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COHERENT STATES FOR OSCILLATORS OF NON-CONVENTIONAL STATISTICS

Dao Vong Duc
Institute of Physics, P.O. Box 429 Bo Ho, Hanoi 10 000, Vietnam[1]
and
The Abdus Salam International Centre for Theoretical Physics, Trieste, Italy

and

Nguyen Ba An
Institute of Physics, P.O. Box 429 Bo Ho, Hanoi 10 000, Vietnam.

Abstract

In this work we consider systematically the concept of coherent states for oscillators of non-conventional statistics - parabose oscillator, infinite statistics oscillator and generalised q-deformed oscillator. The expressions for the quadrature variances and particle number distribution are derived and displayed graphically. The obtained results show drastic changes when going from one statistics to another.

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[1]Permanent address.

1 INTRODUCTION

The formalism of coherent states is an important part of quantum theory. Coherent states play a special role not only in optics but also in condensed matter physics and particle physics. They describe physically realizable systems such as light emitted from a laser operating far above threshold or Bose-Einstein condensated excitons/biexcitons, etc. ... Another fact worth mentioning is that in some nonlinear interaction processes a coherent state may evolve into a non-classical state with properties having no classical counterparts such as squeezing, antibunching, ...

Coherent states are classical states governed by an external pumping source. Unlike Fock states, in which the particle number is definite while the phase is completely random, coherent states have small phase fluctuation but contain an uncertain particle number. For this reason coherent states can be mathematically viewed as eigenstates of the oscillator annihilation operator.

A good deal of work has been devoted to the study of coherent states for bosons and fermions. During the last few years the concept of coherent states has been extended [1, 2] to the q-deformed oscillators. These have been originally introduced [3, 4] in connection with the description of quantum groups and algebras [5, 6, 7]. The study of q-deformed oscillators has also been stimulated during recent years by the increasing interest in particles obeying statistics different from Bose and Fermi.

The aim of this work is to study systematically the concept of coherent states for oscillators of non-conventional statistics, namely oscillators of parabose statistics [8, 9], infinite statistics [10, 11] and generalised q-deformed oscillators. The obtained results include those for conventional Bose statistics at special values of the parameters involved.

2 OSCILLATORS OF PARASTATISTICS

As has been shown in Refs. [12, 13], the parabose oscillator of order p obeys the commutation relations

$$[a, a^+] = 1 + (p-1)(-1)^N$$

$$[N, a] = -a \tag{1}$$

where N is oscillator number operator having the form

$$N = \frac{1}{2}(a^+a + aa^+ - p) \tag{2}$$

The algebra (1) can be realised in the Fock space spanned by the orthonormalised eigenstates of the operator N defined as follows

$$|n>=\frac{1}{\sqrt{n_{(p)}!}}a^{+n}|0> \tag{3}$$

$$n_{(p)}=\begin{cases} n, & n \quad \text{even} \\ n+p-1, & n \quad \text{odd} \end{cases}$$

$$n_{(p)}! \equiv 1_{(p)}2_{(p)}...n_{(p)}.$$

This leads to the commutation relation

$$[P,Q]=-i\hbar(1+(p-1)(-1)^N) \tag{4}$$

between coordinate and momentum operators defined as

$$Q \equiv (\frac{\hbar}{2m\omega})^{1/2}(a^+ + a)$$

$$P \equiv i(\frac{m\hbar\omega}{2})^{1/2}(a^+ - a) \tag{5}$$

with m and ω being mass and frequency of the oscillator.

Instead of (5) the following dimensionless quantities are often used

$$\tilde{Q}=(\frac{m\omega}{2\hbar})^{1/2}Q=\frac{1}{2}(a^+ + a)$$

$$\tilde{P}=(2m\hbar\omega)^{-1/2}P=\frac{i}{2}(a^+ - a).$$

The coherent states $|z>$ defined as eigenstates of the annihilation operator a

$$a|z>=z|z> \tag{6}$$

can be constructed in the following manner

$$|z>=C(z)e_{(p)}^{za^+}|0>=C(z)\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n_{(p)}!}}|n> \tag{7}$$

where $C(z)$ is the normalisation constant,

$$C(z)=\frac{1}{\sqrt{e_{(p)}^{|z|^2}}} \tag{8}$$

and $e_{(p)}^x$ is the "para-exponential" function defined through the formula

$$e_{(p)}^x \equiv \sum_{n=0}^{\infty}\frac{x^n}{n_{(p)}!}. \tag{9}$$

For the quadrature variances the calculations give the following result

$$< z|(\Delta\tilde{Q})^2|z> = < z|(\Delta\tilde{P})^2|z> = \frac{1}{4}F \tag{10}$$

where

$$F = 1 + (p-1)e_{(p)}^{-|z|^2}\left(e_{(p)}^{|z|^2}\right)^{-1}. \tag{11}$$

So, the uncertainty relation takes the form

$$< (\Delta\tilde{Q})^2 >< (\Delta\tilde{P})^2 > = \frac{1}{16}F^2. \tag{12}$$

The average particle number in the coherent state (7) is found to be

$$< z|N|z > = |z|^2 + \frac{1}{2}(p-1)\left[e_{(p)}^{-|z|^2}(e_{(p)}^{|z|^2})^{-1} - 1\right] \tag{13}$$

and the probability of the coherent state being in the n-particle state is

$$W_n = |< n|z >|^2 = \left(e_{(p)}^{|z|^2}\right)^{-1}\frac{|z|^{2n}}{n_{(p)}!}. \tag{14}$$

It is straightforward to check that by putting $p = 1$ in the above obtained formulae we recover the familiar results for coherent states of conventional bosonic oscillators.

The results (11), (13) and (14) are graphically presented in Figs. 1, 2 and 3 respectively.

Figure 1 shows that the non-conventional Shot Noise Limit $F/4$ is always greater than the conventional one because $F > 1$ for $p > 1$.

For a given coherent state (with given $|z|$ or $< z|N|z >$) it increases with the parastatistics order p. Especially the vacuum fluctuations verify a simple rule

$$< 0|(\Delta\tilde{Q})^2|0 >) = < 0|(\Delta\tilde{P})^2|0 > = \frac{p}{4}.$$

Figure 2 shows that the characteristic $|z|^2$ of the non-conventional coherent state is less then the average particle number and the difference increases with the parastatistics order p.

From Fig. 3 it is seen that the particle number distribution differs notably from the Poisson's when $p > 1$. The whole distribution is shifted to the side of smaller n and exhibits oscillations which are more pronounced at higher values of p.

3 OSCILLATORS OF INFINITE STATISTICS

The concept of infinite statistics has been first introduced by Greenberg [14, 15] in considering the oscillator obeying the relation

$$aa^+ = 1. \tag{15}$$

The particle number operator satisfying the commutation relation $[N, a] = -a$ now has the form

$$N = a^+a + a^+a^+aa + ... \equiv \sum_{k=1}^{\infty} a^{+k}a^k. \tag{16}$$

The orthonormalised eigenstates of the operator N are

$$|n>= (a^+)^n|0> . \tag{17}$$

It is worth noting that all the states $|n>$ with $n > 0$ are eigenstates of a^+a with the same eigenvalue 1, i.e.

$$a^+a|n>= |n>, \qquad n > 0. \tag{18}$$

From (15) we have

$$[P, Q] = -i\hbar(1 - a^+a). \tag{19}$$

The coherent states are of the form:

$$|z>= C(z)\sum_{n=0}^{\infty}(za^+)^n|0>= C(z)\sum_{n=0}^{\infty} z^n|n>,$$

$$C(z) = \left(\sum_{n=0}^{\infty}|z|^{2n}\right)^{-1/2} \tag{20}$$

Note that at $|z| \geq 1$ the sum $\sum_{n=0}^{\infty}|z|^{2n}$ diverges, and therefore the norm $C(z)$ vanishes. Hence physically realizable could be only the states with $|z| < 1$. In this case we can rewrite (20) as

$$|z>= (1 - |z|^2)^{\frac{1}{2}}\sum_{n=0}^{\infty} z^n|n> . \tag{21}$$

The condition $|z| < 1$ can also be considered as a direct consequence of infinite statistics. In fact, from (15), (16) and (20) one can derive that

$$|z|^2 = \frac{< z|N|z >}{< z|N|z > +1} \tag{22}$$

and hence $|z|^2 < 1$, and

$$< z|N|z >= \frac{|z|^2}{1 - |z|^2}. \tag{23}$$

The quadrature variances turn out to be

$$< z|(\Delta\tilde{Q})^2|z >=< z|(\Delta\tilde{P})^2|z >= \frac{1}{4}(1 - |z|^2) \tag{24}$$

and hence

$$< (\Delta\tilde{Q})^2 >< (\Delta\tilde{P})^2 >= \frac{1}{16}(1-|z|^2)^2. \tag{25}$$

It is noted that in contrast with the parastatistics, the infinite statistics yields a smaller Shot Noise Limit than conventional statistics does. Moreover, the system tends to become purely classical with

$$< z|(\Delta\tilde{Q})^2|z >=< z|(\Delta\tilde{P})^2|z >\to 0 \tag{26}$$

in the limit $|z| \to 1$, in which $< z|N|z >\to \infty$.

The probability for the coherent state (21) to be in the n particle state is

$$W_n = (1-|z|^2)|z|^{2n} = \frac{< z|N|z >^n}{(< z|N|z > +1)^{n+1}}. \tag{27}$$

The particle distribution (27) is a monotonically decreasing function of n, a fact completely different from the case of conventional coherent states.

4 GENERALISED q-DEFORMED OSCILLATORS

The relation (15) for oscillator of infinite statistics may be considered as a special case of the commutation relation for the generalised q-deformed oscillator of the form [16]

$$aa^+ - qa^+a = q^{cN} \tag{28}$$

when going to the limit: first $c \to 0$, then $q \to 0$. Here q and c are some parameters. The usual q-deformation [3, 4]

$$aa^+ - qa^+a = q^{-N} \tag{29}$$

corresponds to the value $c = -1$.

The algebra (28) can be realised in the Fock space spanned by the orthonormalised eigenstates of N,

$$|n >\equiv \frac{1}{[n]_q^{(c)}!}a^{+n}|0 > \tag{30}$$

where the general notation

$$[x]_q^{(c)} \equiv \frac{q^x - q^{cx}}{q - q^c} \tag{31}$$

is used, and

$$[n]_q^{(c)}! \equiv [1]_q^{(c)}[2]_q^{(c)}...[n]_q^{(c)}.$$

In the Fock space (30) the following relations hold

$$a^+a = [N]_q^{(c)}, \qquad aa^+ = [N+1]_q^{(c)} \tag{32}$$

and the algebra (28) leads to the commutation relation between coordinate and momentum of the form

$$[P,Q] = -i\hbar\left([N+1]_q^{(c)} - [N]_q^{(c)}\right). \tag{33}$$

For the calculations it is helpful to make use of the identities

$$[x+y]_q^{(c)} = q^y[x]_q^{(c)} + q^{cx}[y]_q^{(c)}$$

and

$$[-x]_q^{(c)} = -q^{-(c+1)}[x]_{q^{-1}}^{(c)}. \tag{34}$$

The coherent states are constructed as follows

$$|z> = \frac{1}{\sqrt{[e]^{|z|^2}}}[e]^{za^+}|0> = \frac{1}{\sqrt{[e]^{|z|^2}}}\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{[n]_q^{(c)}!}}|n> \tag{35}$$

where $[e]^x$ is the "generalised q-deformed exponential" defined through the formula

$$[e]^x \equiv \sum_{n=0}^{\infty}\frac{x^n}{[n]_q^{(c)}!}. \tag{36}$$

The calculations give the following result for the quadrature variances

$$< z|(\Delta\tilde{Q})^2|z> = < z|(\Delta\tilde{P})|z> = \frac{1}{4}F \tag{37}$$

where

$$F = (q-1)|z|^2 + \left([e]^{|z|^2}\right)^{-1}[e]^{q^c|z|^2}. \tag{38}$$

In particular, for the case $c=0$ this gives

$$< z|(\Delta\tilde{Q})^2|z> = < z|(\Delta\tilde{P})^2|z> = \frac{1}{4}\left((q-1)|z|^2+1\right). \tag{39}$$

This formula recovers the result (24) of the previous section in the limit $q=0$.

The average particle number in the coherent state (35) and the probability for this state to be in the n-particle state are

$$< z|N|z> = |z|^2\left([e]^{|z|^2}\right)^{-1}\left([e]^{(1)|z|^2}\right) \tag{40}$$

$$W_n = ([e]^{|z|^2})^{-1}\frac{|z|^{2n}}{[n]_q^{(c)}!} \tag{41}$$

where we denote

$$[e]^{(1)x} \equiv \frac{d}{dx}[e]^x \equiv \sum_{n=0}^{\infty}\frac{(n+1)}{[n+1]_q^{(c)}!}x^n. \tag{42}$$

The above obtained analytical results are graphically presented in Figs. 4 to 9. It is noted from these figures that the system behaves differently depending on whether $c < 0$ or $c \geq 0$.

In the case $c < 0$ we observe from Figs. 4 and 5 that $F \geq 1$ and $< z|N|z > \leq |z|^2$. The deviation from the case of conventional statistics with $q = 1$ is not simply a monotonic function of q. It seems to be governed by $|q-1|^\alpha$ with α different for $q < 1$ and $q > 1$. This property is gained when F is viewed as a function of $|z|^2$ and q in Fig. 6. Figure 7 shows the shift of the particle number distribution for $q \neq 1$ as compared to the case $q = 1$. A single-peak structure of W_n occurs for all values of q. Particularly, the same sequence of $q = \{0.8, 1.2, 0.9, 1.1, 1\}$ appearing in Figs. 4, 5 and 7 reveals the $|q-1|^\alpha$-like law of the q-deformation for $c < 0$.

For $c \geq 0$ the physical values of q and $|z|$ have to be bound to each other to guarantee the non-negativity of the quadrature variances. Figure 8, for example, shows the allowed and unallowed domains of the values of q and $|z|^2$ for the case $c = 1$. In the special case $c = 0$ it turns out that for a given value $q < 1$, $|z|$ must satisfy the condition $|z|^2 < (1-q)^{-1}$, which at $q = 0$ reproduces the result obtained in the previous section that $|z| < 1$ for the infinite statistics.

The influence of the q-deformation on the particle number distribution is illustrated in Fig. 9 for $c = 0$. Similar figures are also obtained for $c > 0$.

A qualitative distinction between the cases $c < 0$ and $c \geq 0$ should be noticed. As seen from Fig. 7, for $c < 0$ any deviation from $q = 1$ (both $q < 1$ and $q > 1$) shifts the particle number distribution to the left. For $c \geq 0$, however, the $q > 1$ distribution is shifted to the left, while the $q < 1$ distribution is shifted to the right, as clearly recognized in Fig. 9.

5 CONCLUSIONS

In this work we have considered the coherent states for parabose oscillators, infinite statistics oscillators and generalised q-deformed oscillators. The expressions for quadrature variances, average particle number and particle number distribution have been derived analytically and displayed graphically. The behavior of these characteristics has been discussed at different values of the parameters involved emphasizing on specific features of each statistics. In particular, for the generalised q-deformed oscillators some interesting features are due to the variation of the value of the parameter c.

We hope the results presented here might be useful for the study of the structure of coherent states as well as various nonclassical states for non-conventional statistics.

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FIGURE CAPTIONS

Fig. 1: The dependence of F on $|z|^2$ for different value of parastatistics order $p = 1, 2, 3, 5$ and 9 upwards.

Fig. 2: The dependence of $< z|N|z >$ on $|z|^2$ for different values of parastatistics order $p = 1, 2, 3, 5$ and 9 downwards.

Fig. 3: Deformation of the particle number distribution with parastatistics order p for $|z|^2 = 9$. The closed circles are for $p = 1$, while the long-, medium- , short-dashed and solid curves are respectively for $p = 2, 3, 5$ and 9.

Fig. 4: The dependence of F on $|z|^2|$ for $c = -1$ at different values of $q = 0.8, 1.2, 0.9, 1.1$ and 1 downwards.

Fig. 5: The dependence of $< z|N|z >$ on $|z|^2$ for c $= -1$ at different values of $q = 0.8, 1.2, 0.9, 1.1$ and 1 upwards.

Fig. 6: F as a function of $|z|^2$ and q for $c = -1$.

Fig. 7: Deformation of the particle number distribution with $|z|^2 = 9$, $c = -1$ and different q. The solid, long-, medium- , short-dashed curves and closed circles are respectively for $q = 0.8, 1.2, 0.9, 1.1$ and 1.

Fig. 8: Allowed and unallowed domains of $|z|^2$ and q in the case $c = 1$.

Fig. 9: Deformation of the particle number distribution with $|z|^2 = 9$, $c = 0$ and different q. The closed circles are for $q = 1$, the solid curve for $q = 1.2$, the long-dashed curve for $q = 1.1$, the medium-dashed curve for $q = 0.96$, and the short-dashed curve for $q = 0.94$.

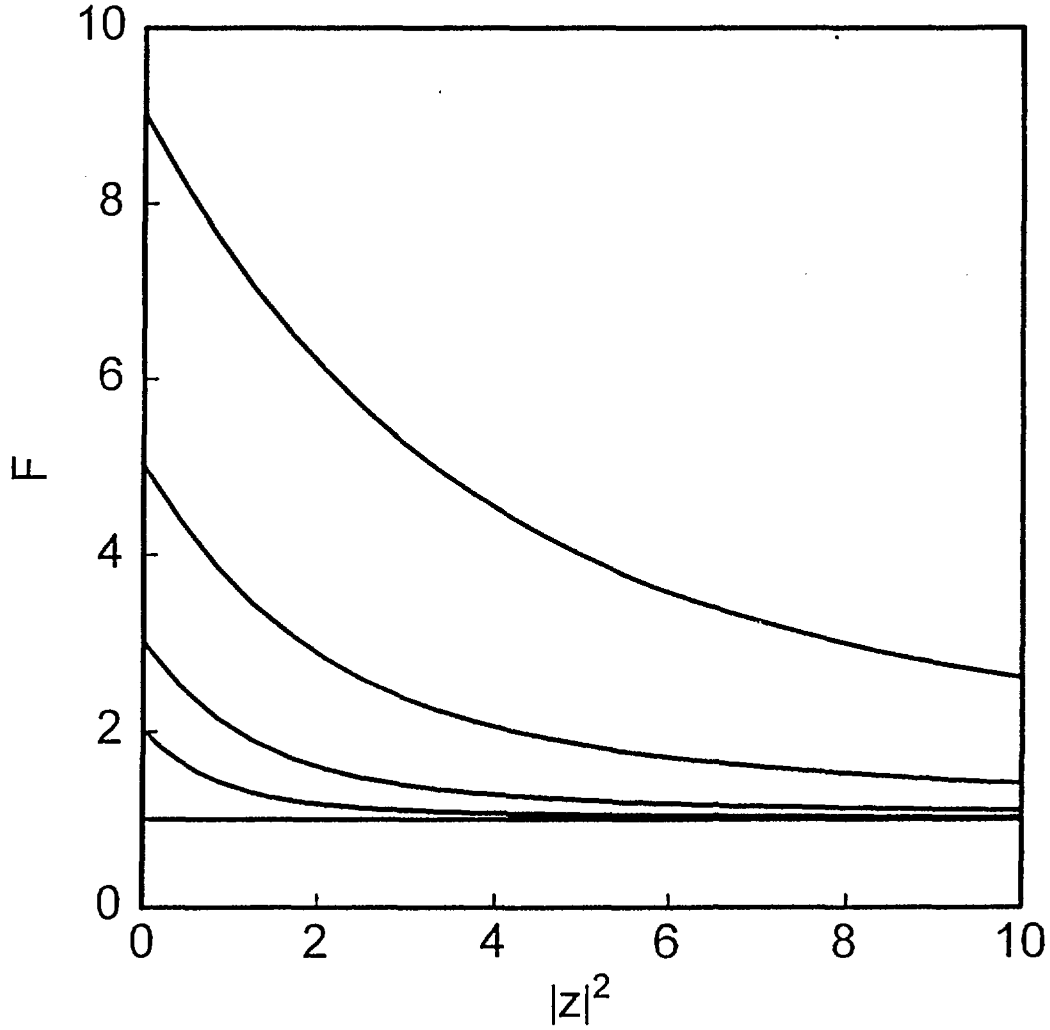

Fig.1

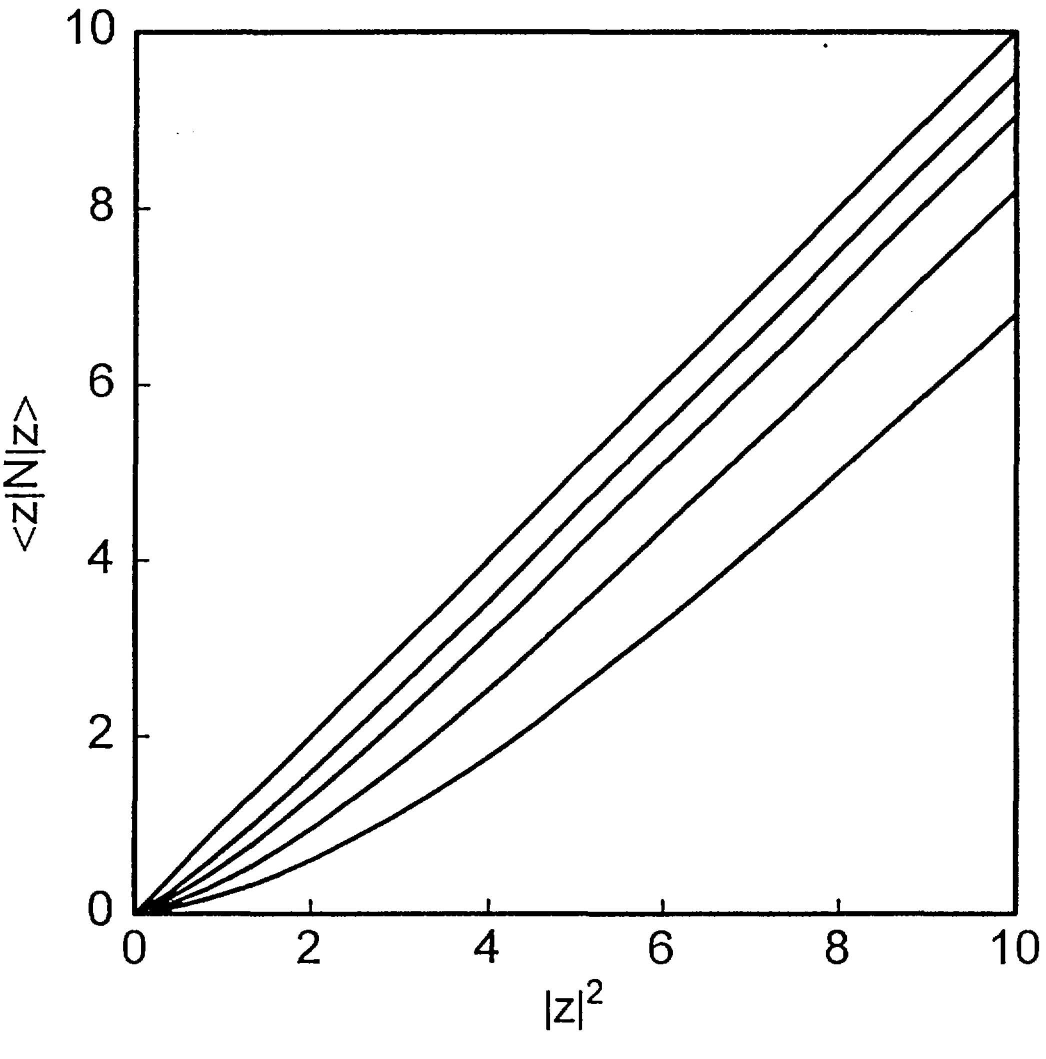

Fig.2

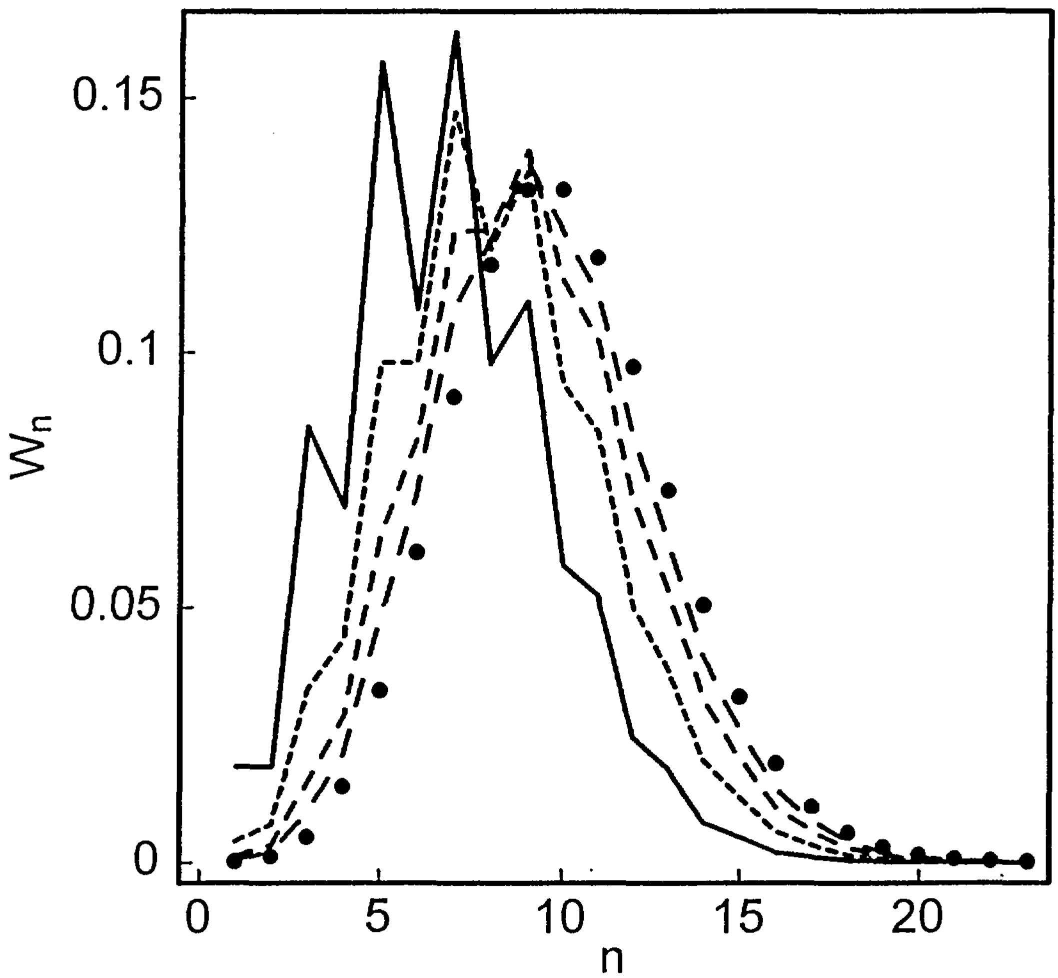

Fig.3

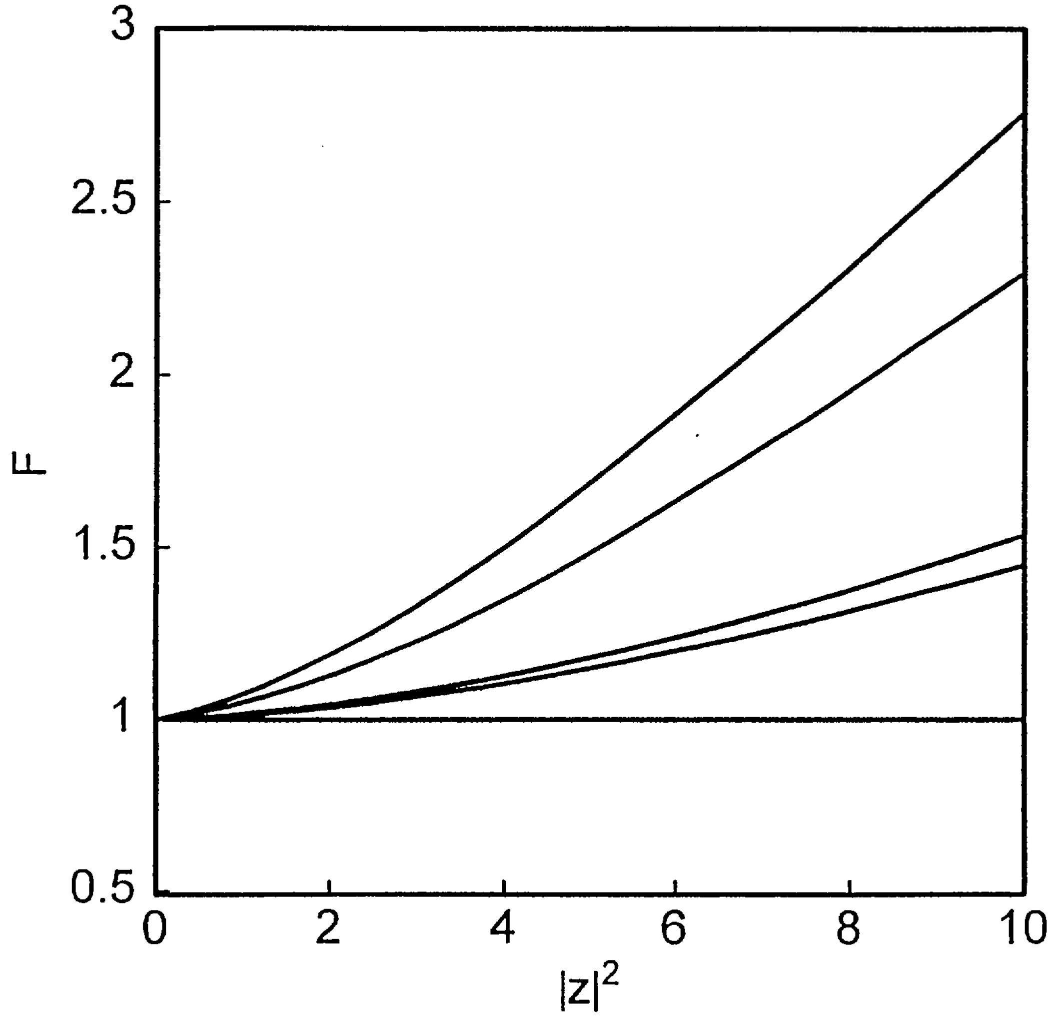

Fig.4

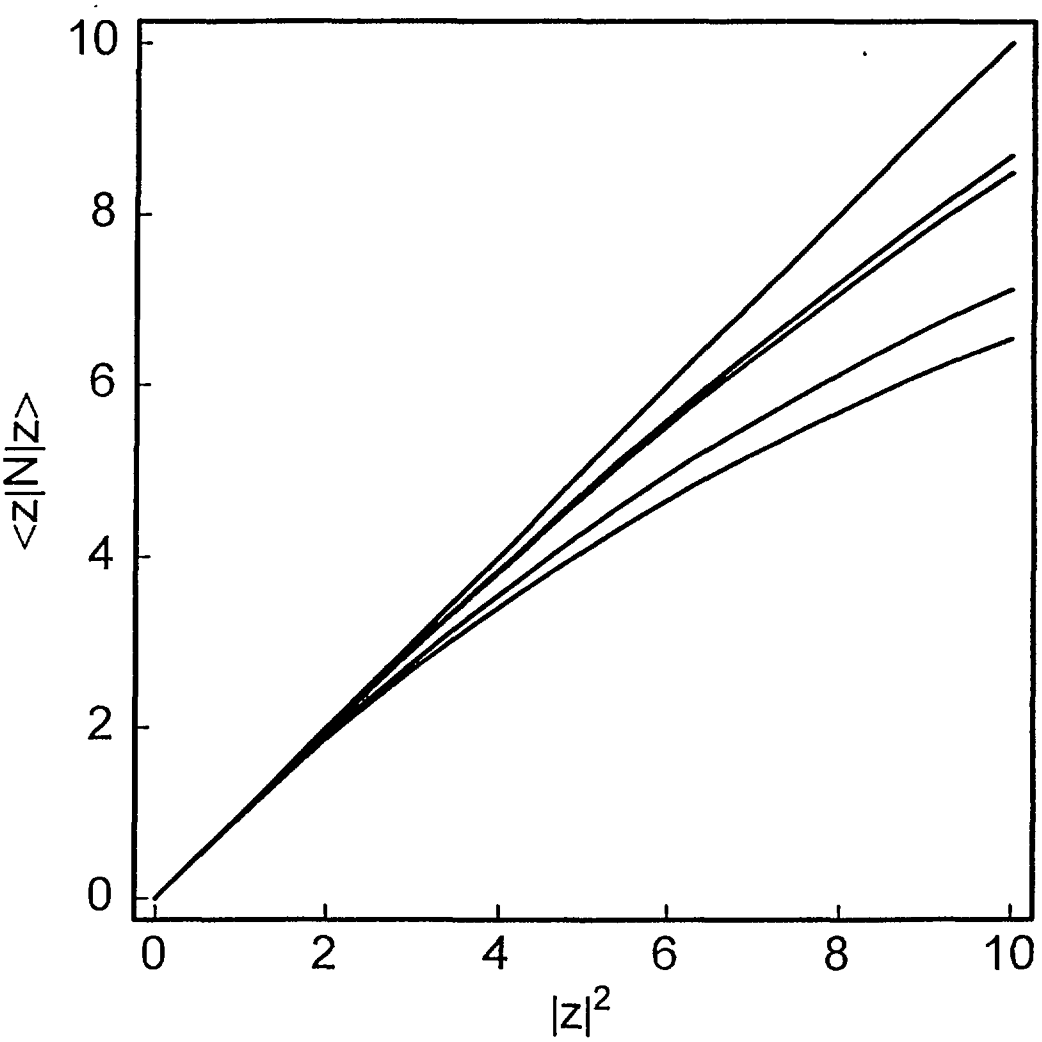

Fig.5

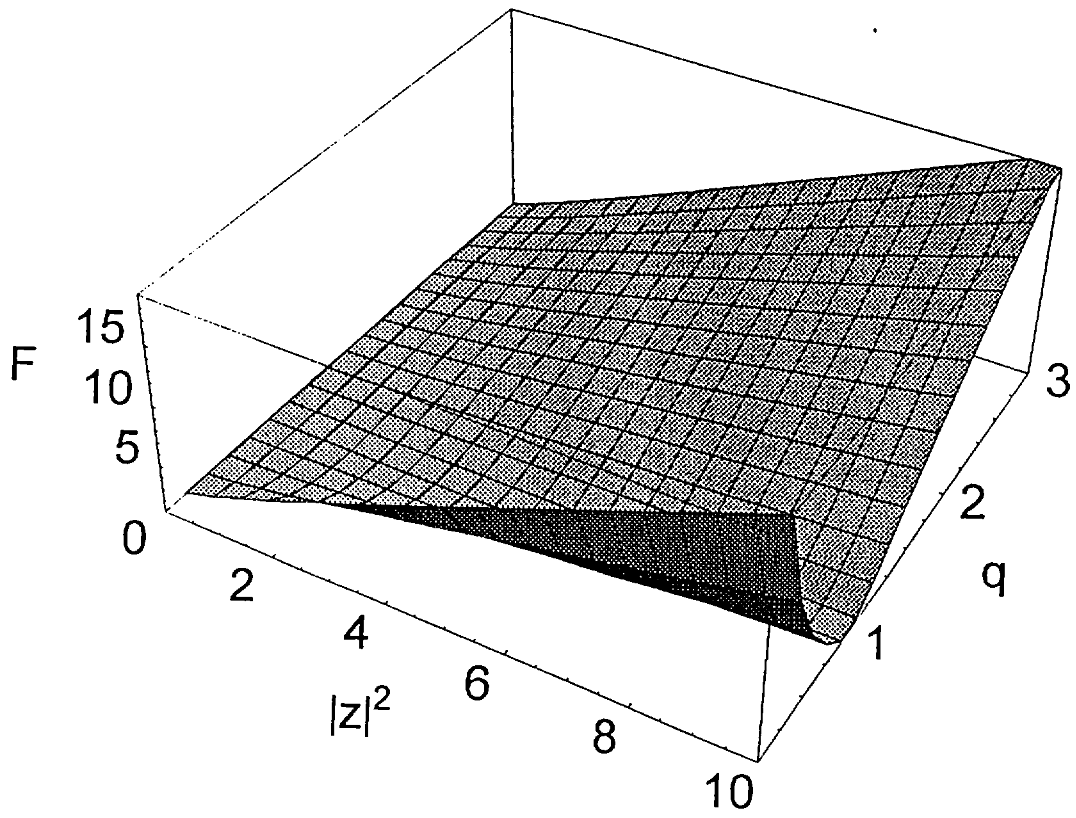

Fig.6

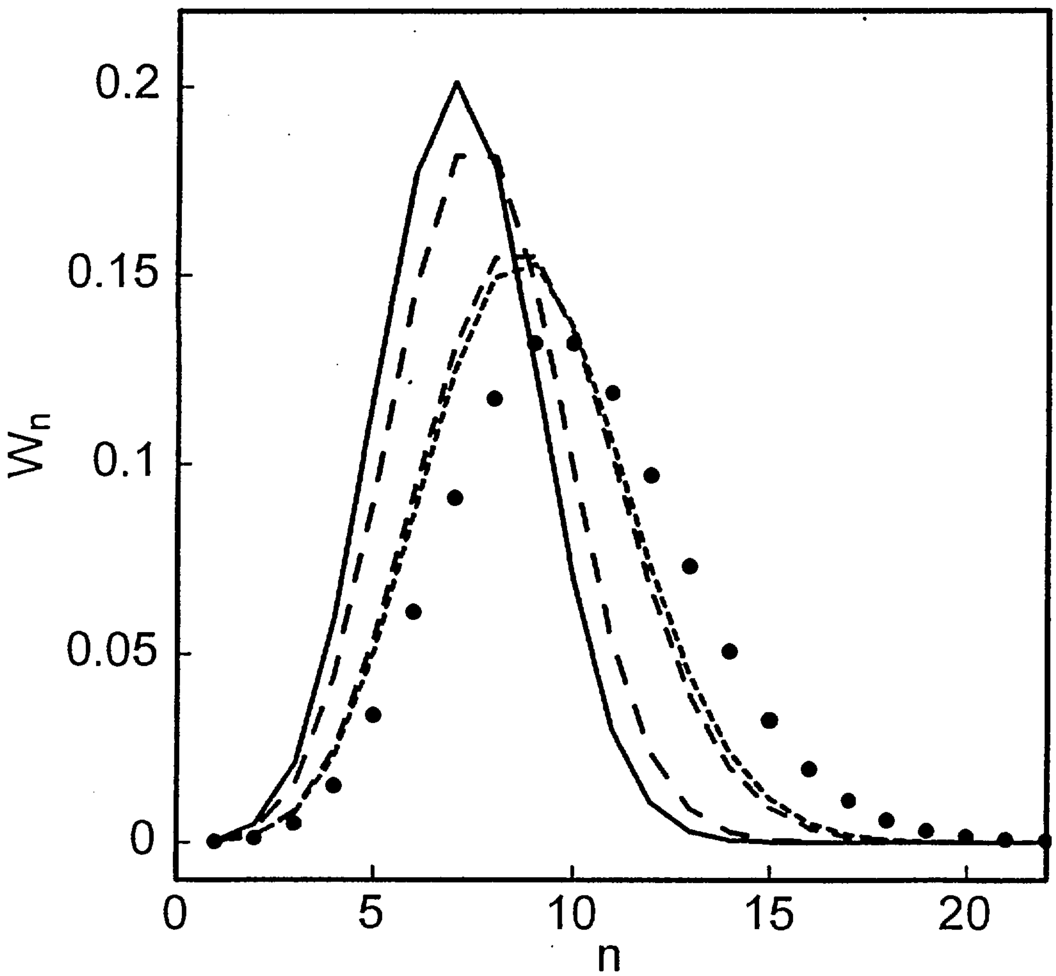

Fig.7

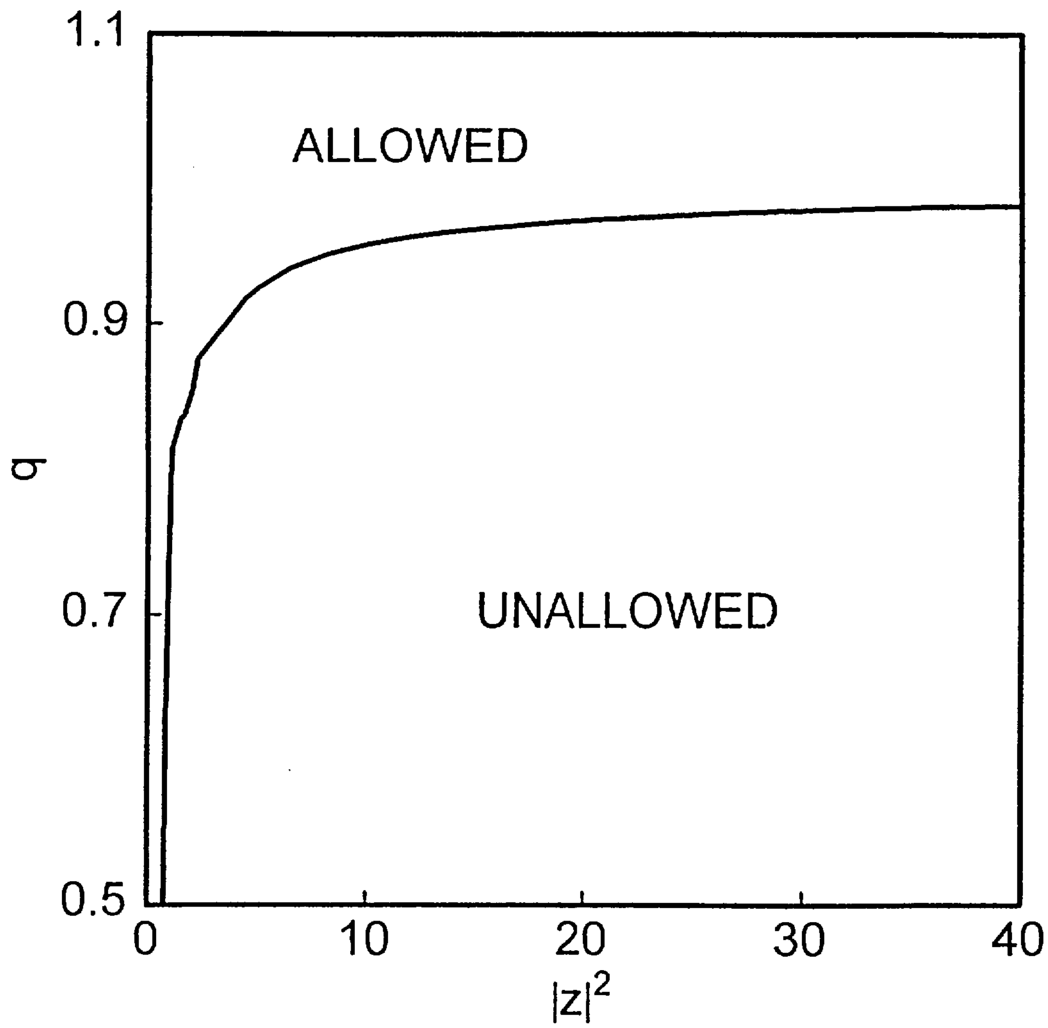

Fig.8

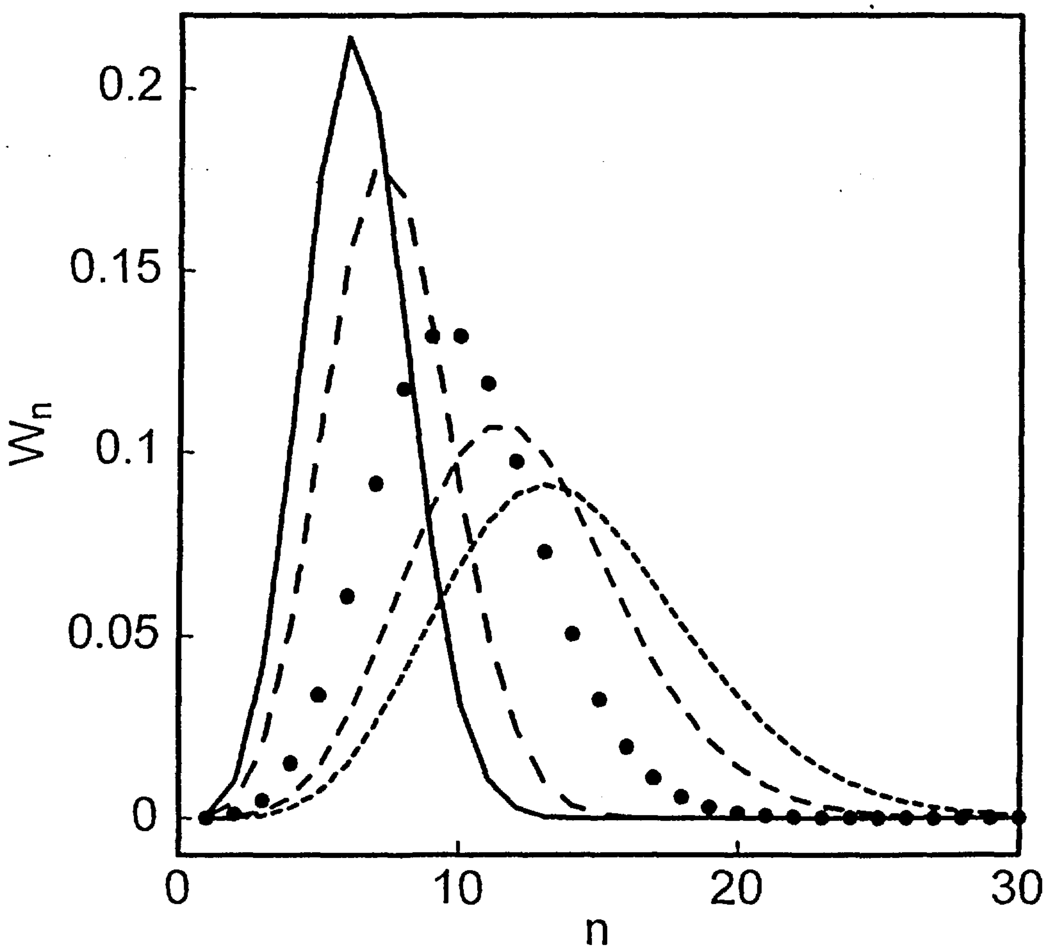

Fig.9