given in both one and two dimensions. It will be seen that they are in complete agreement with the earlier results. In the next section we treat the one dimensional problem of a slab reactor in which a strong absorber of thickness $2d$ and absorption cross section Σ_r is located away from the centre. We then study the flux perturbations induced by the vibrations of this strong absorber, hereafter also referred to as a control rod. It is shown that the flux perturbations are solutions of a Helmholtz equation with a modified (frequency dependent) buckling. The problem can be solved analytically and we give explicit expressions for the flux perturbations. We also show in the Appendix that in the limit $d \rightarrow 0$, $\Sigma_r \rightarrow \infty$ and $\Sigma_r d \rightarrow finite limit$ the results of the ϵ/d-method go over to those of the FGW model.

In section III we consider a bare cylindrical reactor with a central control rod. It is to be noted that vibrations of the central rod force the flux to be cylindrically asymmetric and the problem is essentially two-dimensional. In this case we employ a logarithmic gradient boundary condition at the rod surface. We once again show that the perturbations satisfy a frequency dependent Helmholtz equation. We then derive the boundary conditions to be imposed on the "flux perturbations" at the the surface of the control rod. The boundary conditions involve the unperturbed static flux near the rod, besides its extrapolation distance. With these new boundary conditions it is possible to treat vibrations of an eccentic control rod as well.

II. SOLUTION IN ONE DIMENSION

As mentioned in the introduction, we will employ (a) the ϵ/d model and (b) the FGW model for representing the control rod. In both models, we consider a bare, homogeneous slab reactor of size $2a$ in which a control absorber is located away from the central position. The two models will differ in the way the rod is represented. In the ϵ/d model, a rod of finite thickness $2d$ will be used, whereas in the FGW model, the rod will be represented by a δ-function. As will also be seen in the Appendix, the FGW model can be derived from the ϵ/d one by letting the rod thickness tend to zero and rod the strength tend to infinity such that the product remains finite.

We are interested in studying the flux perturbations induced by small vibrations of the absorber. We will employ one group diffusion theory with one group of delayed neutron precursors for analysing the time variation of the flux in the reactor. Thus we have to solve the equations

$$\frac{1}{v}\frac{\partial\Phi(\mathbf{r},t)}{\partial t} = D\nabla^2\Phi(\mathbf{r},t) + [\nu\Sigma_f(1-\beta) - \Sigma_a]\Phi(\mathbf{r},t) + \lambda C(\mathbf{r},t) \tag{1}$$

and

$$\frac{\partial C(\mathbf{r},t)}{\partial t} = \nu\Sigma_f\beta\Phi(\mathbf{r},t) - \lambda C(\mathbf{r},t) \tag{2}$$

The position variable $\mathbf{r}$ in a slab reactor is characterised by a single co-ordinate $x \in (-a,a)$, where a, the half-thickness of the reactor, includes the extrapolation distance. Thus we assume that the flux $\Phi(x,t)$, the solution of Eqs. (1) and (2), satisfies the boundary conditions

$$\Phi(-a,t) = \Phi(a,t) \equiv 0 \qquad \forall \quad t \tag{3}$$

Let the stationary location of the central point of the absorber be $x = x_p$. We assume that it vibrates with a fixed frequency ω, so that at any given instant of time its position is given by $x_p + \epsilon e^{i\omega t}$. We observe that the absorber divides the reactor into two regions A and B extending over $x \in (-a, x_p - d + \epsilon e^{i\omega t})$ and $x \in (x_p + d + \epsilon e^{i\omega t}, a)$, respectively. Here, for simplicity, we treat both models together, so the above formulas may be interpreted for the FGW model with taking $d = 0$. In region A we have to solve Eqs. (1) and (2) subject to boundary condition $\Phi(-a,t) = 0$ and some condition at $x = x_p - d + \epsilon e^{i\omega t}$. In region B, on the other hand, we have to impose the boundary condition $\Phi(a,t) = 0$ and another condition at $x = x_p + d + \epsilon e^{i\omega t}$.

In the previous paper, Garis et al. (1996), we had used the logarithmic gradient boundary condition at the moving boundary. This was possible because we had to do with an outer boundary of the system. In the present problem, we cannot use this boundary condition at the two surfaces of the absorber. With the present method, a similar approach is only possible for the FGW model. For the ϵ/d model, such a logarithmic gradient condition treats the criticality of the regions A and B separately. Actually, the two regions are individually subcritical and the system becomes critical only because of their mutual coupling which is mediated by the flux within the rod.

A. ϵ/d Model

In this model we will use the diffusion equation within the rod along with the conditions of continuity of flux and current at its surfaces. However, it should be noted that the flux $\Phi_r(x,t)$ within the absorber is of no interest at all and one can use an adjusted absorption cross section Σ_r of the rod to yield the correct reactivity worth of the absorber. The boundary conditions at the two surfaces of the absorber are

$$\Phi(x_p - d + \epsilon e^{\imath\omega t}, t) = \Phi_r(x_p - d + \epsilon e^{\imath\omega t}, t)$$
$$D\frac{\partial\Phi(x,t)}{\partial x}\Big|_{x=x_p-d+\epsilon e^{\imath\omega t}} = D_r\frac{\partial\Phi_r(x,t)}{\partial x}\Big|_{x=x_p-d+\epsilon e^{\imath\omega t}}$$
$$\Phi(x_p + d + \epsilon e^{\imath\omega t}, t) = \Phi_r(x_p + d + \epsilon e^{\imath\omega t}, t)$$
$$D\frac{\partial\Phi(x,t)}{\partial x}\Big|_{x=x_p+d+\epsilon e^{\imath\omega t}} = D_r\frac{\partial\Phi_r(x,t)}{\partial x}\Big|_{x=x_p+d+\epsilon e^{\imath\omega t}} \tag{4}$$

where D_r denotes the diffusion coefficient in the absorber.

In either model it is difficult to incorporate the boundary conditions on a moving surface. We therefore seek co-ordinate transformations that will render all the boundaries at fixed values of the variables. In the ϵ/d model we introduce the variable $y \in (0,1)$ defined by the equations

$$x = -ay + (1-y)(x_p - d + \epsilon e^{\imath\omega t}), \quad x \in A \tag{5a}$$

and

$$x = ay + (1-y)(x_p + d + \epsilon e^{\imath\omega t}), \quad x \in B \tag{5b}$$

We also define

$$\phi_\alpha(y,t) = \Phi(x(y,t),t),$$
$$c_\alpha(y,t) = C(x(y,t),t), \tag{6}$$

where

$$\alpha = A \quad if \quad x \in (-a, x_p - d + \epsilon e^{\imath\omega t})$$
$$= B \quad if \quad x \in (x_p + d + \epsilon e^{\imath\omega t}, a) \tag{7}$$

We now use the chain rule for partial differentiation to affect this transformation. The details are quite similar to those used by Garis et al. (1996). Thus (note that $\nabla^2 = \frac{\partial^2}{\partial x^2}$)

$$\frac{\partial\phi_A(y,t)}{\partial y} = \frac{\partial(x(y,t))}{\partial y}\cdot\frac{\partial\Phi_A(x(y,t),t)}{\partial x} \equiv -(a + x_p + d + \epsilon e^{\imath\omega t})\cdot\frac{\partial\Phi_A(x(y,t),t)}{\partial x}$$
$$\frac{\partial^2\phi_A(y,t)}{\partial y^2} = (a + x_p + d + \epsilon e^{\imath\omega t})^2\cdot\frac{\partial^2\Phi_A(x(y,t),t)}{\partial x^2}$$
$$\frac{\partial\phi_A(y,t)}{\partial t} = \frac{\partial\Phi_A}{\partial t} + \frac{\partial(x(y,t))}{\partial t}\cdot\frac{\partial\Phi_A}{\partial x} \equiv \frac{\partial\Phi_A}{\partial t} + (1-y)\epsilon\imath\omega e^{\imath\omega t}\cdot\frac{\partial\Phi_A}{\partial x} \tag{8}$$

Eqs. (1) and (2) then transform as

$$\frac{1}{v}\frac{\partial\phi_A(y,t)}{\partial t} + \frac{(1-y)\epsilon\imath\omega e^{\imath\omega t}}{v(a + x_p - d + \epsilon e^{\imath\omega t})}\frac{\partial\phi_A(y,t)}{\partial y}$$
$$= D\frac{1}{(a + x_p - d + \epsilon e^{\imath\omega t})^2}\frac{\partial^2\phi_A(y,t)}{\partial^2 y} + [\nu\Sigma_f(1-\beta) - \Sigma_a]\phi_A(y,t) + \lambda c_{A,0}(y,t) \tag{9a}$$

$$\frac{\partial c_A(y,t)}{\partial t} + \frac{(1-y)\epsilon\imath\omega e^{\imath\omega t}}{(a + x_p - d + \epsilon e^{\imath\omega t})}\frac{\partial c_A(y,t)}{\partial y} = \nu\Sigma_f\beta\phi_A(y,t) - \lambda c_A(y,t) \tag{10a}$$

and the boundary condition on $\phi_A(y,t)$ is

$$\phi_A(1,t) = 0 \qquad \forall \quad t \tag{11a}$$

In region B we have the following equations for $\phi_B(y,t)$ and $c_B(y,t)$

$$\frac{1}{v}\frac{\partial \phi_B(y,t)}{\partial t} + \frac{(1-y)\epsilon\imath\omega e^{\imath\omega t}}{v(x_p - a + d + \epsilon e^{\imath\omega t})}\frac{\partial \phi_B(y,t)}{\partial y}$$
$$= D\frac{1}{(x_p - a + d + \epsilon e^{\imath\omega t})^2}\frac{\partial^2 \phi_B(y,t)}{\partial^2 y} + [\nu\Sigma_f(1-\beta) - \Sigma_a]\phi_B(y,t) + \lambda c_B(y,t) \tag{9b}$$

$$\frac{\partial c_B(y,t)}{\partial t} + \frac{(1-y)\epsilon\imath\omega e^{\imath\omega t}}{(x_p - a + d + \epsilon e^{\imath\omega t})}\frac{\partial c_B(y,t)}{\partial y} = \nu\Sigma_f\beta\phi_B(y,t) - \lambda c_B(y,t) \tag{10b}$$

and the boundary condition

$$\phi_B(1,t) = 0 \qquad \forall \quad t \tag{11b}$$

Within the absorber $x \in (x_p - d + \epsilon e^{\imath\omega t}, x_p + d + \epsilon e^{\imath\omega t})$ we have the time-dependent diffusion equation

$$\frac{1}{v}\frac{\partial \Phi_r(x,t)}{\partial t} = D_r\frac{\partial^2 \Phi_r(x,t)}{\partial x^2} - \Sigma_r\Phi_r(x,t) \tag{12}$$

With the transformations

$$x = x_p + \epsilon e^{\imath\omega t} + (2y-1)d$$
$$\Phi_r(x,t) = \Phi_r(x_p + \epsilon e^{\imath\omega t} + (2y-1)d, t) \equiv \phi_r(y,t) \tag{13}$$

we have

$$\frac{1}{v}\left[\frac{\partial \phi_r(y,t)}{\partial t} - \frac{\epsilon\imath\omega e^{\imath\omega t}}{2d}\frac{\partial \phi_r(y,t)}{\partial y}\right] = \frac{D_r}{4d^2}\frac{\partial^2 \phi_r(y,t)}{\partial y^2} - \Sigma_r\phi_r(y,t) \tag{14}$$

The conditions of continuity of flux and current yield

$$\phi_A(0,t) = \phi_r(0,t) \tag{15a}$$

$$-\frac{D}{a + x_p - d + \epsilon e^{\imath\omega t}}\frac{\partial \phi_A(y,t)}{\partial y}\Big|_{y=0} = \frac{D_r}{2d}\frac{\partial \phi_r(y,t)}{\partial y}\Big|_{y=0} \tag{16a}$$

$$\phi_B(0,t) = \phi_r(1,t) \tag{15b}$$

$$-\frac{D}{x_p - a + d + \epsilon e^{\imath\omega t}}\frac{\partial \phi_B(y,t)}{\partial y}\Big|_{y=0} = \frac{D_r}{2d}\frac{\partial \phi_r(y,t)}{\partial y}\Big|_{y=1} \tag{16b}$$

In the above formulation all the boundaries occur at fixed values of the variable y, $y = 0$ or $y = 1$.

We now wish to approximate the set of Eqs. (9-16) by using a perturbation series in ϵ, the amplitude of the vibrations. For this purpose we assume

$$\phi_\alpha(y,t) = \phi_{\alpha,0}(y) + \epsilon\phi_{\alpha,1}(y,t) + \ldots\ldots, \qquad \alpha \equiv A,\ B \ or \ r \tag{17a}$$

$$c_\alpha(y,t) = c_{\alpha,0}(y) + \epsilon c_{\alpha,1}(y,t) + \ldots\ldots, \qquad \alpha \equiv A,\ B \tag{17b}$$

In writing Eqs. (17) we have used the fact that when the amplitude ϵ vanishes, i.e. the rod is stationary, the reactor is critical. Then we have a stationary flux profile $\phi_{\alpha,0}(y)$ in the system, where y_0 is a symbolic indication of the fact that also $\epsilon = 0$ must be assumed. It can be shown that the first order perturbations $\phi_{\alpha,1}(y,t)$ and $c_{\alpha,1}(y,t)$ also oscillate with the frequency ω. Thus $\phi_{\alpha,1}(y,t) = \phi_{\alpha,1}(y)e^{\imath\omega t}$ and $c_{\alpha,1}(y,t) = c_{\alpha,1}(y)e^{\imath\omega t}$.

An important feature of the present method, and thus of Eqs. (17a) and (17b), must be mentioned here. As long as the function $\phi_\alpha(y,t)$ is considered as a function of the parameter y, a power series expansion like (17a) separates the various powers of ϵ. However, when transforming back to the original variable space x, one needs to take account of the fact that the transformation between the variables x and y depends on ϵ, see (5a) and (5b). Thus, after transforming back, one needs another expansion w.r.t. to ϵ to separate the various powers, and, in particular, extract the static term and the term linear in ϵ (i.e. the induced neutron noise). We need to keep this in mind in order to be consistent.

Substituting Eqs. (17) into Eqs. (9-14) and collecting the zeroth order terms (independent of ϵ) we obtain

$$D\frac{1}{(x_p \pm (a-d))^2}\frac{\partial^2\phi_{\alpha,0}(y)}{\partial^2 y} + [\nu\Sigma_f - \Sigma_a]\phi_{\alpha,0}(y) = 0, \qquad \alpha \equiv A,\ B, \tag{18}$$

$$c_{\alpha,0}(y) = \frac{\beta\nu\Sigma_f}{\lambda}\phi_{\alpha,0}(y), \qquad \alpha \equiv A,\ B \tag{19}$$

where the upper sign in $\pm$ in Eq. (18) is to be used if $\alpha = A$ and the lower for $\alpha = B$. In the absorber region we have

$$\frac{D_r}{4d^2}\frac{\partial^2\phi_{r,0}(y)}{\partial y^2} - \Sigma_r\phi_r(y) = 0. \tag{20}$$

The solution of Eqs. (19) and (20) can be written as

$$\phi_{A,0}(y) = E_{A,0}\sin\left[B_0(x_p + a - d)(1-y)\right]$$

$$\phi_{B,0}(y) = E_{B,0}\sin\left[B_0(a - x_p - d)(1-y)\right]$$

$$\phi_{r,0}(y) = E_{r,0}^{(1)}\exp\left[\kappa_r(x_p - d + 2dy)\right] + E_{r,0}^{(2)}\exp\left[-\kappa_r(x_p - d + 2dy)\right] \tag{21}$$

where we have used the fact that both $\phi_{A,0}(y)$ and $\phi_{B,0}(y)$ vanish for $y = 1$. Here,

$$B_0^2 = \frac{\nu\Sigma_f - \Sigma_a}{D}$$

$$\kappa_r^2 = \frac{\Sigma_r}{D_r} \tag{22}$$

and $E_{A,0}$, $E_{B,0}$, $E_{r,0}^{(1)}$ and $E_{r,0}^{(2)}$ are arbitrary constants that will be determined by the interface conditions.

The interface conditions for $\phi_{A,0}(y)$, $\phi_{B,0}(y)$ and $\phi_{r,0}(y)$ can be obtained by substituting the expansion (17a) into Eqs. (15a,b) and (16a,b) and equating the powers of ϵ^0 (i.e. those independent of ϵ). These yield a set of homogeneous equations for the determination of the arbitrary constants $E_{A,0}$, $E_{B,0}$, $E_{r,0}^{(1)}$ and $E_{r,0}^{(2)}$

$$-E_{A,0}\sin B_0(x_p + a - d) + E_{r,0}^{(1)}\exp\left(\kappa_r(x_p - d)\right) + E_{r,0}^{(2)}\exp\left(-\kappa_r(x_p - d)\right) = 0$$

$$-DB_0E_{A,0}\cos B_0(x_p + a - d) + D_r\kappa_r[E_{r,0}^{(1)}\exp\left(\kappa_r(x_p - d)\right) - E_{r,0}^{(2)}\exp\left(-\kappa_r(x_p - d)\right)] = 0$$

$$-E_{B,0}\sin B_0(a - x_p - d) + E_{r,0}^{(1)}\exp\left(\kappa_r(x_p + d)\right) + E_{r,0}^{(2)}\exp\left(-\kappa_r(x_p + d)\right) = 0$$

$$DB_0E_{B,0}\cos B_0(a - x_p - d) + D_r\kappa_r[E_{r,0}^{(1)}\exp\left(\kappa_r(x_p + d)\right) - E_{r,0}^{(2)}\exp\left(-\kappa_r(x_p + d)\right)] = 0 \tag{23}$$

The determinant of these homogeneous equations must vanish, and that gives the static criticality condition for obtaining the material buckling B_0. We can then determine three out of four constants $E_{A,0}$, $E_{B,0}$, $E_{r,0}^{(1)}$ and $E_{r,0}^{(2)}$ leaving one of them for normalisation.

When transforming back to variable x, as mentioned earlier, we need to perform another expansion in ϵ. Doing this and retaining up to linear terms in ϵ only, we have from Eqs. (21)

$$\Phi_{A,0}(x,t) = E_{A,0}[\sin B_0(a+x) - B_0\frac{(a+x)\epsilon e^{\imath\omega t}}{a+x_p-d}\cos B_0(a+x)]$$

$$\Phi_{B,0}(x,t) = E_{B,0}[\sin B_0(a-x) + B_0\frac{(a-x)\epsilon e^{\imath\omega t}}{a-x_p-d}\cos B_0(a-x)]$$

$$\Phi_{r,0}(x,t) = [E_{r,0}^{(1)}\exp(\kappa_r x)+E_{r,0}^{(2)}\exp(-\kappa_r x)]-\kappa_r\epsilon e^{\imath\omega t}[E_{r,0}^{(1)}\exp(\kappa_r x)-E_{r,0}^{(2)}\exp(-\kappa_r x)] \quad (24)$$

The above solution, however, does not include all those terms that are linear in ϵ. These also come from the first order term $\phi_{\alpha,1}(y,t)$, $\alpha = A,\ B\ or\ r$. As was remarked earlier, $\phi_{\alpha,1}(y,t)$ also oscillates with the same frequency ω. We, therefore, turn our attention to the determination of $\phi_{\alpha,1}(y)$. Equating the first order terms in ϵ in Eqs. (9-14), (after substituting Eqs. (17)), we get for the region A

$$\frac{(1-y)\imath\omega}{v(a+x_p-d)}\frac{\partial\phi_{A,0}(y)}{\partial y} + 2D\frac{1}{(a+x_p-d)^3}\frac{\partial^2\phi_{A,0}(y)}{\partial^2 y}$$

$$= D\frac{1}{(a+x_p-d)^2}\frac{\partial^2\phi_{A,1}(y)}{\partial^2 y} + [\nu\Sigma_f(1-\beta) - (\Sigma_a + \frac{\imath\omega}{v})]\phi_{A,1}(y) + \lambda c_{A,1}(y) \quad (25a)$$

$$\frac{(1-y)\imath\omega}{(a+x_p-d)}\frac{\partial c_{A,0}(y)}{\partial y} = \nu\Sigma_f\beta\phi_{A,1}(y) - [\lambda+\imath\omega]c_{A,1}(y) \quad (26a)$$

with the boundary condition $\phi_{A,1}(1) = 0$. We can also write down the interface conditions at the absorber surface. However, before imposing the interface conditions we will obtain explicit solutions of Eqs. (25a) and (26a). Using the expressions for $\phi_{A,0}(y)$ and $c_{A,0}(y)$ from Eqs. (21) and (19) we have

$$c_{A,1}(y) = \frac{\beta\nu\Sigma_f}{(\lambda+\imath\omega)}\phi_{A,1}(y) + \imath\omega(1-y)\frac{\beta\nu\Sigma_f}{\lambda(\lambda+\imath\omega)}E_{A,0}B_0\cos[B_0(x_p+a-d)(1-y)] \quad (27a)$$

Using (27a) in addition to (21) in (25a) we obtain

$$\frac{1}{(a+x_p-d)^2}\frac{\partial^2\phi_{A,1}(y)}{\partial^2 y} + B^2(\omega)\phi_{A,1}(y) = -2\frac{1}{(a+x_p-d)}B_0^2E_{A,0}\sin[B_0(x_p+a-d)(1-y)]$$

$$+(1-y)E_{A,0}B_0[B^2(\omega)-B_0^2]\cos[B_0(x_p+a-d)(1-y)] \quad (28a)$$

where

$$B^2(\omega) = \frac{1}{D}[(\nu\Sigma_f - \Sigma_a) - \imath\omega(\frac{1}{v} + \frac{\beta\nu\Sigma_f}{\lambda+\imath\omega})] \quad (29)$$

Eq. (28a) is an inhomogeneous equation. Its solution will contain a particular integral and a complementary function, the solution of its homogeneous part. It can be verified by direct substitution that the particular integral is given by

$$\phi_{A,1}^{P.I.}(y) = (1-y)E_{A,0}B_0\cos[B_0(x_p+a-d)(1-y)] \quad (30)$$

It is also seen that $\phi_{A,1}^{P.I.}(1) = 0$. The complementary function, therefore, must also vanish at $y = 1$. Hence we can write the solution of Eq. (28a) as

$$\phi_{A,1}(y) = (1-y)E_{A,0}B_0 \cos\left[B_0(x_p + a - d)(1-y)\right] + E_{A,1} \cdot \sin\left[B(\omega)(a + x_p - d)(1-y)\right] \quad (31a)$$

where $E_{A,1}$ is an arbitrary constant. Similarly, we can show that the solution $\phi_{B,1}(y)$ is given by

$$\phi_{B,1}(y) = -(1-y)E_{B,0}B_0 \cos\left[B_0(a - x_p - d)(1-y)\right] + E_{B,1} \cdot \sin\left[B(\omega)(a - x_p - d)(1-y)\right] \quad (31b)$$

Transforming back to variable x in Eq. (31) (note that in this transformation we now need retain only zeroeth order terms in ϵ) and combining it with Eq. (24), we see that the above particular integral, along with the factor $\epsilon e^{\imath\omega t}$, cancels the 2nd term in RHS of (24). Thus the solution $\Phi_A(x,t)$ and $\Phi_B(x,t)$ up to linear tems in ϵ is given by

$$\Phi_A(x,t) = E_{A,0} \sin B_0(a+x) + \epsilon e^{\imath\omega t} E_{A,1} \cdot \sin B(\omega)(a+x) \equiv E_{A,0}\Psi_0(x) + \epsilon e^{\imath\omega t} E_{A,1}\Psi_\omega(x) \quad (32a)$$

and

$$\Phi_B(x,t) = E_{B,0} \sin B_0(a-x) + \epsilon e^{\imath\omega t} E_{B,1} \cdot \sin B(\omega)(a-x) \equiv E_{B,0}\Psi_0(x) + \epsilon e^{\imath\omega t} E_{B,1}\Psi_\omega(x) \quad (32b)$$

In Eqs. (32), $\Psi_0(x)$ and $\Psi_\omega(x)$ are the solutions of the Helmholtz equations

$$\nabla^2\Psi_0(x) + B_0^2\Psi_0(x) = 0$$

$$\nabla^2\Psi_\omega(x) + B^2(\omega)\Psi_\omega(x) = 0 \quad (33)$$

This representation of the time-dependent flux distribution, as a sum of two Helmholtz functions, has been obtained earlier by Pázsit (1984, 1988) and Pázsit and Karlsson (1997). It is also not restricted to any particular geometry and is a general property of the time dependent diffusion and precursor equations (1) and (2). Thus the present co-ordinate transformation technique yields results in agreement with earlier works.

Expressions (32) vanish at the extrapolated boundary. We now turn our attention to the interface conditions that will specify the arbitrary constants $E_{A,1}$ and $E_{B,1}$. For this purpose we need to study the function $\phi_{r,1}(y,t)$, Eq. (17a), that determines the perturbed flux in the absorber. Using Eq. (17a) in Eq. (14) we see that $\phi_{r,1}(y,t)$ also oscillates with the frequency ω. Defining $\phi_{r,1}(y)$ by the relation $\phi_{r,1}(y,t) = \epsilon e^{\imath\omega t}\phi_{r,1}(y)$ we see from Eq. (14) that $\phi_{r,1}(y)$ satisfies the equation

$$\frac{D_r}{4d^2}\frac{\partial^2\phi_{r,1}(y)}{\partial y^2} - [\Sigma_r + \frac{\imath\omega}{v}]\phi_{r,1}(y) = -\frac{\imath\omega}{2vd}\frac{\partial\phi_{r,0}(y)}{\partial y} \quad (34)$$

The general solution of Eq. (34) is given by

$$\phi_{r,1}(y) = E_{r,1}^{(1)} \exp\left[\kappa_\omega(x_p - d + 2dy)\right] + E_{r,1}^{(2)} \exp\left[-\kappa_\omega(x_p - d + 2dy)\right]$$

$$+\kappa_r[E_{r,0}^{(1)} \exp\left[\kappa_r(x_p - d + 2dy)\right] - E_{r,0}^{(2)} \exp\left[-\kappa_r(x_p - d + 2dy)\right]] \quad (35)$$

where

$$\kappa_\omega^2 = \kappa_r^2 + \frac{\imath\omega}{D_r v} \quad (36)$$

Transforming back to (x,t) co-ordinates

$$\Phi_r(x,t) = [E_{r,0}^{(1)} \exp(\kappa_r x) + E_{r,0}^{(2)} \exp(-\kappa_r x)] + \epsilon e^{\imath\omega t}[E_{r,1}^{(1)} \exp[\kappa_\omega x] + E_{r,1}^{(2)} \exp[-\kappa_\omega x]] \quad (37)$$

Substituting the expansion (17a) into Eqs. (15a,b) and (16a,b) and equating the coefficients of ϵ, we get the following interface conditions for $\phi_{A,1}(y)$, $\phi_{B,1}(y)$ and $\phi_{r,1}(y)$

$$\phi_{A,1}(0) = \phi_{r,1}(0)$$

$$-\frac{D}{a+x_p-d}\frac{\partial \phi_{A,1}(y)}{\partial y}|_{y=0} \quad +\frac{D}{(a+x_p-d)^2}\frac{\partial \phi_{A,0}(y)}{\partial y}|_{y=0} \quad = \frac{D_r}{2d}\frac{\partial \phi_{r,1}(y)}{\partial y}|_{y=0}$$

$$\phi_{B,1}(0) = \phi_{r,1}(1)$$

$$\frac{D}{a-x_p-d}\frac{\partial \phi_{B,1}(y)}{\partial y}|_{y=0} \quad +\frac{D}{(a-x_p-d)^2}\frac{\partial \phi_{B,0}(y)}{\partial y}|_{y=0} \quad = \frac{D_r}{2d}\frac{\partial \phi_r(y,t)}{\partial y}|_{y=1} \qquad (38)$$

In Eqs. (38) we now substitute for $\phi_{A,0}(y)$, $\phi_{B,0}(y)$ and $\phi_{r,0}(y)$ using Eq. (21), Eq. (31) for $\phi_{A,1}(y)$, and a similar relation for $\phi_{B,1}(y)$ and Eq. (35) for $\phi_{r,1}(y)$. After some simplification, using Eq. (22), we have the following 4 linear equations for the determination of 4 arbitrary constants $E_{A,1}$, $E_{B,1}$, $E_{r,1}^{(1)}$ and $E_{r,1}^{(2)}$.

$$E_{A,1}\sin\left(B(\omega)(a+x_p-d)\right) + E_{A,0}B_0\cos\left(B_0(a+x_p-d)\right) =$$

$$\kappa_r\{E_{r,0}^{(1)}\exp\left[\kappa_r(x_p-d)\right]-E_{r,0}^{(2)}\exp\left[-\kappa_r(x_p-d)\right]\}+\{E_{r,1}^{(1)}\exp\left[\kappa_\omega(x_p-d)\right]+E_{r,1}^{(2)}\exp\left[-\kappa_\omega(x_p-d)\right]\}$$

$$\begin{aligned} DB(\omega)E_{A,1}\cos\left(B(\omega)(a+x_p-d)\right) - E_{A,0}DB_0^2\sin\left(B_0(a+x_p-d)\right) \quad &= \\ D_r\kappa_r^2\{E_{r,0}^{(1)}\exp\left[\kappa_r(x_p-d)\right] + E_{r,0}^{(2)}\exp\left[-\kappa_r(x_p-d)\right]\} \\ +D_r\kappa_\omega\{E_{r,1}^{(1)}\exp\left[\kappa_\omega(x_p-d)\right] - E_{r,1}^{(2)}\exp\left[-\kappa_\omega(x_p-d)\right]\} \end{aligned}$$

$$E_{B,1}\sin\left(B(\omega)(a-x_p-d)\right) - E_{B,0}B_0\cos\left(B_0(a-x_p-d)\right) =$$

$$\kappa_r\{E_{r,0}^{(1)}\exp\left[\kappa_r(x_p+d)\right]-E_{r,0}^{(2)}\exp\left[-\kappa_r(x_p+d)\right]\}+\{E_{r,1}^{(1)}\exp\left[\kappa_\omega(x_p+d)\right]+E_{r,1}^{(2)}\exp\left[-\kappa_\omega(x_p+d)\right]\}$$

$$\begin{aligned} -DB(\omega)E_{B,1}\cos\left(B(\omega)(a-x_p-d)\right) - E_{B,0}DB_0^2\sin\left(B_0(a-x_p-d)\right) \quad &= \\ D_r\kappa_r^2\{E_{r,0}^{(1)}\exp\left[\kappa_r(x_p+d)\right] + E_{r,0}^{(2)}\exp\left[-\kappa_r(x_p+d)\right]\} \\ +D_r\kappa_\omega\{E_{r,1}^{(1)}\exp\left[\kappa_\omega(x_p+d)\right] - E_{r,1}^{(2)}\exp\left[-\kappa_\omega(x_p+d)\right]\} \qquad (39) \end{aligned}$$

The solution of the linear algebric equations (39) completes the specification of the perturbations in the flux profiles $\Phi_A(x,t)$ and $\Phi_B(x,t)$, Eqs. (32a,b). In fact, both sets of algebric equations (24) and (39) can also be obtained by using Eqs. (32) and (37) in the boundary conditions (4) and equating the coefficients of ϵ^0 and ϵ^1 separately. This is possible in the present one-dimensional problem wherein we can write down the explicit solutions, Eqs. (32) and (37). The co-ordinate transformation technique enables us to extract the equations satisfied by the perturbation part of the flux, as well as boundary conditions to be imposed on them, before obtaining explicit solutions. These can then be solved numerically if desired. This procedure is quite general and can be used also in two-dimensional problems, as illustrated in section III.

B. FGW Model

As was mentioned in the introduction, in this model the localised absorber is represented by a spatial δ-function of strength γ. Thus with an absorber vibrating with an amplitude ϵ (and frequency ω) around its central position $x = x_p$, the diffusion equation (1) reduces to

$$\frac{1}{v}\frac{\partial \Phi(x,t)}{\partial t} = D\frac{\partial^2 \Phi(x,t)}{\partial x^2} + [\nu\Sigma_f(1-\beta)-\Sigma_a]\Phi(x,t) - \gamma\Phi(x,t)\delta(x-x_p-\epsilon e^{\imath\omega t}) + \lambda C(x,t) \quad (40)$$

We have to solve Eq. (40) along with the precursor equation (2) and the boundary conditions (3). The absorber once again divides the domain $x \in (-a, a)$ into 2 regions A and B, corresponding to $x \in (-a, x_p + \epsilon e^{\imath\omega t})$ and $x \in (x_p + \epsilon e^{\imath\omega t}, a)$, respectively. Since the absorber thickness is zero (there is no absorber region), we need to get the interface conditions in place of Eqs. (4). Integrating Eq. (40) over an infinitesimal region around the absorber $(x = x_p + \epsilon e^{\imath\omega t})$ we see that the flux is continuous at the absorber location while its derivative is discontinuous. Thus the new interface conditions are

$$\Phi(x_p + \epsilon e^{\imath\omega t}, t) \equiv \Phi_A(x,t) = \Phi_B(x,t) \quad |_{x=x_p+\epsilon e^{\imath\omega t}} \quad (41)$$

and

$$D\left[\frac{\partial \Phi_B(x,t)}{\partial x}\,|_{x=x_p+\epsilon e^{\imath\omega t}} - \frac{\partial \Phi_A(x,t)}{\partial x}\,|_{x=x_p+\epsilon e^{\imath\omega t}}\right] = \gamma\Phi(x_p + \epsilon e^{\imath\omega t}, t) \quad (42)$$

We note that Eq. (41) also follows from the 1st and 3rd conditions of Eqs. (4) if we let $d \to 0$. However, Eq.(42) can not be directly obtained from Eqs. (4). To achieve this, one first needs to obtain the flux solution within the rod and then take the limit $d \to 0$, $\Sigma_r \to \infty$ and $\Sigma_r 2d \to \gamma$.

The co-ordinate transformations and the definitions of transformed fluxes and precursor densities, $\phi_A(y,t)$, $\phi_B(y,t)$ etc., Eqs. (5-7), are the same if we set $d = 0$. The transformed diffusion and precursor equations along with the boundary conditions, Eqs. (9-11), are also same. The interface conditions, Eqs. (15,16), however have to be changed to

$$\phi_A(0,t) = \phi_B(0,t) \equiv \phi(0,t) \quad (43)$$

$$\frac{D}{a - x_p - \epsilon e^{\imath\omega t}}\frac{\partial \phi_B(y,t)}{\partial y}\,|_{y=0} + \frac{D}{a + x_p + \epsilon e^{\imath\omega t}}\frac{\partial \phi_A(y,t)}{\partial y}\,|_{y=0} = \gamma\phi(0,t) \quad (44)$$

The perturbation expansion, Eq. (17), is unchanged and the continuity condition (42) implies the continuity of the zeroth order and first order terms separately. Equating the coefficients of ϵ^0 in Eqs. (43) and (44) we get

$$\phi_{A,0}(0) = \phi_{B,0}(0) \equiv \phi_0(0)$$
$$\frac{D}{a - x_p}\frac{\partial \phi_{B,0}(y)}{\partial y}\,|_{y=0} + \frac{D}{a + x_p}\frac{\partial \phi_{A,0}(y)}{\partial y}\,|_{y=0} = \gamma\phi_0(0) \quad (45)$$

In view of these continuity relations the zeroth order solutions $\phi_{A,0}(y)$ and $\phi_{B,0}(y)$ are given by

$$\phi_{A,0}(y) = E_0 \sin[B_0(a - x_p)] \sin[B_0(a + x_p)(1 - y)]$$
$$\phi_{B,0}(y) = E_0 \sin[B_0(a + x_p)] \sin[B_0(a - x_p)(1 - y)] \quad (46)$$

where E_0 is the constant for normalisation. The static criticality condition is obtained from Eq. (45) as

$$B_0 \sin(2B_0 a) + \frac{\gamma}{D}\sin[B_0(a + x_p)]\sin[B_0(a - x_p)] = 0 \quad (47)$$

This equation replaces the condition of vanishing of the determinant of Eq. (23). It has been derived earlier by Pázsit (1988). The first order equations for $\phi_{A,1}(y)$, $\phi_{B,1}(y)$ $c_{A,1}(y)$, and $c_{B,1}(y)$, Eqs. (25a,b) and (26a,b) are the same with d set equal to 0. Thus the solutions, that vanish at the extrapolated boundary $y = 1$, are obtained from Eqs. (31a,b) as

$$\phi_{A,1}(y) = (1-y)B_0E_0 \sin[B_0(a-x_p)] \cos[B_0(x_p+a)(1-y)] + E_{A,1} \cdot \sin[B(\omega)(a+x_p)(1-y)] \tag{48a}$$

and

$$\phi_{B,1}(y) = -(1-y)B_0E_0 \sin[B_0(a+x_p)] \cos[B_0(a-x_p)(1-y)] + E_{B,1} \cdot \sin[B(\omega)(a-x_p)(1-y)] \tag{48b}$$

The arbitrary constants $E_{A,1}$ and $E_{B,1}$ can be evaluated from the interface conditions of $\phi_{A,1}(y)$ and $\phi_{B,1}(y)$ at $y = 0$. These can be inferred from (44) by equating the coefficients of ϵ and are given by

$$\phi_{A,1}(0) = \phi_{B,1}(0) \equiv \phi_1(0)$$

$$\frac{D}{a-x_p}\frac{\partial \phi_{B,1}(y)}{\partial y}\Big|_{y=0} + \frac{D}{(a-x_p)^2}\frac{\partial \phi_{B,0}(y)}{\partial y}\Big|_{y=0} + \frac{D}{a+x_p}\frac{\partial \phi_{A,1}(y)}{\partial y}\Big|_{y=0} - \frac{D}{(a+x_p)^2}\frac{\partial \phi_{A,0}(y)}{\partial y}\Big|_{y=0} = \gamma\phi_1(0) \tag{49}$$

Using Eqs. (48a,b) and (46) in Eqs. (49) we get

$$E_{B,1} \sin[B(\omega)(a-x_p)] - E_{A,1} \sin[B(\omega)(a+x_p)] = E_0B_0 \sin(2B_0a)$$

$$-DB(\omega)E_{B,1} \cos[B(\omega)(a-x_p)] - DB(\omega)E_{A,1} \cos[B(\omega)(a+x_p)] = \gamma\Big[\frac{E_{A,1} \sin[B(\omega)(a+x_p)] + E_{B,1} \sin[B(\omega)(a-x_p)]}{2} - E_0B_0 \sin(2B_0x_p)\Big] \tag{50}$$

Transforming back to (x,t) co-ordinates and picking up the contributions of $\phi_{A,1}(y)$ and $\phi_{A,0}(y)$ for $\Phi_A(x,t)$ and of $\phi_{B,1}(y)$ and $\phi_{B,0}(y)$ for $\Phi_B(x,t)$ we get

$$\Phi_A(x,t) = E_0 \sin[B_0(a-x_p)] \sin[B_0(a+x)] + \epsilon e^{i\omega t}E_{A,1} \cdot \sin B(\omega)(a+x) \tag{51a}$$

and

$$\Phi_B(x,t) = E_0 \sin[B_0(a+x_p)] \sin B_0(a-x) + \epsilon e^{i\omega t}E_{B,1} \cdot \sin B(\omega)(a-x) \tag{51b}$$

Eqs. (51) give Φ_A and Φ_B in the same form as Eqs. (32) before. The coefficients $E_{A,1}$ and $E_{B,1}$ are determined by Eqs. (50), leaving the coefficient E_0 for normalisation.

Solution of equations (50) and substitution into (51a,b) yields the expressions for the induced noise. Because of the periodic perturbations, one can use a general condensed notation for (51a) and (51b) in the form

$$\Phi(x,\omega) = \Phi_0(x) + \delta\Phi(x,\omega)$$

where $\Phi(x,\omega)$, as well as $\Phi_0(x)$ and $\delta\Phi(x,\omega)$, can stand for either of region A or B. They are all defined by (51a) and (51b). Identifying the neutron noise component $\delta\Phi(x,\omega)$ in this form makes it possible to make comparison with earlier results where the noise was determined directly in the frequency domain.

Comparing the solution above with the corresponding expression obtained by Pázsit (1984) for $\delta\Phi(x,\omega)$ shows that the two results are equivalent. This derivation is, however, very lengthy and will not be given here.

III. SOLUTION IN TWO DIMENSIONS

We now consider the case of a bare, homogeneous cylindrical reactor of radius R with a concentric control absorber of radius a. We will assume that the absorption cross section Σ_a of the reactor includes the axial leakage $D(\frac{\pi}{H_e})^2$, where H_e is the extrapolated height of the reactor. Thus the position vector $\mathbf{r}$ is characterised by only 2 co-ordinates r and θ of the cylindrical co-ordinates $\mathbf{r} = (r, \theta, z)$. With a concentric absorber the static flux distribution $\Phi(\mathbf{r})$ depends only on r. However, with a vibrating absorber the flux $\Phi(\mathbf{r}, t)$ depends both on r and θ, i.e. $\Phi(\mathbf{r}, t) = \Phi(r, \theta, t)$. Hence the problem is two-dimensional.

Let us assume that the absorber undergoes small oscillations with a fixed frequency ω and amplitude ϵ. As the amplitude ϵ is small, we will neglect all quadratic and higher order terms in ϵ throughout our analysis. Moreover, without any loss of generality we can assume that the rod vibrates along the $\theta = 0$ direction. This will have the consequence that the time-dependent flux will be a symmetric function of θ, which will be utilized later.

According to the above, the location of the centre of the rod at any time is $\epsilon e^{\imath\omega t}$. We have to consider, once again, Eqs. (1) and (2) which now take the form

$$\frac{1}{v}\frac{\partial\Phi(r,\theta,t)}{\partial t} = D[\frac{\partial^2\Phi(r,\theta,t)}{\partial r^2} + \frac{1}{r}\frac{\partial\Phi(r,\theta,t)}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi(r,\theta,t)}{\partial\theta^2}]$$
$$+[\nu\Sigma_f(1-\beta) - \Sigma_a]\Phi(r,\theta,t) + \lambda C(r,\theta,t) \tag{52}$$

and

$$\frac{\partial C(r,\theta,t)}{\partial t} = \nu\Sigma_f\beta\Phi(r,\theta,t) - \lambda C(r,\theta,t) \tag{53}$$

For a given direction θ and time t, the variable r ranges over $r \in (a + \epsilon e^{\imath\omega t}\cos\theta, R)$, if we neglect higher order terms in ϵ. We have thus to solve Eqs. (52) and (53) for the range of variables $r \in (a + \epsilon e^{\imath\omega t}\cos\theta, R)$, $\theta \in (0, 2\pi)$, subject to the boundary condition

$$\Phi(R,\theta,t) = 0 \qquad \forall \quad t,\theta \tag{54}$$

and another condition at the absorber surface. In the previous section we had observed that the logarithmic gradient condition, Garis et al. (1996), can not be imposed on the absorber surface. This is because in slab geometry the absorber divides the whole reactor in 2 subregions. The logarithmic gradient condition treats the criticality of 2 subregions separately. This is not the case in cylindrical geometry. Hence we employ a logarithmic gradient condition at any point $\mathbf{r}_s$, on the rod surface. Thus we have

$$\frac{\partial\Phi(\mathbf{r},t)}{\partial n}\,|_{\mathbf{r}=\mathbf{r}_s} = \mu\Phi(\mathbf{r}_s,t) \tag{55}$$

where n denotes the distance along the direction normal to the rod at the point $\mathbf{r}_s$ and $\frac{1}{\mu}$ is the extrapolation distance of the rod. In terms of (r,θ) co-ordinates Eq. (55) yields

$$[\frac{\partial\Phi(r,\theta,t)}{\partial r} + \frac{\epsilon e^{\imath\omega t}\sin\theta}{ar}\frac{\partial\Phi(r,\theta,t)}{\partial\theta}]\,|_{r=a+\epsilon e^{\imath\omega t}\cos\theta} = \mu\Phi(a+\epsilon e^{\imath\omega t}\cos\theta,\theta,t) \tag{56}$$

We now effect a co-ordinate transformation

$$r = yR + (1-y)(a+\epsilon e^{\imath\omega t}\cos\theta) \tag{57}$$

and also define

$$\Phi(r,\theta,t) = \Phi(yR + (1-y)(a+\epsilon e^{\imath\omega t}\cos\theta,\theta,t) \equiv \phi(y,\theta,t)$$

$$C(r,\theta,t) = C(yR + (1-y)(a + \epsilon e^{\imath\omega t}\cos\theta), \theta, t) \equiv c(y,\theta,t) \tag{58}$$

Using the chain rule of differentiation we get

$$\frac{\partial \Phi(r,\theta,t)}{\partial r} = \frac{1}{R - a - \epsilon e^{\imath\omega t}\cos\theta}\frac{\partial \phi(y,\theta,t)}{\partial y}$$

$$\frac{\partial \Phi(r,\theta,t)}{\partial t} = \frac{\partial \phi(y,\theta,t)}{\partial t} - \frac{\imath\omega(1-y)\epsilon e^{\imath\omega t}\cos\theta}{R - a - \epsilon e^{\imath\omega t}\cos\theta}\frac{\partial \phi(y,\theta,t)}{\partial y}$$

$$\frac{\partial \Phi(r,\theta,t)}{\partial \theta} = \frac{\partial \phi(y,\theta,t)}{\partial \theta} + \frac{(1-y)\epsilon e^{\imath\omega t}\sin\theta}{R - a - \epsilon e^{\imath\omega t}\cos\theta}\frac{\partial \phi(y,\theta,t)}{\partial y}$$

$$\frac{\partial^2 \Phi(r,\theta,t)}{\partial r^2} = \frac{1}{[R - a - \epsilon e^{\imath\omega t}\cos\theta]^2}\frac{\partial^2 \phi(y,\theta,t)}{\partial y^2}$$

$$\frac{\partial^2 \Phi(r,\theta,t)}{\partial \theta^2} = \frac{\partial^2 \phi(y,\theta,t)}{\partial \theta^2} + \frac{(1-y)\epsilon e^{\imath\omega t}\cos\theta}{R - a - \epsilon e^{\imath\omega t}\cos\theta}\frac{\partial \phi(y,\theta,t)}{\partial y} + \frac{(1-y)\epsilon e^{\imath\omega t}\sin\theta}{R - a - \epsilon e^{\imath\omega t}\cos\theta}\cdot$$

$$[2\frac{\partial^2 \phi(y,\theta,t)}{\partial y\partial \theta} + \frac{(1-y)\epsilon e^{\imath\omega t}\cos\theta}{R - a - \epsilon e^{\imath\omega t}\cos\theta}\frac{\partial^2 \phi(y,\theta,t)}{\partial y^2} - 2\frac{(1-y)\epsilon e^{\imath\omega t}\sin\theta}{R - a - \epsilon e^{\imath\omega t}\cos\theta}\frac{\partial \phi(y,\theta,t)}{\partial y}] \tag{59}$$

Substituting these expressions for partial derivatives, Eqs. (59), in Eq. (52) and neglecting all quadratic and higher order terms in ϵ we get

$$\frac{1}{v}\frac{\partial \phi(y,\theta,t)}{\partial t} - \frac{\imath\omega(1-y)\epsilon e^{\imath\omega t}\cos\theta}{v(R-a)}\frac{\partial \phi(y,\theta,t)}{\partial y} =$$

$$[\nu\Sigma_f(1-\beta) - \Sigma_a]\phi(y,\theta,t) + \lambda c(y,\theta,t)$$

$$+\frac{D}{(R-a)^2}[1 + \frac{2\epsilon e^{\imath\omega t}\cos\theta}{R-a}]\frac{\partial^2 \phi(y,\theta,t)}{\partial y^2}$$

$$+\frac{D}{(R-a)(a+(R-a)y)}[1 + \frac{\epsilon e^{\imath\omega t}\cos\theta}{R-a}]\frac{\partial \phi(y,\theta,t)}{\partial y}$$

$$+\frac{D}{(a+(R-a)y)^2}[1 - \frac{2\epsilon e^{\imath\omega t}\cos\theta}{a+(R-a)y}]\frac{\partial^2 \phi(y,\theta,t)}{\partial \theta^2}$$

$$+\frac{D}{(a+(R-a)y)^2}\frac{2(1-y)\epsilon e^{\imath\omega t}\sin\theta}{R-a}\frac{\partial^2 \phi(y,\theta,t)}{\partial y\partial \theta} \tag{60}$$

Similarly Eqs. (53), (54) and (56) are transformed to

$$\frac{\partial c(y,\theta,t)}{\partial t} - \frac{\imath\omega(1-y)\epsilon e^{\imath\omega t}\cos\theta}{(R-a)}\frac{\partial c(y,\theta,t)}{\partial y} = \nu\Sigma_f\beta\phi(y,\theta,t) - \lambda c(y,\theta,t) \tag{61}$$

$$\phi(1,\theta,t) = 0 \tag{62}$$

and

$$[\frac{1}{R-a} + \frac{\epsilon e^{\imath\omega t}\cos\theta}{(R-a)^2}]\frac{\partial \phi(y,\theta,t)}{\partial y}\Big|_{y=0} + \frac{\epsilon e^{\imath\omega t}\sin\theta}{a^2}\frac{\partial \phi(y,\theta,t)}{\partial y}\Big|_{y=0} = \mu\phi(0,\theta,t) \tag{63}$$

Substituting a perturbation series expansion

$$\phi(y,\theta,t) = \phi_0(y,\theta) + \epsilon e^{\imath\omega t}\phi_1(y,\theta) + ...$$

$$c(y,\theta,t) = c_0(y,\theta) + \epsilon e^{\imath\omega t} c_1(y,\theta) + ... \tag{64}$$

for $\phi(y,\theta,t)$ and $c(y,\theta,t)$ in Eqs. (60-63), we get for the zeroth order terms

$$[\frac{1}{(R-a)^2}\frac{\partial^2}{\partial y^2} + \frac{1}{(R-a)(a+(R-a)y)}\frac{\partial}{\partial y} + \frac{1}{(a+(R-a)y)^2}\frac{\partial^2}{\partial\theta^2}]\phi_0(y,\theta) + B_0^2\phi_0(y,\theta) = 0$$

$$c_0(y,\theta) = \frac{\beta\nu\Sigma_f}{\lambda}\phi_0(y,\theta) \tag{65}$$

along with the boundary condition $\phi_0(1,\theta) = 0$ and

$$\frac{1}{(R-a)}\frac{\partial\phi_0(y,\theta)}{\partial y}\mid_{y=0} = \mu\phi_0(0,\theta). \tag{66}$$

The general solution of Eq. (65) that vanishes at $y = 1$ and is a symmetric function of θ is given by

$$\phi_0(y,\theta) = \sum_{m=0}^{\infty} A_m[J_m(B_0\{a+(R-a)y\}).Y_m(B_0R) - Y_m(B_0\{a+(R-a)y\}).J_m(B_0R)]\cos(m\theta) \tag{67}$$

In Eq. (67) J_m and Y_m are the Bessel functions of integral order m, of first and second kind, respectively. In fact, the terms for each individual value of m separately satisfy Eq. (65) and also vanish at $y = 1$. We have formed the general solution by superposition with the arbitrary constants A_m, to be determined by the boundary condition at $y = 0$. Further, the static flux in a homogeneous cylindrical reactor with a concentric control rod is cylindrically symmetric. Thus $A_m = 0$ for $\forall m \neq 0$. The boundary condition (66), at $y = 0$, then determines the buckling B_0. However, for the moment we will not be exploiting this symmetry of the static flux. Transforming back to (r,θ,t) co-ordinates and retaining up to linear terms in ϵ we have

$$\Phi_0(r,\theta,t) = \Psi_0(r,\theta) - \frac{\epsilon e^{\imath\omega t}(R-r)\cos\theta}{R-a}\frac{\partial\Psi_0(r,\theta)}{\partial r} \tag{68}$$

where the function $\Psi_0(r,\theta)$, defined by the relation

$$\Psi_0(r,\theta) = \sum_{m=0}^{\infty} A_m[J_m(B_0r).Y_m(B_0R) - Y_m(B_0r)J_m(B_0R)]\cos(m\theta), \tag{69}$$

is a solution of the Helmholtz equation

$$\nabla^2\Psi_0(r,\theta) + B_0^2\Psi_0(r,\theta) = 0 \tag{70}$$

It will be seen below that the second term on the RHS of Eq. (69) cancels a term resulting from $\phi_1(y,\theta)$.

Let us now consider the contribution of the first order terms, $\phi_1(y,\theta)$ to the solution $\Phi(r,\theta,t)$. Equating the linear terms in ϵ in Eqs. (60-63) it is seen that $\phi_1(y,\theta)$ satisfies the equation

$$[\frac{1}{(R-a)^2}\frac{\partial^2}{\partial y^2} + \frac{1}{(R-a)(a+(R-a)y)}\frac{\partial}{\partial y} + \frac{1}{(a+(R-a)y)^2}\frac{\partial^2}{\partial\theta^2}]\phi_1(y,\theta) + B^2(\omega)\phi_1(y,\theta) =$$

$$[B^2(\omega) - B_0^2]\frac{(1-y)\cos\theta}{R-a}\frac{\partial\phi_0(y,\theta)}{\partial\theta} - \frac{2(1-y)\sin\theta}{(R-a)(a+(R-a)y)}\frac{\partial^2\phi_0(y,\theta)}{\partial y\partial\theta}$$

$$-(\cos\theta)[\frac{2}{(R-a)^3}\frac{\partial^2}{\partial y^2}+\frac{1}{(R-a)^2(a+(R-a)y)}\frac{\partial}{\partial y}-\frac{2}{(a+(R-a)y)^3}\frac{\partial^2}{\partial\theta^2}]\phi_0(y,\theta) \quad (71)$$

$$c_1(y,\theta)=\frac{\beta\nu\Sigma_f}{\lambda+\imath\omega}\phi_1(y,\theta)+\frac{\imath\omega(1-y)\cos\theta}{R-a}\frac{\beta\nu\Sigma_f}{\lambda(\lambda+\imath\omega)}\frac{\partial}{\partial y}\phi_0(y,\theta) \quad (72)$$

The general solution of the inhomogeneous Eq. (71) will contain a particular integral and a complementary solution of the corresponding homogeneous equation. Thus the most general solution which is a symmetric function of θ and vanishes at $y=1$ is given by

$$\phi_1(y,\theta)=\frac{(1-y)\cos\theta}{R-a}\frac{\partial}{\partial y}\phi_0(y,\theta)+$$

$$\sum_{m=0}^{\infty}F_m[J_m(B(\omega)\{a+(R-a)y\}).Y_m(B(\omega)R)-Y_m(B(\omega)\{a+(R-a)y\}).J_m(B(\omega)R)]\cos(m\theta) \quad (73)$$

Once again the terms corresponding to each individual m satisfy the homogeneous part of Eq. (71) and we have formed the solution by superposition with arbitrary constants F_m to be determined by the boundary condition at $y=0$. We now transform the first order solution $\phi_1(y,\theta)$ back to (r,θ,t) co-ordinates. We observe that we need retain only terms of the order of ϵ^0 as $\phi_1(y,\theta)$ itself is a first order term. Thus we have

$$\Phi_1(r,\theta,t)=\frac{(R-r)\epsilon e^{\imath\omega t}\cos\theta}{R-a}\frac{\partial\Psi_0(r,\theta)}{\partial r}+\epsilon e^{\imath\omega t}\Psi_\omega(r,\theta) \quad (74)$$

where $\Psi_\omega(r,\theta)$, defined by the series

$$\Psi_\omega(r,\theta)=\sum_{m=0}^{\infty}B_m[J_m(B(\omega)r).Y_m(B(\omega)R)-Y_m(B(\omega)r).J_m(B(\omega)R)]\cos(m\theta), \quad (75)$$

satisfies the Helmholtz equation

$$\nabla^2\Psi_\omega(r,\theta)+B^2(\omega)\Psi_\omega(r,\theta)=0 \quad (76)$$

Adding Eqs. (68) and (74) we get the solution $\Phi(r,\theta,t)$ which can be written as a sum of two Helmholtz functions that vanish at the extrapolated boundary of the reactor, i.e.

$$\Phi(r,\theta,t)=\Psi_0(r,\theta)+\epsilon e^{\imath\omega t}\Psi_\omega(r,\theta) \quad (77)$$

We now turn our attention to the boundary conditions for $\phi_1(y,\theta)$ at $y=0$. Equating the terms proportional to ϵ in Eq. (63) we get

$$\frac{1}{R-a}\frac{\partial\phi_1(y,\theta)}{\partial y}\Big|_{y=0}-\mu\phi_1(0,\theta)=-\frac{\cos\theta}{(R-a)^2}\frac{\partial\phi_0(y,\theta)}{\partial y}\Big|_{y=0}\ -\frac{\sin\theta}{a^2}\frac{\partial\phi_0(y,\theta)}{\partial\theta}\Big|_{y=0} \quad (78)$$

Transforming back to (r,θ) co-ordinates we get the following boundary condition for $\Psi_\omega(r,\theta)$ at $r=a$

$$\frac{\partial\Psi_\omega(r,\theta)}{\partial r}\Big|_{r=a}=\mu\Psi_\omega(r,\theta)-(\cos\theta)[\frac{\partial^2\Phi_0(r,\theta)}{\partial r^2}\Big|_{r=a}\ -\mu^2\Phi_0(a,\theta)]\ -\frac{\sin\theta}{a^2}\frac{\partial\Phi_0(r,\theta)}{\partial\theta}\Big|_{r=a} \quad (79)$$

Eq. (77) provides a simple way of obtaining the flux perturbations produced by a vibrating control rod. One first obtains the static flux $\Psi_0(r,\theta)$ by solving Eq. (70), subject to usual static boundary conditions. One then solves for the spatial profile of the perturbation $\Psi_\omega(r,\theta)$ by solving the Helmholtz equation (76), with its frequency dependent buckling $B^2(\omega)$, subject to the boundary condition (79) at the rod surface and usual condition, e.g. vanishing of the flux, at the outer extrapolated boundary of the reactor.

With this interpretation we can also work out the perturbations induced by an eccentric control rod. Following Lamarsh (1966) one introduces two sets of cylindrical co-ordinates, one (ρ,χ) centred along the axis of the reactor and the other (r,θ) along the axis of the rod to specify the position vector $\mathbf{r}$ of any point in the reactor. One can then write down the solutions, Ψ_0 of Helmholtz equation (70) as a superposition of Bessel functions $J_m(B_0\rho)$ (and $Y_m(B_0\rho)$ in case the solution is not regular along the axis $\rho = 0$) along with $\sin m\chi$ and $\cos m\chi$ as one term and another term involving $J_m(B_0 r)$ and $Y_m(B_0 r)$, along with $\sin m\theta$ and $\cos m\theta$. Similarly the solution Ψ_ω of Eq. (76) will contain a term which is a linear combination of $J_m(B(\omega)\rho)$ and $Y_m(B(\omega)\rho)$, $\sin m\chi$ and $\cos m\chi$ on one hand and another term that is a superposition of $J_m(B(\omega)r)$ and $Y_m(B(\omega)r)$ along with $\sin m\theta$ and $\cos m\theta$. The expansion coefficients can be determined from the boundary conditions that are the same for both Ψ_0 and Ψ_ω at the extrapolated boundary of the reactor. One has also to impose the conditions at the rod surface in its static location. This condition is well known for the static flux Ψ_0. For the perturbation Ψ_ω is given by Eq. (79), where (r,θ) co-ordinates refer to the co-ordinate system centred along the rod axis. Again, this calculation is quite involved and will not be pursued any further in this paper.

IV. CONCLUSIONS

It was shown that the time-dependent diffusion equations in reactor cores containing vibrating components of regular geometry can be handled effectively with the use of the transformation technique elaborated in this paper. The technique was originally developed for solving 1-D problems such as a vibrating boundary or a vibrating absorber rod represented as a 1-D δ function in a slab reactor. Nevertheless, it was shown here that the method can be employed even in the rather non-trivial case of an eccentric cylindrical control rod in a 2-D cylindrical reactor. The neutron fluctuations, induced by 1-D and 2-D control rod vibrations, were calculated and shown to be equivalent with those obtained from other models.

The variable transformation technique developed in this paper lends an effective tool in solving diffusion problems wherever moving boundaries are encountered. Such cases occur, besides the case of control rod vibrations treated in this paper, also in several other areas. Examples are systems with a time-varying volume such as fluidized bed cores and gaseous core reactors, future accelerator driven systems of the molten salt type, models treating core meltdown, re-flooding and other transient processes etc. In this respect a great advantage of the method is that the transformation itself, by which the boundary conditions are transformed into static ones, do not use the smallness of the perturbation. Thus, some further use of the method in various dynamical and noise problems can be expected.

V. REFERENCES

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VI. APPENDIX

We now wish to show that the results of the ϵ/d model reduce to the FGW model if the absorber thickness $2d \to 0$ and the absorber absorption cross section $\Sigma_r \to \infty$ such that $2d\Sigma_r = 2dD_r\kappa_r^2 \to \gamma$. We have thus to show that the expressions for the static fluxes $\phi_{A,0}(y)$ and $\phi_{B,0}(y)$, Eqs. (21), reduce to Eq. (46). Likewise, the flux perturbations $\phi_{A,1}(y)$ and $\phi_{B,1}(y)$, as given in Eqs. (31), must reduce to Eq. (48). It is easily seen from the functional form of Eqs. (21) and (31) that if we let $d \to 0$ we obtain Eqs. (46) and (48). Only the static flux constants $E_{A,0}$ and $E_{B,0}$, and the perturbation flux constants $E_{A,1}$ and $E_{B,1}$ have to be identified. Thus we need to show that the linear algebraic equations (23) and (39) result in (47) and (50) if we let $d \to 0$, $\Sigma_r \to \infty$ and $2d\Sigma_r \to \gamma$.

Let us first consider the static flux. Letting $d \to 0$ in the 1st and 3rd equations of the set (23) we see that

$$E_{A,0}\sin B_0(x_p + a) = E_{B,0}\sin B_0(a - x_p) \equiv E_{r,0}^{(1)}\exp(\kappa_r b) + E_{r,0}^{(2)}\exp(-\kappa_r b) \qquad (A.1)$$

which allows us to identify

$$E_{A,0} = E_0 \sin B_0(a - x_p)$$

$$E_{B,0} = E_0 \sin B_0(a + x_p)$$

$$E_{r,0}^{(1)}\exp(\kappa_r b) + E_{r,0}^{(2)}\exp(-\kappa_r b) = E_0\sin B_0(a - x_p)\sin B_0(a + x_p) \qquad (A.2)$$

The 2nd and 4th equations of the set (23) are obtained by imposing the condition of continuity of current on the two surfaces of the absorber separately. With the absorber thickness $d \to 0$, what is of interest is the difference in the currents in the two regions of the reactor at the absorber location. Subtracting the 2nd equation from the 4th and using a Taylor series expansion of $\exp(\pm\kappa_r d)$ up to linear terms in d, before letting $d \to 0$, we have

$$DB_0E_{A,0}\cos B_0(a + x_p) + DB_0E_{B,0}\cos B_0(a - x_p) = 2dD_r\kappa_r^2[E_{r,0}^{(1)}\exp(\kappa_r b) + E_{r,0}^{(2)}\exp(-\kappa_r b)]$$

$$\equiv \gamma E_0 \sin B_0(a - x_p)\sin B_0(a + x_p) \qquad (A.3)$$

Using the expressions (A.1) for $E_{A,0}$ and $E_{B,0}$ in (A.3) and dividing by the multiplicative constant E_0 we get Eq. (47) of the text.

The above procedure can also be applied to the flux perturbations $\phi_{A,1}(y)$ and $\phi_{B,1}(y)$ and the constants $E_{A,1}$ and $E_{B,1}$. If $d \rightarrow 0$ Eqs. (31a,b) reduce to Eqs. (48a,b). Setting $d \rightarrow 0$ in the 1st and 3rd of the equations of the set Eq. (39), and substituting for $E_{A,0}$ and $E_{B,0}$ from Eq. (A.2) we get

$$E_{A,1} \sin\left(B(\omega)(a+x_p)\right) + E_0 B_0 \sin\left(B_0(a-x_p)\right) \cos\left(B_0(a+x_p)\right) =$$

$$E_{B,1} \sin\left(B(\omega)(a-x_p)\right) - E_0 B_0 \sin\left(B_0(a+x_p)\right) \cos\left(B_0(a-x_p)\right)$$

$$\equiv \kappa_r \{E_{r,0}^{(1)} \exp[\kappa_r b] - E_{r,0}^{(2)} \exp[-\kappa_r b]\} + \{E_{r,1}^{(1)} \exp[\kappa_\omega b] + E_{r,1}^{(2)} \exp[-\kappa_\omega b]\} \tag{A.4}$$

Subtracting the 2nd equation from the 4th of the set (39), using Taylor series expansions of $\exp(\pm\kappa_r d)$ and $\exp[-\kappa_\omega b]$ up to linear terms in d, and then setting $d \rightarrow 0$, we have

$$-DB(\omega)E_{B,1} \cos\left(B(\omega)(a-x_p)\right) - DB(\omega)E_{A,1} \cos\left(B(\omega)(a+x_p)\right) =$$

$$2D_r\kappa_\omega^2 d\{E_{r,1}^{(1)} \exp[\kappa_\omega b] + E_{r,1}^{(2)} \exp[-\kappa_\omega b]\} + 2D_r\kappa_r^3 d\{E_{r,0}^{(1)} \exp[\kappa_r b] - E_{r,0}^{(2)} \exp[-\kappa_r b]\} \tag{A.5}$$

With $d \rightarrow 0$, we have

$$2D_r\kappa_\omega^2 d = 2D_r\kappa_r^2 d \equiv \gamma$$

Using this relation and Eq. (A.4) in (A.5) we get Eq. (50) of text.