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VORTICES, SEMI-LOCAL VORTICES IN GAUGED LINEAR SIGMA MODEL

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Abstract

We consider the static (2+1)D gauged linear sigma model. By analyzing the governing system of partial differential equations, we investigate various aspects of the model. We show the existence of energy finite vortices under a partially broken symmetry on $\mathbf{R}^2$ with the necessary condition suggested by Y. Yang[13]. We also introduce generalized semi-local vortices and show the existence of energy finite semi-local vortices under a certain condition. The vacuum manifold for the semi-local vortices turns out to be graded. Besides, with a special choice of a representation, we show that the $O(3)$ sigma model of which target space is nonlinear is a singular limit of the gauged linear sigma model of which target space is linear.

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1 Introduction

The (2+1)D gauged linear sigma model developed in [9] can be thought as a natural extension of several other field theories containing Abelian Higgs model[6] and two-Higgs extended electroweak theory of Bimonte and Lozano[1]. It is also believed that the model can contain the nonlinear gauged sigma model[9] as a certain limit. The model is for n Higgs field with corresponding n gauge fields. The model can be derived by arguments of supersymmetry or just requiring simple-minded $U(1)^n$ gauge invariance to contain Fayet-Iliopoulos D-terms[3]. The merit of this model is that it is self-dual therefore we can reduce the static equations of motion to a set of first order equations by following typical Bogomol'nyi process. In this paper, we are concerned with the solutions of the reduced system of first order equations which are called vortices or semi-local vortices when we do not have Fayet-Iliopoulos D-terms fully. Recently, Y. Yang announced in [13] that the existence of vortices of the system has been obtained on compact Riemann surfaces[14]. He also announced the existence on $\mathbf{R}^2$ in the case of the completely broken symmetry and a necessary condition for the existence of the vortices on $\mathbf{R}^2$ for several cases of partially broken symmetry. From his result, different from the fully broken symmetry case, the system on $\mathbf{R}^2$ is known to be rather subtle under partially broken symmetry. In this paper, our aim is to understand the static feature of the model rigorously and discuss the possibility whether it can contain other model which is not linear as a singular limit. We consider the case of partially broken symmetry on $\mathbf{R}^2$ at first and show that the necessary condition Y. Yang obtained is sufficient for the existence in its full generality. We also show through a singular limit of a certain parameter the system converges to the nonlinear $O(3)$ sigma model in terms of suitable variables with a special choice of a representation of the gauge group. The system also contains certain kind of vortices of which vacuum manifolds is simply connected. In this case, the actual system of governing equations turns out to be different from the vortices. Indeed, it contains strong coupling terms among Higgs fields. We approach the problems by changing the system of partial differential equations to an equivalent one which admit a variational structure and applying variational methods to them. While we deal with the singular limit of the system of elliptic partial differential equations, we reformulate the variational problem so that we can use a specific reference solution, which with the maximum principle enables us to obtain a certain kind of compactness on $\mathbf{R}^2$.

2 Gauged linear sigma model

To write the Lagrangian of the (2+1)D gauged linear sigma model on the Minkowski space with metric diag $(1,-1,-1)$, we first choose a representation of $U(1)^n$ so that the generators τ_a are integer-valued $n \times n$ diagonal matrices. From now on, we rely on the summation convention of the repeated index and μ, ν, λ run over $0, 1, 2$ and a, b, c run over $1, \cdots, n$. Also, the range of index will not be mentioned often if it is clear. Let $A = A_\mu dx^\mu = A_\mu^a \tau_a dx^\mu$ be the $U(1)^n$-valued vector potential and $\phi = (\phi_a)$ be C^n valued function. The Lagrangian density is then written

$$\mathcal{L}(A,\phi) = \frac{1}{2}(D_\mu\phi)^\dagger D^\mu\phi - \frac{1}{4e_a^2}F^a_{\mu\nu}F^{a\mu\nu} - \frac{e_a^2}{8}(R_a - \phi^\dagger\tau_a\phi)^2 \tag{1}$$

Here, the covariant derivative $D_\mu = \partial_\mu - iA_\mu$, the curvature $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu$, and R_a, e_a are positive constants. For the static case, we can derive the variational equations of the Lagrangian by extrimizing the following proposed energy functional;

$$\mathcal{E}(A,\phi) = \frac{1}{2}\int |D\phi|^2 + \frac{1}{2e_a^2}|F^a_{12}|^2 + \sum_a \frac{e_a^2}{8}(R_a - \phi^\dagger\tau_a\phi)^2 \tag{2}$$

with the Gauss' law constraints

$$\phi^\dagger\tau_a^\dagger\tau_a\phi A^a_0 - \frac{1}{e_a^2}\Delta A^a_0 = 0 \tag{3}$$

for each a. (3) is given by taking a variation of A^a_0. Since $\phi^\dagger\tau_a^\dagger\tau_a\phi$ is non-negative definite, $A^a_0 = 0$ if $\nabla A^a_0 \in L^2$. Otherwise, we arrive that

$$\int \mathcal{L} = \int (\frac{1}{2e_a^2}|\nabla A_0|^2 + |A_0\phi|^2) - \mathcal{E} = +\infty$$

since we ask $\mathcal{E}$ to be finite. Therefore, for the Lagrangian to be a meaningful action, $\nabla A^a_0 \in L^2$. Note that $A_0 = 0$ implies the system carries no charge in all components of ϕ^a which is consistent with the Abelian-Higgs system[6]. Then, for (2) to be finite, we have

$$D_j\phi,\ \ F^a_{12},\ \ R_a - \phi^\dagger\tau_a\phi \in L^2$$

for $j = 1, 2$ and $a = 1, \cdots, n$. Since τ_a are linearly independent, $R_a - \phi^\dagger\tau_a\phi \in L^2$ for all $a = 1, \cdots, n$ implies $C - |\phi|^2 \in L^2$ for a suitable constant C. Then, (2) can be rewritten as follows by Bogomol'nyi type reduction;

$$\begin{aligned}\mathcal{E} &= \frac{1}{2}\int |D_1\phi \pm iD_2\phi|^2 + |\frac{1}{e_a}F^a_{12} \mp \frac{e_a}{2}(R_a - \phi^\dagger\tau_a\phi)|^2 \\ &\pm\frac{1}{2}R_a\int F^a_{12} \pm \frac{1}{2}\lim_{s\to\infty}\int_{\partial B_s} i\phi^\dagger[(D_2\phi)dx_1 - (D_1\phi)dx_2].\end{aligned} \tag{4}$$

Here, the boundary term appeared by using the divergence theorem, which is justified since $D_i\phi \in L^2$. The boundary term is zero since $D_j\phi \in L^2$ and ϕ is bounded as $|x| \to \infty$, which comes from the fact $C - |\phi|^2 \in L^2$. The minimum of the energy is then

$$\mathcal{E} \geq \frac{1}{2}R_a|\int F^a_{12}|$$

which is a topological quantity depends only on the degree of A^a as a map to the vacuum manifold, a torus if $R_a > 0$ for all $a = 1, \cdots, n$. Therefore, it is constant among A having the same degree if $R_a > 0$ for all $a = 1, \cdots, n$. From this philosophy, The realization of the minimum is called Bogomol'nyi equations;

$$D_1\phi \pm iD_2\phi = 0 \tag{5}$$

$$F^a_{12} \mp \frac{e_a^2}{2}(R_a - \phi^\dagger\tau_a\phi) = 0. \tag{6}$$

We follow the upper sign since the other case can be obtained by a simple transform of the one. Following [9], we denote $(\tau_a)_{bc} = \tau_{ab}\delta_{bc}$, $T = (\tau_{ab})$ and introduce $\overline{A} = AT$, $\overline{F}_{12} = F_{12}T$, and $r = T^{-1}R$ with $R = (R_a)$. Then, we can rewrite (5) and (6) as follow;

$$(\partial_1 + i\partial_2 - i(\overline{A}_1^a + i\overline{A}_2^a))\phi_a = 0, \quad \forall a \tag{7}$$

$$\overline{F}_{12}^a - \frac{1}{2}M_{ab}(r_b - |\phi_b|^2) = 0 \tag{8}$$

where $M_{ab} = e_c^2\tau_{ca}\tau_{cb}$. Note M is symmetric positive definite and can be written in the matrix form T^tQ^2T by introducing the matrix $Q =$ diag $(e_1, \cdots, e_n)$. We call the system is partially symmetry broken if not all of $r_j > 0$ and fully symmetry broken otherwise. Due the finite energy condition, we have $\overline{F}_{12}, r_a - |\phi_a|^2 \in L^2$. It is standard for self-dual field theory models to fix the gauge degree of freedom by setting $\phi_a = |\phi_a|e^{\frac{i}{2}\theta_a}$ with

$$\theta_a = n_a^i Arg(x - x_a^i)$$

for positive integers n_a^i, $i = 1, \cdots, m_a$ by applying the $\overline{\partial}$-Poincaré lemma[6] to (7). Therefore, by denoting $|\phi_a|^2 = e^{u_a}$, (8) is reduced to

$$\Delta u_a = M_{ab}(e^{u_b} - r_b) + \sum_i 4\pi n_a^i \delta_{x_a^i}. \tag{9}$$

(7) can then be considered as the defining equation for $\overline{A}$ and the integrability $\nabla\overline{A} \in L^2$ obtained automatically from the condition $r_a - |\phi_a|^2 \in L^2$. For convenience's sake, we denote $N_a = \sum_i n_a^i$ from now on.

3 Function Space

In this section, we introduce the following space, $\mathcal{D}_n$;

$$\mathcal{D}_n = \{v \ : \ \mathbf{R}^n \to \mathbf{R} \mid \int |\nabla v|^n, \int \frac{v^n}{(1+|x|^2)^n} < \infty\}$$

with the norm

$$\|\cdot\|_{\mathcal{D}_n}^n = \int \frac{|v|^n}{(1+|x|^2)^n}dx + \sum_i \int |\nabla_i v|^n$$

and its subspace

$$\mathcal{D}_n^0 = \{w \in \mathcal{D}_n | \int \frac{w}{(1+|x|^2)^n} = 0.\}$$

It turns out $\mathcal{D}_2^0$ is quite natural for the problem. Now, we will give some inequalities that the elements in $\mathcal{D}_n$ satisfy and its proof for the purpose of completeness. We will use as a $W^{1,n}(S^n)$ norm the following

$$\|v\|_{W^{1,n}(S^n)}^n = \int_{S^n} |v|^n dx + \sum_i \int |\nabla_{e_i} v|^n$$

where e_i is the standard orthonormal basis of the tangent space, TS^n. This definition is equivalent to the usual norm which has $(\sum_i |\nabla_{e_i} v|^2)^{\frac{n}{2}}$ instead of $\sum_i |\nabla_{e_i} v|^n$.

Proposition 1 *$\mathcal{D}_n$, $\mathcal{D}_n^0$ are Banach spaces and is equivalent to the Sobolev space $W^{1,n}(S^n)$, $W_0^{1,n}(S^n)$ respectively therefore $v \in \mathcal{D}_n$ satisfies the following Poincaré inequality*

$$\int_{\mathbf{R}^n} \frac{|v-\overline{v}|^n}{(1+|x|^2)^n} dx \leq \frac{1}{2^n C_n} \|\nabla v\|_{L^n}^n \tag{10}$$

where C_n is the Poincaré constant for S^n and

$$\overline{v} = \frac{1}{\pi} \int_{\mathbf{R}^n} \frac{v}{(1+|x|^2)^n} dx.$$

In particular, $v \in \mathcal{D}_2^0$ satisfies the following Moser-Trudinger type inequality

$$\log \int_{\mathbf{R}^2} \frac{e^v}{(1+|x|^2)^\alpha} dx \leq \frac{1}{16\pi} \|\nabla v\|_{L^2}^2 + C \tag{11}$$

where $\alpha \geq 2$.

Proof) We show it by the stereographic projection which can be expressed in terms of spherical coordinates

$$\begin{aligned} \rho \,:\, (|x|, \theta) \in \mathbf{R} \times S^{n-1} = \mathbf{R}^n \to (\theta_1, \cdots, \theta_n) \in S^n \\ \rho(x) = \rho(|x|, \theta) = (\sin^{-1}(\frac{2|x|}{1+|x|^2}), \theta) \in S^n. \end{aligned} \tag{12}$$

The canonical measure ds_n on S^n is then[4]

$$\begin{aligned} ds_n &= \sin^{n-1}\theta_1 \cdots \sin\theta_{n-1} d\theta_1 \cdots d\theta_n \\ &= \sin^{n-1}\theta_1 ds_{n-1}(\theta_2, \cdots, \theta_n) \end{aligned} \tag{13}$$

and thus the measure $\frac{1}{(1+|x|^2)^n} dx$ can be transformed as follows.

$$\begin{aligned} d\rho(\frac{1}{(1+|x|^2)^n} dx) &= d\rho(\frac{|x|^{n-1}}{(1+|x|^2)^n} d|x| ds_{n-1}) \\ &= \frac{|x|^{n-1}}{(1+|x|^2)^n} \frac{d|x|}{d\theta_1} d\theta_1 ds_{n-1} \\ &= \frac{1}{2}(\frac{|x|}{1+|x|^2})^{n-1} d\theta_1 ds_{n-1} = \frac{1}{2^n} ds_n \end{aligned} \tag{14}$$

where we have used (12) and (13). Therefore, for any function $v \in \mathcal{D}_n(\mathbf{R}^n)$

$$\int_{\mathbf{R}^n} \frac{v}{(1+|x|^2)^n} dx = \frac{1}{2^n} \int_{S^n} v ds_n.$$

Now, we calculate ∇v on $\mathbf{R}^n$.

$$\nabla_{\mathbf{R}^n} v = (\frac{\partial}{\partial |x|} v, \frac{1}{|x|} \nabla_{S^{n-1}} v)$$

in spherical coordinates. Then,

$$\begin{aligned} |\nabla_{\mathbf{R}^n} v|^n &= \left|\frac{\partial v}{\partial |x|}\right|^n + \frac{1}{|x|^n} |\nabla_{S^{n-1}} v|^n \\ &= 2^n |1+|x|^2|^{-n} \left|\frac{\partial v}{\partial \theta_1}\right|^n + \frac{1}{|x|^n} |\nabla_{S^{n-1}} v|^n \\ &= 2^n |1+|x|^2|^{-n} \left[\left|\frac{\partial v}{\partial \theta_1}\right|^n + \left|\frac{1+|x|^2}{2|x|}\right|^n |\nabla_{S^{n-1}} v|^n\right] \\ &= 2^n |1+|x|^2|^{-n} |\nabla_{S^n} v|^n \end{aligned}$$

by (12). By the same process as in the proof of (14) and the above equation, we have

$$\begin{aligned}\int_{\mathbf{R}^n} |\nabla_{\mathbf{R}^n} v|^n dx &= \int_{\mathbf{R}^n} 2^n |1+|x|^2|^{-n} |\nabla_{S^n} v|^n dx \\ &= \int_{S^n} |\nabla_{S^n} v|^n ds_n \end{aligned} \tag{15}$$

Thus $\mathcal{D}_n$ is equivalent to $W^{1,n}(S^n)$ and $\overline{v}$ is nothing but the average of the function on S^n. From the Poincaré and Moser-Trudinger inequality[8] on $W^{1,2}(S^n)$, we have (10) and (11) by transforming them back. □

Remark

1. It is interesting (11) is sharp even for $\alpha > 2$ in the sense that the exponent $1/16\pi$ cannot be improved. Indeed, otherwise we would have an improved Moser-Trudinger inequality for functions supported in unit ball which is impossible.
2. we can extend (11) for the case $1 < \alpha < 2$ with the cost of larger exponent. Indeed,

$$\begin{aligned}\int \frac{e^u}{(1+|x|^2)^\alpha} &= \int \frac{e^u}{(1+|x|^2)^\beta}(1+|x|^2)^{\beta-\alpha} \\ &\leq \left(\int e^{2u/\beta}(1+|x|^2)^{-2}\right)^{\beta/2} \|(1+|x|^2)^{\beta-\alpha}\|_{L^{2/(2-\beta)}} \\ &\leq C_\beta e^{\|\nabla u\|^2_{L^2}/(8\pi\beta)} \end{aligned} \tag{16}$$

for any $\beta < 2\alpha - 2$ by the Hölder's inequality.
3. $u \in \mathcal{D}_n$ grows slower than any polynomial due to the Hardy's inequality[7] and satisfies the divergence theorem $\int \mathbf{R}^n \Delta u = 0$[2].

From now on, any integrals and spaces are assumed to be over $\mathbf{R}^2$ unless otherwise mentioned.

4 Existence of vortices

Since we are interested in the situation under partially broken symmetry, we suppose $r_1, \cdots, r_{a_1} > 0$ and $r_{a_1+1} = \cdots = r_n = 0$ for some $1 \leq a_1 < n$. To be an energy finite solution, $u_a \to \log r_a$ as $x \to \infty$ for $a = 1, \cdots, a_1$ and $u_a = o(|x|^{-1})$ for $a > a_1$. We set $\alpha_a = 0$ for $a \leq a_1$ and $\alpha_a > 1$ otherwise, and set $v_a = u_a - f_a$ where

$$f_a = n_a^i \log \frac{|x - x_a^i|^2}{1+|x|^2} - \alpha_a \log(1+|x|^2). \tag{17}$$

We note that f_a is a solution of the following equation

$$\Delta f_a = n_a^i \delta_{x_a^i} - g_a$$

with

$$g_a = \frac{4(N_a + \alpha_a)}{(1+|x|^2)^2}.$$

(9) is now reduced to

$$\Delta v_a = M_{ab}(e^{v_b + f_b} - r_b) + g_b. \tag{18}$$

To satisfy the energy finite condition, we will try to find a solution of (18) satisfying $v_a - \log r_a \in H^1$, the usual Sobolev space, for $a = 1, \cdots, a_1$ and $v_a \in \mathcal{D}_2$ for $a = a_1 + 1, \cdots, n$. We have the identity

$$\int \Delta v = 0$$

for $v \in \mathcal{D}_2$ by the divergence theorem[2]. Therefore, by integrating both sides of (18), we apriorily have

$$M_{ab} \int (e^{v_b + f_b} - r_b) = -4\pi(N_a + \alpha_a).$$

By multiplying by the inverse of M we obtain

$$\int e^{v_a + f_a} = -4\pi M_{ab}^{-1}(N_b + \alpha_b)$$

for $b > a_1$. Then, we decompose $v_a = w_a + \overline{v}_a$, $w_a \in \mathcal{D}_2^0$ for $a > a_1$ to have

$$e^{\overline{v}_a} = -\frac{4\pi M_{ab}^{-1}(N_b + \alpha_b)}{\int e^{w_a + f_a}}.$$

Since the right hand side of the above equation is positive, it is necessary that $M_{ab}^{-1}(N_b + \alpha_b) < 0$ to have a solution in the space. For convenience, we will denote $w_a = v_a - \log r_a$ for $a \le a_1$ and $w_a = v_a - \overline{v}_a$ for $a > a_1$. Then, $(w_a) \in (H^1)^{a_1} \times (\mathcal{D}_2^0)^{n-a_1}$ which we will denote by $\mathcal{T}^{a_1}$. (18) are reduced further to

$$M_{ab}^{-1} \Delta w_b = r_a(e^{w_a + f_a} - 1) + M_{ab}^{-1} g_b \quad \text{for } a \le a_1 \tag{19}$$

$$M_{ab}^{-1} \Delta w_b = \frac{V_a e^{w_a + f_a}}{\int e^{w_a + f_a}} + M_{ab}^{-1} g_b \quad \text{otherwise} \tag{20}$$

where $V_a = -4\pi M_{ab}^{-1}(N_b + \alpha_b) > 0$.

Theorem 1 *For a symmetric positive definite $n \times n$ matrix (M_{ab}), positive integers n_a^i, $x_a^i \in \mathbf{R}^2$, and α_a satisfying $\alpha_1 = \cdots = \alpha_{a_1} = 0$ and $\alpha_{a_1+1}, \cdots, \alpha_n > 1$, there is a unique energy finite solution $v \in \mathcal{T}^{a_1}$ of (18) on $\mathbf{R}^2$ if and only if (n_a^i) and α_a satisfy the following condition;*

$$M_{ab}^{-1}(N_b + \alpha_b) < 0 \quad \textit{for } a > a_1. \tag{21}$$

Proof) (21) is necessary for the existence due to the above analysis and it is the same as that appeared in [14] for some special cases. It is enough to show the existence of the solution of (19) and (20). Let us consider the following functional on $\mathcal{T}^{a_1}$;

$$\mathcal{F}(w_a) = \frac{1}{2} \int M_{ab}^{-1} \nabla w_a \cdot \nabla w_b + \sum_{a > a_1} V_a \log(\int e^{w_a + f_a})$$
$$+ \sum_{a \le a_1} \int \left[r_a(e^{f_a}(e^{w_a} - 1) - w_a - 1) + M_{ab}^{-1} w_a g_b \right]. \tag{22}$$

The functional $\mathcal{F}$ is well defined due to (10), (11) and the fact $\alpha_{a_1+1}, \cdots, \alpha_n > 1$. (19), (20) can be identified as the Frechét derivative of $\mathcal{F}$. Indeed, since $g_a \sim (1 + r^2)^{-2}$ and M is symmetric, by the Lagrange's multiplier principle, the variational equation with respect to w_a, $a > a_1$ is

$$M_{ab}^{-1} \Delta w_b = \frac{V_a e^{w_a + f_a}}{\int e^{w_a + f_a}} + \Lambda(1 + r^2)^{-2}$$

for the Lagrange multiplier Λ. By integrating the above equation, we arrive $\Lambda = 4\pi M^{-1}_{ab}(N_b+\alpha_b)$ which says that $\Lambda(1+r^2)^{-2} = M^{-1}_{ab} g_b$ proving (20). (19) can be retrieved by direct differentiation with respect to w_a, $a \le a_1$. We use (A.1) to show that the functional

$$\mathcal{F}_1(w_a) = \int \left(\frac{1}{2}|\nabla w_a|^2 + e^{f_a}(e^{w_a} - 1) - w_a + hw_a\right)$$

is coercive lower semi-continuous in H^1 for any $h \in L^2$. Indeed, $D_{w_a}\mathcal{F}_1(w_a)$ is of the same form as in (A.1) except the term hw_a and thus $D_{w_a}\mathcal{F}_1(w_a) > C(\|w_a\|^{4/3}_{H^1} - 1)$ concluding $\mathcal{F}_1$ is coercive. Therefore, if we show that

$$\mathcal{F}_2(w) = \int |\nabla w|^2 + C \log \int e^{w+f_a}$$

is coercive lower semi-continuous in $\mathcal{D}_2$ for any $C > 0$, the functional $\mathcal{F}$ itself becomes coercive since M is positive definite, which proves the theorem by identifying the global minimum of the functional as a solution. In fact, $\mathcal{F}_2$ is even coercive for $0 \ge C > -\frac{1}{16\pi(\alpha_a - 1)}$ by (16). Given any $q > 1$, we have

$$\int e^{f_a} = \int e^{w/q - w/q + f_a} \le (\int e^{w+f_a})^{1/q} (\int e^{-w/(q-1)+f_a})^{(q-1)/q}$$

by the Hölder's inequality. By substituting $\alpha = \alpha_a$ and $u = \frac{w}{q-1}$ in (16), we have

$$\begin{aligned}\int e^{w+f_a} &\ge C_a (\int e^{-w/(q-1)+f_a})^{1-q} \\ &\ge C_a e^{-|\nabla w|^2/8\pi\beta(q-1)}\end{aligned}$$

for any $\beta < \alpha_a - 1$. Here, $C_a = (\int e^{f_a})^q$. If we take q large enough such that $\frac{C}{8\pi\beta(q-1)} < 1$, we arrive that the functional $\mathcal{F}_2$ is coercive for any $C > 0$. The solution we have found is of finite energy if $D_j\phi, R_a - \phi^\dagger \tau_a \phi \in L^2$ by (4) or equivalently if $\partial_j \phi, r_a - |\phi_a|^2 \in L^2$ by the definition of r_a and (7). This is exactly what we have found.

On the uniqueness, supposing there are two solutions (v^1_a) and (v^2_a) for (18), we multiply the equation by the inverse matrix M^{-1} and subtract the equation of v^2_a from that of v^1_a. Finally, integrating after multiplying it by $v^1_a - v^2_a$, we have

$$\int M^{-1}_{ab} \nabla(v^1_a - v^2_a) \cdot \nabla(v^1_b - v^2_b) + \int (v^1_a - v^2_a)(e^{v^1_a} - e^{v^2_a}) = 0$$

by the Green's theorem. The positive-definiteness of the matrix M and the mean value theorem applied to the second term imply $v^1_a = v^2_a$. □

Although we deal with $a < n$, the proof still holds for the fully broken symmetry case, $a_1 = n$.

5 Semi-local vortices

In the proof of the theorem 1, the positive definiteness of M has been used crucially however similar existence results can be shown even if M is not positive definite. When $A_a, e_a = 0$ for certain $n-k$ number of a, we call the finite energy solution of (18) k semi-local vortices which can be thought as a generalization of $U(1)^2$ semi-local vortices[5, 10]. In this case, the finite energy conditions $r_a - |\phi_a|^2 \in L^2$ cannot be deduced any more. Instead we have $R_a - \phi^\dagger \tau_a \phi \in L^2$ for certain k number of a since we do not have Fayet-Iliopoulos D-term fully in (1). We also need $D\phi \in L^2$ in this case. We can assume, by relabeling the index a, $e_a = 0$ for $a = k+1, \cdots, n$. Although we do not have Fayet-Iliopoulos D-term fully in (1), we can still follow the Bogomol'nyi reduction in the section 2 by simply replacing $A^a, F^a = 0$ for $a = k+1, \cdots, n$. Then, (5) and (6) are deduced. We can proceed to the transformation leading to (7) and (8) by completing the vector (R_a) by choosing arbitrary R_a for $a = k+1, \cdots, n$. The remaining procedures in the section 2 can be applied in this case too, which justify (18). Therefore, $v \in \mathcal{T}^0$, a solution of (9) satisfying the condition $R_a - \tau_{ab} e^{v_b + f_b} \in L^2$, $a = 1, \cdots, k$ is a k semi-local vortices for (1). Note $M = T^t Q^2 T$ is of rank k in this case. Introducing new variables $\tilde{v} = (T^t)^{-1} v$, we have

$$\Delta(\tilde{v}) = Q^2 T (e^{T^t \tilde{v} + f} - r) - T^{-t} g$$

where T^{-t} is the inverse of T^t and $e^{T^t \tilde{v} + f} - r$ is the column vector of elements $e^{T^t_{ab} \tilde{v}_b + f_a} - r_a$. We notice the $k+1, \cdots, n$-th components of the above equation. They do not have the nonlinear term since $e_a = 0$ for $a = k+1, \cdots, n$. If $v \in \mathcal{T}^0$, $\tilde{v}_a = T^{-t} v \in \mathcal{D}_2$ too and since the divergence theorem holds in $\mathcal{D}_2$, we have $\int T^{-t} g = \tau_{ca}^{-1}(N_c + \alpha_c) = 0$ for all $k+1 \le a \le n$ by integrating them, which implies $\Delta \tilde{v} = 0$ for $k+1 \le a \le n$. Thus $\tilde{v}_a = C_a$, a constant using the remark of the proposition 1. The equations are then reduced to

$$\Delta \tilde{v}_a = e_a^2 \tau_{ab} (G_b \exp(\sum_{c \le k} \tau_{cb} \tilde{v}_c + f_b)) - e_a^2 R_a - \tau_{ba}^{-1} g_b \tag{23}$$

for $a \le k$ since $\tau_{ab} r_b = R_a$. Here, we denoted $\exp(\sum_{b>k} \tau_{ba} C_b)$ by G_a. Therefore, we do not have the dependence on R_a, $a > k$ which we put artificially to derive the self-dual equations of the semi-local vortices. Since the k number of row vectors, $(\tau_{a \cdot})$, $a = 1, \cdots, k$ are linearly independent, we can choose k columns, $1 \le t_1 < \cdots < t_k \le n$ such that the matrix Z with $Z_{ab} = \tau_{a t_b}$, $a, b = 1, \cdots, k$ is a $k \times k$ nonsingular matrix. Next, denoting the remaining columns in increasing order by $s_1 < \cdots < s_{n-k}$, we can express them uniquely by t_b, $b = 1, \cdots, k$ columns. That is,

$$\tau_{a s_b} = Z_{ac} J_{cb} = \tau_{a t_c} J_{cb}$$

for some $k \times (n-k)$ matrix J. Thus $\sum_{b \le k} \tau_{ba} \tilde{v}_b$ can be represented by $Z^t \tilde{v}|_j$ if $a = t_j$, $j = 1, \cdots, k$ and $J^t Z^t \tilde{v}|_j$ if $a = s_j$, $j = 1, \cdots, n-k$. We again introduce new variables $\tilde{w} = Z^t \tilde{v}$ and end up with the equation

$$\begin{aligned} \Delta \tilde{w}_a &= Z_{ca} e_c^2 \tau_{c t_b} (G_{t_b} e^{\tilde{w}_b + f_{t_b}}) \\ &+ Z_{ca} e_c^2 \tau_{c s_b} (G_{s_b} e^{J_{db} \tilde{w}_d + f_{s_b}}) - Z_{ca} e_c^2 R_c - Z_{ca} \tau_{bc}^{-1} g_b. \end{aligned} \tag{24}$$

(24) are of the same form as (18) with positive e_a, $a = 1, \cdots, k$ except the coupling terms of the form $e^{J_{db} \tilde{w}_d}$. Note that the matrix $\overline{M}_{ab} \equiv Z_{ca} e_c^2 \tau_{c t_b} = Z^t_{ac} e_c^2 Z_{cb}$ is symmetric positive definite by

definition. The integrability conditions also become

$$
\begin{aligned}
&R_a - \tau_{ab} G_b \exp(\sum_{c\le k} \tau_{cb}\tilde{v}_c + f_b) \\
&= R_a - \tau_{at_b} G_{t_b} e^{\tilde{w}_b + f_{t_b}} - \tau_{as_b} G_{s_b} e^{J_{db}\tilde{w}_d + f_{s_b}} \in L^2
\end{aligned}
$$

for $a = 1, \cdots, k$. Therefore, if we define $\overline{r} = ZR$, the conditions is reduced to

$$
\overline{r}_a - G_{t_a} e^{\tilde{w}_a + f_{t_a}} - J_{ab} G_{s_b} e^{J_{db}\tilde{w}_d + f_{s_b}} \in L^2 \tag{25}
$$

for all $1 \le a \le k$. And (24) is reduced to

$$
\begin{aligned}
\overline{M}^{-1}_{ab} \Delta \tilde{w}_b &= G_{t_a} e^{\tilde{w}_a + f_{t_a}} + J_{ab} G_{s_b} e^{J_{db}\tilde{w}_d + f_{s_b}} \\
&\quad - \overline{r}_a + Z_{ab} e_b^{-2} T^{-t}_{bc} g_c.
\end{aligned} \tag{26}
$$

Theorem 2 *Let T be a representation and $Z_{ab} \equiv (\tau_{at_b})$, $1 \le t_1 < \cdots < t_k \le n$ be its nonsingular $k \times k$ submatrix. For a given distribution of zeros of ϕ_a, x^i_a with its multiplicities, n^i_a, the charges at infinity, $\alpha_a > 1$ or $\alpha_a = 0$, there exists a $(n-k)$ parameter family of k semi-local vortices for (1) satisfying (25) if the charges at infinity, $\alpha_{t_a} = 0$, $\overline{r}_a = Z_{ab} R_b > 0$ for $a = 1, \cdots, k$, and $\tau^{-1}_{ca}(N_c + \alpha_c) = 0$ for any $k+1 \le a \le n$.*

Proof) Under the same notations as before, we shall find a solution of (26), $\tilde{w} \in (\mathcal{D}_2)^k$ for any positive values of G_a, $a = 1, \cdots, n$. Suppose $m \le n-k$ number of α are zero among $\alpha_{s_1}, \cdots, \alpha_{s_{n-k}}$. Without loss of generality, we can assume $\alpha_{s_1} = \cdots = \alpha_{s_{n-k-m}} = 0$ which we call the directions of broken symmetry. Let $B = \{s_1, \cdots, s_{n-k-m}\}$ be the the set of indexes corresponding to the directions of broken symmetry. From the lemma A.2, we have k number of constants, K_a satisfying (A.2). From this, we can define $y_a = \tilde{w}_a - K_a$ and rewrite (26) in terms of y;

$$
\begin{aligned}
\overline{M}^{-1}_{ab} \Delta y_b &= S_{t_a}(e^{y_a + f_{t_a}} - 1) + \sum_{b\in B} J_{ab} S_{s_b}(e^{J_{db} y_d + f_{s_b}} - 1) \\
&\quad + \sum_{b\notin B} J_{ab} S_{s_b} e^{J_{db} y_d + f_{s_b}} + Z_{ab} e_b^{-2} T^{-t}_{bc} g_c
\end{aligned} \tag{27}
$$

where $S_{t_a} = G_a e^{K_a}$ and $S_{s_b} = G_{s_b} e^{J_{db} K_d}$. (27) is in fact the Frechét derivative of the functional $\int \overline{\mathcal{F}}$ on $(H^1)^k$ where

$$
\begin{aligned}
\overline{\mathcal{F}}(\tilde{w}) &= \frac{1}{2} \overline{M}^{-1}_{ab} \tilde{w}_a \tilde{w}_b + S_{t_a}(e^{y_a + f_{t_a}} - y_a - 1) \\
&\quad + \sum_{b\in B} S_{s_b}(e^{J_{db} y_d + f_{s_b}} - J_{db} y_d - 1) \\
&\quad + \sum_{b\notin B} S_{s_b} e^{J_{db} y_d + f_{s_b}} + Z_{ab} e_b^{-2} T^{-t}_{bc} g_c \tilde{w}_a.
\end{aligned} \tag{28}
$$

The functional $\overline{\mathcal{F}}$ is well defined on $(H^1)^k$ and continuous by the usual Trudinger inequality. To check the coercivity condition, we calculate $D_y \overline{\mathcal{F}}$ which is only differ from the functional $\mathcal{F}_1$ in the proof of theorem 1 by

$$\sum_{b\in B} S_{s_b}(J_{db}y_d e^{J_{db}y_d+f_{s_b}} - J_{db}y_d) + \sum_{b\notin B} S_{s_b}y_d J_{db}e^{J_{db}y_d+f_{s_b}} = I + II$$

The first term is of the form $\int \overline{y}_b(e^{\overline{y}_b+f_{s_b}}-1)$ if we apply the change of variables, $\tilde{y} = J^t y$. Thus,

$$I \geq \sum_{b\in B}\int S_{s_b}e^{f_{s_b}}\tilde{y}_b(e^{\tilde{y}_b}-1) - C\int|\tilde{y}_b||e^{f_{s_b}}-1| \geq -C\|y_b\|_{L^2}$$

since $e^{f_{s_b}}-1 \in L^2$ by (17) and $\tilde{y}_a(e^{\tilde{y}_a}-1)\geq 0$. The second term is

$$II = \sum_{b\notin B} S_{s_b}\tilde{y}_b e^{f_{s_b}}(e^{\tilde{y}_b}-1) + S_{s_b}e^{f_{s_b}}\tilde{y}_b \geq S_{s_b}e^{f_{s_b}}\tilde{y}_b$$

which is again bounded by $-C\|y\|_{L^2}$ since $e^{f_{s_b}} \in L^2$ for $s_b \notin B$. Applying (A.1), we finish the proof for the existence.

Next, supposing there are two solutions $\tilde{w}^1, \tilde{w}^2 \in (\mathcal{D}_2)^k$ for (26), we have

$$\begin{aligned}\overline{M}_{ab}^{-1}\Delta(\tilde{w}_b^1-\tilde{w}_b^2) &= G_{t_a}e^{f_{ta}}(e^{\tilde{w}_a^1}-e^{\tilde{w}_a^2})\\ &\quad +J_{ab}G_{s_b}e^{f_{s_b}}(e^{J_{db}\tilde{w}_d^1}-e^{J_{db}\tilde{w}_d^2})\end{aligned}$$

Applying the mean value theorem to the right hand side and the divergence theorem to the left hand side after multiplying $\tilde{w}_a^1 - \tilde{w}_a^2$, summing and integrating, we arrive a contradiction which show there are only one 1 semi-local vortices for a fixed (G_a). Since (τ_{ab}), $a > k$, $1 \leq b \leq n$ is of rank $n-k$, only $n-k$ number of G_a are linearly independent and the theorem is proved. ∎

Remark

1. When $k = n-1$ and τ_1 is the identity, the above theorem states that solutions satisfy $\sum_{a\leq k}|\phi_a|^2 = R_1$ at infinity according to the number of directions of broken symmetry $1 \leq k \leq n$. Therefore, the vacuum manifold of 1 semi-local vortices is the $n-1$-dimensional complex sphere, CS^{n-1}. Similarly, as k varies from $n-1$ to 1, we have as a vacuum manifold a k dimensional hypersurface of CS^{n-1}. As a totality, the vacuum manifold for semi-local vortices is a graded sum of these hypersurfaces. The one described in [9] corresponds to $CS^1 = S^2$ with $n = 2$.
2. Here, we assumed $r_a > 0$ for $a \leq k$ which means nondegenerate directions are broken. We do not know whether this condition is necessary, however, with special assumptions on T, we can show similar existence theorem without this condition. A *systematic* approach to this case is open to the author.

6 Limiting Behavior

In this section, we consider the solution of (18) when $n = 2$ and $a_1 = 1$. In this case, we have two parameters $e_1, e_2 > 0$. We fix $e_2 = 1$, the representation τ as $\tau_1 = \text{diag}\ (1,1)$, $\tau_2 = \text{diag}$

$(1,-1)$ and investigate the behavior of the vortices when $e_1 \to +\infty$. Note that T is symmetric in this case. For simplicity, we change the notations r_1 by r, e_1 by $\frac{1}{\epsilon}$ so that

$$M = \begin{bmatrix} 1+\epsilon^{-2} & -1+\epsilon^{-2} \\ -1+\epsilon^{-2} & 1+\epsilon^{-2} \end{bmatrix}, \quad M^{-1} = \frac{1}{4}\begin{bmatrix} 1+\epsilon^{2} & -1+\epsilon^{2} \\ -1+\epsilon^{2} & 1+\epsilon^{2} \end{bmatrix}.$$

The existence condition (21) then becomes $N_1(-1+\epsilon^2)+(N_2+\alpha_2)(1+\epsilon^2) < 0$ which is satisfied for small enough ϵ provided $N_1 > N_2+\alpha_2$. Therefore, for any positive integers N_1, N_2, and a real number $\alpha_2 > 1$ satisfying $N_1 > N_2+\alpha_2$, we can study the behavior of the solution as $\epsilon \to 0$. We thus assume $N_1 > N_2+\alpha_2$ and $0 < \epsilon < \frac{N_1-N_2-\alpha_2}{N_1+N_2+\alpha_2}$ in this section. This condition under which we can study the limit of the solution is exactly the necessary and sufficient condition of the existence of solutions of $O(3)$ sigma model in $\mathcal{D}_2$[12];

$$\Delta v_\sigma = \frac{4re^{v_\sigma+f_2^\sigma-f_1^\sigma}}{1+e^{v_\sigma+f_2^\sigma-f_1^\sigma}} + g_2^\sigma - g_1^\sigma \quad \text{on } \mathbf{R}^2. \tag{29}$$

Here, f_a^σ, g_a^σ, $a=1,2$ are

$$f_a^\sigma = \sum_i \log\frac{|x-x_a^i|^2}{\kappa^2+|x-x_a^i|^2} - \alpha_a\log(\kappa^2+|x|^2) \tag{30}$$

$$g_a^\sigma = \sum_i \frac{4n_a^i\kappa^2}{(\kappa^2+|x-x_a^i|^2)^2} + \frac{4\alpha_a\kappa^2}{(\kappa^2+|x|^2)^2} \tag{31}$$

which differ from f_a and g_a in (17) up to smooth L^2 functions. We are going to show $w_2^\epsilon - w_1^\epsilon$ converges to the solution of $O(3)$ sigma model. To be consistent with literatures we use $w_1^\epsilon = v_1 - \log r + f_1 - f_1^\sigma$ and $w_2^\epsilon = v_2 - \log r + f_2 - f_2^\sigma$ and we assume the positions of vortices of the two fields do not coincide $x_1^i \neq x_2^j$ for any i,j in this section. The merit of new reference solution, f_a^σ is that we have an inequality

$$e^{f_a^\sigma} + g_a^\sigma - 1 \le 0 \tag{32}$$

for large enough κ which is the lemma 3.7 in [6] while the former reference solution, f_a enables us get rid of inhomogeneous terms in the functional $\mathcal{F}$ and reduce $\tau_{ba}^{-1}v_b = C_a$, $a > k$ in the formulation of the semi-local vortices. For the purpose of simplicity, we denote f_a^σ and g_a^σ by the same symbol f_a and g_a in this section. We also fix κ large enough so that (32) and lemma A.3 hold. Then the correct system of equations are

$$\Delta w_1^\epsilon = r(1+\epsilon^{-2})(e^{w_1^\epsilon+f_1}-1) + r(e^{-2}-1)e^{w_2^\epsilon} + g_1 \tag{33}$$

$$\Delta w_2^\epsilon = r(\epsilon^{-2}-1)(e^{w_1^\epsilon+f_1}-1) + r(e^{-2}+1)e^{w_2^\epsilon} + g_2 \tag{34}$$

Lemma 1 *$(w_1^\epsilon, w_2^\epsilon)$ satisfy*

$$e^{w_1^\epsilon+f_1},\ e^{w_2^\epsilon+f_2},\ e^{w_1^\epsilon+f_1} + \frac{M_{12}}{M_{11}}e^{w_2^\epsilon+f_2} \le 1 \tag{35}$$

and

$$\int|1-e^{w_1^\epsilon+f_1}|,\ \int e^{w_2^\epsilon+f_2},\ \epsilon^2\int|1-e^{w_1^\epsilon+f_1}-e^{w_2^\epsilon+f_2}| < C \tag{36}$$

uniformly in ϵ.

Proof) Denoting $G \equiv e^{w_1^\epsilon+f_1} + \frac{M_{12}}{M_{11}} e^{w_2^\epsilon+f_2}$, we have

$$\begin{aligned} \Delta G &= \Delta(e^{w_1^\epsilon+f_1} + \frac{M_{12}}{M_{11}} e^{w_2^\epsilon+f_2}) \\ &\geq r(M_{11} e^{w_1^\epsilon+f_1} + \frac{M_{12}^2}{M_{11}} e^{w_2^\epsilon+f_2})(G-1) \end{aligned}$$

using $M_{11} = M_{22}$, $M_{12} = M_{21}$. Remembering $w_1^\epsilon + f_1 \to 1$ and $w_2^\epsilon + f_2 \to -\infty$ as $|x| \to \infty$, $G \leq 1$ is deduced by means of the maximum principle. As a byproduct, we also have $e^{w_1^\epsilon+f_1} \leq 1$. We do the same calculation to $e^{w_2^\epsilon+f_2}$ and obtain

$$\Delta e^{w_2^\epsilon+f_2} \geq e^{w_2^\epsilon+f_2}(M_{12}(e^{w_1^\epsilon+f_1} - 1) + M_{22} e^{w_2^\epsilon+f_2}).$$

Since $e^{w_2^\epsilon+f_2}$ goes to zero as $|x|$ goes to infinity, $e^{w_2^\epsilon+f_2}$ attains its' nonzero maximum and by the maximum principle again,

$$\max e^{w_2^\epsilon+f_2} \leq \frac{M_{12}}{M_{22}}(1 - e^{w_1^\epsilon+f_1}) \leq 1.$$

Next, to get the second estimate, we integrate the transformed equations of (33) and (34)

$$M_{ij}^{-1} \Delta w_j^\epsilon = r(e^{w_i^\epsilon+f_i} - \delta_{1i}) + M_{ij}^{-1} g_j.$$

Then we have $\int e^{w_i^\epsilon+f_i} - \delta_{1i} = M_{ij}^{-1}(N_j + \alpha_j)$. Since M^{-1} is bounded and by the fact $G \leq 1$, the first two inequalities in (36) is derived. We also integrate the sum of (33) and (34);

$$\epsilon^2 \Delta(w_1^\epsilon + w_2^\epsilon) = 2r(e^{w_1^\epsilon+f_1} - 1 + e^{w_2^\epsilon+f_2}) + \epsilon^2(g_1 + g_2)$$

and obtain $\int e^{w_1^\epsilon+f_1} - r + e^{w_2^\epsilon+f_2} = O(\epsilon^2)$. The last in (36) then follows from the positivity in (35), the second inequality in (36), and the fact of $|\frac{M_{11}}{M_{12}} - 1| = O(\epsilon^2)$. $\square$

With these estimates, we can give some bounds of the Sobolev norms of $1 - e^{w_1^\epsilon+f_1}$, $e^{w_2^\epsilon+f_2}$.

Lemma 2 $1 - e^{w_1^\epsilon+f_1}$, $e^{w_2^\epsilon+f_2}$ *are bounded uniformly in* H^1. *As a result,* $1 - e^{w_1^\epsilon+f_1}$, $e^{w_2^\epsilon+f_2}$ *converges in* L^2_{loc} *to some nonnegative functions by taking subsequences if necessary. Besides,* $w_2^\epsilon - w_1^\epsilon - C_\epsilon$ *converges, again by taking subsequences, in* $C^{0,\beta}_{loc}$ *for any* $0 < \beta < 1$ *with* $C_\epsilon = w_2^\epsilon(0) - w_1^\epsilon(0)$.

Proof) Since $e^{w_2^\epsilon+f_2}$, $1 - e^{w_1^\epsilon+f_1}$ are uniformly bounded in L^p, $1 \leq p \leq \infty$ by (35) and (36), it is enough to show that $\nabla(1 - e^{w_1^\epsilon+f_1})$, $\nabla(e^{w_2^\epsilon+f_2})$ are bounded uniformly in L^2. We multiply $e^{2w_1^\epsilon+2f_1}$ and $e^{2w_2^\epsilon+2f_2}$ to (33) and (34) respectively and integrate them giving

$$\begin{aligned} \int 2|\nabla e^{w_1^\epsilon+f_1}|^2 &= -\int (M_{11}(e^{w_1^\epsilon+f_1} - 1) + M_{12} e^{w_2^\epsilon+f_2}) e^{2w_1^\epsilon+2f_1} \\ \int 2|\nabla e^{w_2^\epsilon+f_2}|^2 &= -\int (M_{21}(e^{w_1^\epsilon+f_1} - 1) + M_{22} e^{w_2^\epsilon+f_2}) e^{2w_2^\epsilon+2f_2} \end{aligned}$$

by using the integration by parts. Since $e^{2w_1^\epsilon+2f_1}$, $e^{2w_2^\epsilon+2f_2}$ are uniformly bounded, we have

$$\begin{aligned}\|\nabla e^{w_1^\epsilon+f_1}\|_{L^2}^2 &\le \int \epsilon^2|e^{w_2^\epsilon+f_2}+e^{w_1^\epsilon+f_1}-1| \\ &+C\int|e^{w_1^\epsilon+f_1}-1|+C\int e^{w_2^\epsilon+f_2}.\end{aligned}$$

The right hand side is bounded uniformly by (36). Similar procedure gives estimate for $\nabla e^{w_2^\epsilon+f_2}$. Next, we subtract (33) from (34) and have

$$\Delta(w_2^\epsilon-w_1^\epsilon)=2r(1-e^{w_1^\epsilon+f_1}+e^{w_2^\epsilon+f_2})+g_2-g_1 \tag{37}$$

The right hand side is bounded uniformly in L^p for every $1\le p\le+\infty$ by (36). Therefore, using the Calderon-Zygmund inequality and the Morrey's inequality, we have $\frac{|(w_2^\epsilon-w_1^\epsilon)(x_1)-(w_2^\epsilon-w_1^\epsilon)(x_2)|}{|x_1-x_2|^\beta}$ is bounded uniformly for all $0<\beta<1$. Thus, by taking a subsequence, $w_2^\epsilon-w_1^\epsilon-C_\epsilon$ converges in $C^{0,\beta}_{loc}$ for all $0<\beta<1$ using Ascoli-Azela theorem. □

Lemma 3 *$w_2^\epsilon, w_1^\epsilon$ as before, one of the following holds.*
1) For any ball B centered at the origin

$$\lim_{\epsilon\to0}\int_B|1-e^{w_1^\epsilon+f_1}|, e^{w_2^\epsilon+f_2}\to 0.$$

2) $w_2^\epsilon-w_1^\epsilon$ converges in $C^{0,\beta}_{loc}$ by taking a subsequence if necessary.

Proof) Suppose 1) does not hold. Then there exists a ball B satisfying $\int_B|1-e^{w_1^\epsilon+f_1}|>0$ or $\int_B e^{w_2^\epsilon+f_2}>0$ uniformly. Due to (36), the two are equivalent. Since $\int|1-e^{w_1^\epsilon+f_1}|$ is uniformly bounded by (35), by taking a larger ball which we still denote by B, we can assume $\int_B|1-e^{w_1^\epsilon+f_1}|<\frac{1}{2}|B|$ uniformly. We take a subsequence and assume $1-e^{w_1^\epsilon+f_1}\to\xi_1$, $e^{w_2^\epsilon+f_2}\to\xi_2$ in L^2_{loc}, and $w_2^\epsilon-w_1^\epsilon-C_\epsilon\to w_\sigma$ by using the lemma 2. Then, on the ball B, $\xi_1<1$ on a set of positive measure otherwise we have $\int_B\xi_1=|B|$ which is a contradiction with the convergence of $1-e^{w_1^\epsilon+f_1}$ in L^2_{loc} and the lower semi-continuity of the norm. Also, by the assumption, $\xi_1,\xi_2>0$ on a subset of positive measure. We consider the following quantity

$$\begin{aligned}\int_B e^{w_2^\epsilon+f_1+f_2} &= \int_B e^{\tilde{w}_1+f_1}e^{w_2^\epsilon-w_1^\epsilon+f_2} \\ &= e^{C_\epsilon}\int_B(1-\xi_1)e^{w_\sigma+f_2}+o(1)e^{C_\epsilon}.\end{aligned}$$

We already have

$$\int_B e^{w_2^\epsilon+f_1+f_2}=\int_B e^{f_1}\xi_2+o(1)$$

which imply a contradiction if $C_\epsilon\to\pm\infty$. We arrive that C_ϵ is bounded, which is enough. □

Until now, we depend only on the structure of the equations which comes via the maximum principle and the uniform bound of the integrals of $e^{w_1^\epsilon+f_1}-1$ and $e^{w_2^\epsilon+f_2}$. At this stage we use another property of the solution, that is, minimizing property to get the key estimate. To do that, we recast the variational functional $\mathcal{F}$ in theorem 1. Since the solution is unique, the solution can be characterized by a minimizer of the following variant $\mathcal{F}_{\mathcal{N}}$ of the original functional $\mathcal{F}$ on a Nehari's manifold

$$\mathcal{N} = \{u \in \mathcal{D}_2 | \int e^{u+f_2} = 4\pi(N_1 - N_2 - \alpha_2 + \epsilon^2(N_1 + N_2 + \alpha_2))\},$$

$$\mathcal{F}_{\mathcal{N}}(u) = \int \left[\frac{1}{2}M_{ij}\nabla u_i \cdot \nabla u_j + e^{u_1+f_1} - u_1 - e^{f_1} + M_{ij}^{-1}u_i g_j\right]$$

by the Lagrange's multiplier principle. Indeed, on $\mathcal{N}$,

$$\overline{u}_2 = -\log \int e^{u_2 - \overline{u}_2 + f_2} + C$$

and therefore the two functionals are different by a constant resulting that $\mathcal{F}_{\mathcal{N}}$ is coercive and has a minimizer. The explicit expression of M is plugged in and the explicit formula of $\mathcal{F}_{\mathcal{N}}$ is obtained

$$\begin{aligned}\mathcal{F}_{\mathcal{N}} &= \int \Big[\frac{1}{8}|\nabla(w_1^\epsilon - w_2^\epsilon)|^2 + \frac{\epsilon^2}{8}|\nabla(w_1^\epsilon + w_2^\epsilon)|^2 \\ &\quad + e^{w_1^\epsilon + f_1} - w_1^\epsilon - e^{f_1} + \frac{1}{4}(g_1 - g_2)(w_1^\epsilon - w_2^\epsilon) \\ &\quad + \frac{\epsilon^2}{4}(g_1 + g_2)(w_1^\epsilon + w_2^\epsilon)\Big].\end{aligned}$$

Lemma 4 $\|\nabla(w_1^\epsilon - w_2^\epsilon)\| < +\infty$ *uniformly.*

Proof) We first note that since $w_1^\epsilon - w_2^\epsilon$ is the minimizer, $\mathcal{F}_{\mathcal{N}}(w_1^\epsilon - w_2^\epsilon)$ is bounded above uniformly by plugging $(0, C)$ into $\mathcal{F}_{\mathcal{N}}$ with $C \in \mathcal{N}$. Next, we divide $\mathcal{F}_{\mathcal{N}}$ into two parts $\mathcal{F}_{\mathcal{N}} = \int \mathcal{F}_{\mathcal{N}}^1 + \mathcal{F}_{\mathcal{N}}^2$;

$$\begin{aligned}\mathcal{F}_{\mathcal{N}}^1 &= \frac{1-\epsilon^2}{8}\Big[|\nabla(w_1^\epsilon - w_2^\epsilon)|^2 + 8(e^{w_1^\epsilon + f_1} - w_1^\epsilon - e^{f_1}) \\ &\quad + 2(g_1 - g_2)(w_1^\epsilon - w_2^\epsilon)\Big] \\ \mathcal{F}_{\mathcal{N}}^2 &= \frac{\epsilon^2}{8}\left[|\nabla w_i^\epsilon|^2 + 8(e^{w_1^\epsilon + f_1} - w_1^\epsilon - e^{f_1}) + 4g_i w_i^\epsilon\right].\end{aligned}$$

$\int \mathcal{F}_{\mathcal{N}}^2$ is bounded below uniformly by (A.1). Then $\int \mathcal{F}_{\mathcal{N}}^1 < C$ uniformly. By (35) and lemma A.3,

$$-(g_1 - g_2)w_2^\epsilon = -(g_1 - g_2)(w_2^\epsilon + f_2) + (g_1 - g_2)f_2 \geq (g_1 - g_2)f_2$$

and

$$\begin{aligned}\int (g_1 - g_2)w_1^\epsilon &\geq \int (g_1 - g_2)(w_1^\epsilon + f_1) - C, \\ \int (e^{w_1^\epsilon + f_1} - w_1^\epsilon - e^{f_1}) &\geq \int e^{f_1}(e^{w_1^\epsilon} - w_1^\epsilon - 1) \\ &\quad + \int (e^{f_1} - 1)(w_1^\epsilon + f_1) - C \\ &\geq \int (e^{f_1} - 1)(w_1^\epsilon + f_1) - C\end{aligned}$$

Due to (32), the sum of the above two terms are then bounded below uniformly. All above estimates lead to the conclusion. □

Theorem 3 *Let $w^\epsilon \in \mathcal{T}^1$ be the solution of (33) and (34). $w_2^\epsilon - w_1^\epsilon$ converges in $C_{loc}^{0,\beta}$ to the solution in $\mathcal{D}_2$ of the $O(3)$ sigma model as $\epsilon > 0$ goes to zero*

Proof) Given any test function $\psi \in C_0^\infty$, we have

$$\int \Delta(w_2^\epsilon - w_1^\epsilon)\psi = \int [2r(1 - e^{w_1^\epsilon + f_1} + e^{w_2^\epsilon + f_2}) + g_2 - g_1]\psi \tag{38}$$

by multiplying by ψ and integrating (37). For the case 2) of the lemma 3, due to the lemma 4, we can choose a subsequence of $w_2^\epsilon - w_1^\epsilon$ converging in $H_{loc}^1 \cap C_{loc}^{0,\beta}$ of which limit is denoted by $w_\sigma \in \mathcal{D}_2 \cap C^{0,\beta}$. Then,

$$\int (w_2^\epsilon - w_1^\epsilon)\Delta\psi = \int w_\sigma \Delta\psi + o(1)$$

and

$$\begin{aligned} &1 - e^{w_1^\epsilon + f_1} + e^{w_2^\epsilon + f_2} - \frac{2e^{w_2^\epsilon + f_2}}{e^{w_1^\epsilon + f_1} + e^{w_2^\epsilon + f_2}} \\ &= \frac{(e^{w_1^\epsilon + f_1} - e^{w_2^\epsilon + f_2})(1 - e^{w_1^\epsilon + f_1} - e^{w_2^\epsilon + f_2})}{e^{w_1^\epsilon + f_1} + e^{w_2^\epsilon + f_2}} \end{aligned}$$

by direct calculation. Thus, in view of (36),

$$1 - e^{w_1^\epsilon + f_1} + e^{w_2^\epsilon + f_2} \longrightarrow \frac{2e^{w_2^\epsilon + f_2}}{e^{w_1^\epsilon + f_1} + e^{w_2^\epsilon + f_2}}$$

in L^1. Therefore, by sending $\epsilon \longrightarrow 0$, we have

$$\int w_\sigma \Delta\psi = \int \left[4r\frac{e^{w_\sigma + f_2}}{e^{f_1} + e^{w_\sigma + f_2}} + g_2 - g_1\right]\psi$$

since $w_2^\epsilon - w_1^\epsilon$ converges in $C_{loc}^{0,\beta}$. There is a only one solution in $\mathcal{D}_2$ of (29) and the limit belongs to $\mathcal{D}_2$. Therefore, the original sequence converges to w_σ.
For the case 1) in the lemma 3, if we denote the limit of $w_2^\epsilon - w_1^\epsilon - C_\epsilon$ by η, (38) converges to

$$\int \eta\Delta\psi = \int (g_2 - g_1)\psi.$$

Since $\eta \in \mathcal{D}_2$ by the lemma 4, applying the divergence theorem to the equation $\Delta\eta = g_2 - g_1$, we arrive

$$0 = \int g_2 - g_1 = 4\pi(N_2 + \alpha_2 - N_1)$$

which is a contradiction and proves the theorem. □

Remark

1. It is unknown to the author whether one can derive a correct generalization of $O(3)$ sigma

model for higher dimensional spheres by analyzing the limiting behavior of the solution for $n > 2$ under partially broken symmetry.

2. When we think of (33) and (34) on a flat compact Riemann surface of unit volume and genus greater than 0 with all the assumptions of the above theorem and $\alpha_2 = 0$, the equations can be considered as a Bogomol'nyi reduction of (1) on that surface[13, 14]. The existence of the solution is assured for small enough ϵ if we assume further $\pi(N_1 - N_2) < r$[13, 14]. Then (35) and (36) hold still and, as a result, we can easily deduce that case 2) in the lemma 3 cannot happen without the aid of lemma 4 since the base manifold is compact and we have (36).

3. From the limiting equation, it is clear why we assumed $x_1^i \neq x_2^j$ for all i, j. If some of them coincide, then they annihilate each other and only the term with the higher multiplicity remains.

7 Appendix

Lemma A.1 *Given f defined in (17) with $\alpha = 0$ and u defined on $\mathbf{R}^2$, we have*

$$\int e^f(e^u - 1)u + |\nabla u|^2 \geq C\|u\|_{H^1}^{4/3} - C. \tag{A.1}$$

Proof) We will adapt arguments in [11]. First, we observe $(e^u - 1)u \geq u^2$ if $u \geq 0$ and $\geq \frac{u^2}{1+|u|}$ if $u < 0$ now that $e^{-t} \leq (1+t)^{-1}$ for $t < 0$.

$$\int_{B^c} e^f(e^u - 1)u \geq C\int_{B^c} \frac{u^2}{1+|u|}$$

where B is a ball containing the set $\{x | e^{f(x)} < \frac{1}{2}\}$. Since f goes to zero at finitely many points polynomially, using theorem 4.8 in [11],

$$\int_B e^f(e^u - 1)u \geq \int_B e^f \frac{u^2}{1+|u|} \geq \|\frac{u^2}{1+|u|}\|_{L^\gamma(B)}\|e^f\|_{L^{\gamma'}(B)}$$

for $0 < \gamma < 1$ and $\gamma' < 0$ satisfying $\gamma^{-1} + \gamma'^{-1} = 1$. By taking $|\gamma'|$ small enough, we have

$$\int_B e^f(e^u - 1)u \geq C\|\frac{u^2}{1+|u|}\|_{L^\gamma(B)}.$$

By using the Hölder's inequality, $\gamma < 1$ and the above inequality, we have

$$\begin{aligned} \|u\|_{L^2}^2 &\leq C\|\frac{|u|^2}{1+|u|}\|_{L^1(B^c)}^{2/3}\|u^2(1+|u|)^2\|_{L^1(B^c)}^{1/3} \\ &+C\|\frac{u^2}{1+|u|}\|_{L^\gamma(B)}^{\gamma/2}\|(1+|u|^\gamma)u^{4-2\gamma}\|_{L^1(B)}^{1/2} = I + II \end{aligned}$$

With the notation $\delta = \int e^f(e^u - 1)u$,

$$\begin{aligned} I &\leq C\delta^{2/3}\|u\|_{L^2}^{2/3}(\|u\|_{L^2}^2 + \|u\|_{L^4}^4)^{1/3} \\ &\leq C\delta^{2/3}\|u\|_{L^2}^{2/3}(1 + \|\nabla u\|_{L^2}^{2/3}) \\ &\leq C\delta^{3/2} + \|\nabla u\|_{L^2}^3 + \frac{1}{4}\|u\|_{L^2}^2 + C \end{aligned}$$

by the Sobolev inequality, $\|u\|_{L^4}^2 \leq C\|u\|_{L^2}\|\nabla u\|_{L^2}$. The second term can also be also bounded by

$$\begin{aligned} II &\leq C\delta^{\gamma/2}\||u|^{4-\gamma}\|_{L^1(B)}^{1/2} \leq C\delta^{\gamma/2}(\|u\|_{L^2} + \|u\|_{L^4}^2) \\ &\leq C\delta^{\gamma/2}\|u\|_{L^2}(1 + \|\nabla u\|_{L^2}) \leq C\delta^{3/2} + \|\nabla u\|_{L^2}^{6/(3-2\gamma)} + \frac{1}{4}\|u\|_{L^2}^2 + C \end{aligned}$$

again by the Sobolev inequality and the Young's inequality. If we take γ small enough, $\|u\|_{L^2}^2 \leq C + C\delta^{3/2} + C\|\nabla u\|_{L^2}^3$ which proves the lemma. □

Lemma A.2 *Given J, a $k \times l$ matrix and $G_a, L_b > 0$ for all $a = 1\cdots, k$, $l = 1\cdots, l$, there exists a solution K_d, $d = 1, \cdots, k$ of the following system of equations*

$$\overline{r}_a - G_a e^{K_a} - J_{ab} L_b e^{J_{db}K_d} = 0 \tag{A.2}$$

for all $1 \leq a \leq k$ if $\overline{r}_a > 0$ for all $1 \leq a \leq k$.

Proof) The system of equations is the derivative of the following function

$$U = \overline{r}_a K_a - G_a e^{K_a} - L_b e^{J_{db}K_d}.$$

For each a, we have

$$\overline{r}_a K_a - G_a e^{K_a} \begin{cases} \leq -\overline{r}_m |K_a| & \text{if } K_a \leq 0 \\ \leq \overline{r}_M |K_a| - G_m e^{|K_a|} & \text{otherwise} \end{cases}$$

where G_m is the minimum among G_a and $\overline{r}_M$, $\overline{r}_m$ are the maximum and minimum among $\overline{r}_a$ respectively. Since exponential function grows faster than a linear function, there exists $K_0 > 0$ such that when $K_a > K_0$, $\overline{r}_M|K_a| - G_m e^{|K_a|} < 0$. Therefore, $\overline{r}_a K_a - G_a e^{K_a} \leq r_M K_0$. Now that we have $|K_a| \geq \frac{|K|}{\sqrt{k}}$ for some a, by assuming it be K_1 and $|K|$ large enough, we have

$$U < -r_m \frac{|K|}{\sqrt{k}} + k\overline{r}_M K_0 < -r_m \frac{|K|}{2\sqrt{k}}$$

if $K_1 < 0$. If $K_1 > 0$, we have

$$U < \overline{r}_M \frac{|K|}{\sqrt{k}} - G_m e^{\frac{|K|}{\sqrt{k}}} + k\overline{r}_M K_0 < -\frac{G_m}{2} C e^{\frac{|K|}{\sqrt{k}}}.$$

It means the function U decreases to $-\infty$ as $|K|$ goes to infinity and therefore the solution of (A.2) is realized as a maximum point of the function U. □

Lemma A.3 *Let g_1 and g_2 be g_1^σ and g_2^σ defined in (31). $g_1 - g_2 \geq 0$ if $N_1 > N_2 + \alpha_2$ and κ is large enough.*

Proof) We take $0 < 2\delta < N_1 - N_2 + \alpha_2$ and denote $\alpha_2 = n_2^0$ for convenience. We can rewrite

$$\begin{aligned}\frac{g_1 - g_2}{4\kappa^2} &= \sum_i \frac{n_1^i}{(\kappa^2 + |x - x_1^i|^2)^2} - \sum_j \frac{n_2^j}{(\kappa^2 + |x - x_2^j|^2)^2} \\ &= \sum_i \left(\frac{n_1^i}{(\kappa^2 + |x - x_1^i|^2)^2} - \sum_j \frac{\tilde{n}_2^j}{(\kappa^2 + |x - x_2^j|^2)^2} \right)\end{aligned}$$

with $\sum_j \tilde{n}_2^j < n_1^i - 2\delta$. We use Young's inequality repeatedly to obtain

$$\begin{aligned}(\kappa^2 + |x - x_1^i|^2)^2 &\leq (\kappa^2 + (|x - x_2^j| + |x_2^j - x_1^i|)^2)^2 \\ &\leq (1 + \frac{\delta}{N_1})(\kappa^2 + |x - x_1^j|^2)^2 + C_\delta |x_2^j - x_1^i|^4\end{aligned}$$

which enables us to estimate

$$\frac{g_1 - g_2}{4\kappa^2} \geq \sum_{ij} \frac{\frac{\delta}{N_1}(\kappa^2 + |x - x_2^j|^2)^2 - C_\delta |x_2^j - x_1^i|^4}{(\kappa^2 + |x - x_1^i|^2)^2 (\kappa^2 + |x - x_2^j|^2)^2}.$$

Finally, we take $\kappa^4 > \frac{N_1 C_\delta}{\delta} \sup_{ij} |x_2^j - x_1^i|^4$ and finish the proof. □

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