Monte Carlo Boundary Methods for RF-Heating of Fusion Plasma

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Abstract

A fusion plasma can be heated by launching an electromagnetic wave into the plasma with a frequency close to the cyclotron frequency of a minority ion species. This heating process creates a non-Maxwellian distribution function, that is difficult to solve numerically in toroidal geometry. Solutions have previously been found using a Monte Carlo code FIDO. However the computations are rather time consuming. Therefore methods to speed up the computations, using Monte Carlo boundary methods have been studied.

The ion cyclotron frequency heating mainly perturbs the high velocity distribution, while the low velocity distribution remains approximately Maxwellian. An hybrid model is therefore proposed, assuming a Maxwellian at low velocities and calculating the high velocity distribution with a Monte Carlo method. Three different methods to treat the boundary between the low and the high velocity regime are presented. A Monte Carlo code HYBRID has been developed to test the most promising method, the "Modified differential equation" method, for a one dimensional problem. The results show good agreement with analytical solutions.
1. Introduction

Traditionally parabolic partial differential equations are solved with methods like Finite Elements or Finite Difference, but as the number of dimensions increases these methods become time consuming and require large memory. For these problems the Monte Carlo method offers an alternative. Since the convergence of the Monte Carlo method is slow, proportional to \(1/\sqrt{N}\), where \(N\) is the number of particles (\(N\) is proportional to the computation time), optimizations become important. In some cases one is interested in the solution in a subregion of space, whereas outside this subregion an analytic approximation might be found. This analytic approximation could then be used for speeding up the computation, which is the case for radio frequency (RF) heating in fusion plasmas.

At high temperatures, light nuclei like hydrogen isotopes can be fused together into a heavier ones, releasing energy. This is called thermonuclear fusion and is being developed to become a commercial energy source. In a thermonuclear fusion reactor a temperature of about 100-200 million Kelvin is needed to obtain a burning plasma, where more energy is released than invested in heating. An important mechanism for heating a plasma is the ohmic dissipation of externally induced currents in the plasma. However in order to reach the required temperature many reactor concepts need supplementary heating. This can be obtained by launching RF-waves into the plasma at a frequency close to the cyclotron frequency of one of the ion species. This heating method is called Ion Cyclotron Resonance Heating (ICRH). The background ions that have cyclotron frequencies different from the frequency of the RF-wave, are unaffected by the RF.

ICRH creates a large perturbation in the high velocity region of phase space of the resonant ions, forming a high energy tail, while the low velocity distribution will remain close to Maxwellian. Therefore a hybrid model is introduced, that assumes the low velocity distribution to be Maxwellian and that only uses the Monte Carlo method to calculate the high velocity distribution. When the majority of the particles are in the low velocity regime, the approximation made in the hybrid model at low velocities can be used to speed up the computations.

The hybrid model require a boundary between the low and the high velocity regimes. However, it is not elementary to introduce a boundary in a Monte Carlo method, because of the interactions across the boundary. Three different boundary methods are presented in section 4.1, 4.2 and 4.3. A Monte Carlo code HYBRID is developed to test the “Modified differential equation” method for a one dimensional plasma model.
2. Kinetic description of fusion plasma during ICRH

When studying ICRH macroscopic quantities like density, pressure, temperature, current and so on, are not sufficient of describing the physics, since the distribution function locally cannot be described by a Maxwellian. A better description is given by the particle distribution function in phase space, spanned by both real space and velocity space. This distribution function \( f_i \) can for each particle species \( i \) be described by the Fokker-Planck equation

\[
\frac{df_i}{dt} = C[f_i],
\]

where

\[
\frac{df_i}{dt} = \frac{\partial f_i}{\partial t} + \nabla \cdot \left( f_i \right) + \frac{\partial v}{\partial \mathbf{r}} \cdot \frac{\partial f_i}{\partial v}
\]

and \( C \) is an operator describing the particle-particle interaction due to Coloumb collisions. The effect of the RF-field appears in (2.1) as a Lorenz force.

\[
\frac{\partial v}{\partial t} = \left( \frac{\partial v}{\partial t} \right)_{RF} + \frac{q}{m} \left( \mathbf{E}_{RF} + \nabla \times \mathbf{B}_{RF} \right)
\]

Here \( \left( \frac{\partial v}{\partial t} \right)_{RF} \) is due to forces from the RF and \( \left( \frac{\partial v}{\partial t} \right)_{F} \) is due to other forces, e.g. the force from a stationary magnetic field. Solving these equations together with Maxwell's equations requires some simplifying assumptions. This is usually done by approximating the effect of the RF with the quasi-linear diffusion operator \( Q \). The modification of the velocity distribution of the background plasma species due to the collisional power transfer from the resonant ions is usually neglected. By doing so we need only to solve the Fokker-Planck equation for the resonant ions distribution function \( f \)

\[
\frac{df}{dt} + \nabla \cdot \left( f \right) + \frac{\partial v}{\partial \mathbf{r}} \cdot \frac{\partial f}{\partial v} = C[f] + Q[f].
\]

For low velocities the Coloumb collision frequency is high. As a result the distribution of the resonant ions are mainly determined by the collisions with the background ions for reasonable RF-power. Therefore as a first order approximation at low velocities one can put \( Q = 0 \). This gives a Maxwellian distribution at steady-state.

For higher velocities the Coloumb collision frequency decreases and the affect of the RF-interactions, or \( Q \), becomes more important. \( Q \) is a diffusion operator that flattens the distribution in velocity space. Thereby a high-energy tail is created that decays more slowly at the high velocities, than the Maxwellian described by the temperature of the background ion species, see figure 2.1.

When solving eq. (2.2) with the Monte Carlo method, many particles will have low velocities and will follow a well known Maxwellian distribution. These particles require short time steps because of the high collision-frequency, which will increase the computation time. Further, the heating is usually localized to the centre of the plasma, leaving large regions, in which the distribution function is close to Maxwellian.

A hybrid model could therefore be of interest to model various heating scenarios. But the boundary between the low and the high velocities is ill defined, as the deviation from
2. Kinetic description of fusion plasma during ICRH

Figure 2.1: Deviation from the Maxwell distribution. The solid line is the distribution of a heated plasma and dotted-dashed line is a Maxwellian distribution.

a Maxwellian distribution increases gradually with velocity (see figure 2.1). The appropriate position for the boundary therefore depends on the required accuracy.

In order to study different methods to introduce a boundary between particles with low and high velocities eq. (2.2) is averaged over all directions of velocity, resulting in an equation that only depends on the magnitude of the velocity

\[
\frac{\partial f}{\partial t} = C[f] + Q[f]
\]

\[
C[f] = \frac{1}{v^2} \frac{\partial}{\partial v} \left( -\alpha v^2 f + \frac{1}{2} \frac{\partial}{\partial t} \left( v^2 \beta f \right) \right)
\]

\[
Q[f] = \frac{1}{v^2} \frac{\partial}{\partial v} \left( K v^2 \frac{\partial f}{\partial t} \right).
\]

(2.3)

Here \( \alpha \) and \( \beta \) are functions of velocity describing Coloumb collisions with the background ions and \( K \) describes the ICRH. In this work we only consider heating at the fundamental cyclotron frequency of the resonant ions in the limit of zero gyro orbits, in which case \( K \) is independent of the velocity.

An analytic solution to (2.3) at steady state was presented by Stix [1] given by

\[
f(v) = f(0) e^{I_S(v)}
\]

\[
I_S(v) = - \int_0^v -2\alpha + \frac{1}{v^2} \frac{\partial}{\partial v} (\beta v^2) dv.
\]

(2.4)

(2.5)

Without heating \( (K = 0) \) and when all plasma species are Maxwell distributed having the same temperature \( T \), a thermodynamical equilibrium is obtained, with \( I_S(v) = I_{\text{max}}(v) = -mv^2/(2T) \) giving

\[
mv^2 \frac{T}{mv^2} = -2\alpha + \frac{1}{v^2} \frac{\partial}{\partial v} (\beta v^2).
\]

Equation 2.5 can then be simplified to

\[
I_S(v) = -\frac{mv^2}{2T} + \int_0^v dv \frac{mv}{T} \frac{2K}{1 + 2K/\beta},
\]

from which one can clearly see the deviation from a Maxwell distribution, as shown in figure 2.1.
3. Monte Carlo methods

In the Monte Carlo method the distribution function is described by an ensemble of \(N\) particles located at points \(X_t, i = 1, 2, ..., N\) in coordinate space. At discrete points in time, \(n\), each particle receives an increment which represents physical processes. The increments can be divided into two parts, one deterministic drift term and one stochastic diffusion term, see figure 3.1. The time evolution is for each particle given by

\[
X_i(n + 1) = X_i(n) + R[X_i(n)] \Delta t + \zeta S[X_i(n)] \sqrt{\Delta t},
\]

where the vector \(R\) and the matrix \(S\) are the drift and the diffusion operators, respectively, and \(\zeta\) is a random number vector with zero expectation value and unit variance.

The Monte Carlo method converges as \(1/\sqrt{N}\), implying that double accuracy requires four times as many particles. Thus the method requires many particles for high resolution, making it important to optimize the accuracy as a function of the number of particles.

Because of the complexity of calculating the distribution function during ICRH in a toroidal plasma, a Monte Carlo code FIDO has been developed [2, 3]. FIDO approximates the Coloumb collisions and RF-interactions as perturbations, since the particles in a fusion plasma usually travel long distances during the characteristic time of the Coloumb collisions and the RF-interactions. The unperturbated orbit of the \(i\)-th particle is in FIDO described by three invariants of the unperturbated equation of motion \(X_t = (E, P, \Lambda)\) and a label \(\sigma\). The label \(\sigma\) is required to distinguish between different orbits having the same invariants. The Coloumb collisions and RF-interactions along the unperturbated orbits determines \(R\) and \(S\) in eq. (3.1), which describes the evolution of the invariants.

An ordinary Monte Carlo method like FIDO solves the distribution in the entire phase space and therefore spend much time following particles in the regions where the distribution is close to the Maxwellian, with the same temperature as the background species. With a hybrid model the number of particles could be reduced, which would save computation time.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure3.1.png}
\caption{The distribution function \(P[X_i(n + 1)]\) for the particle position at the time \(n + 1\), assuming the particle position at time \(n\) to be \(X_i(n)\). The particle both drift deterministically (drift= \(R\Delta t\)) and diffuse through random processes (the variance: diffusion= \(S\sqrt{\Delta t}\)), see eq. (3.1).}
\end{figure}
4. Monte Carlo boundary methods for the Hybrid model

A one dimensional model of ICRH eq. (2.3), is used to study different methods to introduce a boundary in a Monte Carlo solver. This model is not appropriate to describe ICRH in a torus, but is well suited as a first test of the boundary method. An analytic solution eq. (2.4), has been obtained for the one dimensional problem, with which the results of the simulations can be compared.

By introducing a boundary between the low and the high velocities it is possible to use a simple model at the low velocities, and to solve the equation with a Monte Carlo method in the higher velocities. Since the Monte Carlo method uses discrete particles and the fluid model uses a continuous distribution function, the interactions across the boundary become non-trivial. The flux from the high to the low velocities will in a time discrete model appear as Monte Carlo particles in the continuous distribution function. Because of the high collision frequency at the low velocities, these particles will quickly thermalize to a Maxwellian distribution. The flux from low to high velocities will in a time discrete model appear as a continuous distribution function in the Monte Carlo distribution. If this distribution function was known, new particles could simply be added according to it, but the distribution is in general not easy to calculate. Instead three different methods to treat these particles have been developed.

4.1 “Sharp boundary” methods

In the first model a boundary at \( v = v_0 \) is introduced.

\[ A = \{ v | 0 < v < v_0 \} \]
\[ B = \{ v | v_0 < v < \infty \} \]

The distribution function can then be separated according to:

\[ f = f_{\text{max}} h + g \]
\[ g = 0, \quad v \in A \]
\[ h = 0, \quad v \in B \]  \hspace{1cm} (4.1)

\[ f_{\text{max}} \]
\[ g \]
\[ v_0 \]
\[ v \]

Figure 4.1: Sharp boundary. Below \( v_0 \) the distribution is given by a Maxwellian \( f_{\text{max}} \), and above by the Monte Carlo distribution \( g \).
4. Monte Carlo boundary methods for the Hybrid model

Figure 4.2: The distribution of $v_p(n+1)$ and $v_q(n+1)$, when knowing $v_p(n)$ and $v_q(n)$, and when the random number $\zeta$ is chosen to be uniformly distributed. The highest possible value of $v_p(n+1)$ is $v_0$, which belongs to $B$. $v_p(n)$ is the smallest velocity from which a particle can travel to $B$ in one time step and $v_p(n)$ sets the lower boundary of $A$. A particle that has a velocity $v_q(n)$ in the figure, cannot travel to $B$ in one time step. Notice that diffusion $= \frac{1}{3} \sqrt{2 \Delta t}$ and drift $= R \Delta t$.

Here $f_{\text{max}}$ is a Maxwellian distribution and $g$ represents the distribution at high velocity, which could be obtained using a Monte Carlo method, see figure 4.1. A Maxwellian in $A$ is obtained by approximating $h(v) = 1$, $v \in A$. When calculating $g(v)$ the particles traveling from $B$ to $A$ are removed from the distribution and new particles are introduced, traveling from $A$ to $B$.

For boundary problems symmetric around the boundary $v_0$, adding and removing of particles can be replaced by reflecting the particles at the boundary. This is not the case in the Fokker-Planck equation of interest, due to the a non zero drift term (see eq. A.6, at page 29) that gives a finite derivative of the distribution function at $v = v_0$. This asymmetry in the flow requires that the particles entering the region $B$ carry information about the conditions in the region $A$.

The change in velocity of a particle during one time step, is a stochastic variable. The shape of this stochastic variables distribution function can be chosen rather arbitrary, with an expectation value and a standard deviation given by eq. (3.1). In the case of a uniform distribution, there will be a maximum possible change in velocity. A particle with sufficiently low velocity will then not be able to travel to $B$ in one time step.

We define the subregion of $A$ from which particles can travel to $B$, to be $\tilde{A}$, see fig 4.2. A method to add particles to $g \in B$ can now be formulated.

1. Randomize particles describing the Maxwell distribution $f_{\text{max}}$ in $\tilde{A}$.

2. Let each particle travel one step according to equation (3.1).

3. Add the particles that enter $B$ to $g$.

For this method to be effective, the region $\tilde{A}$ has to be small compared to $A$. In the Fokker-Planck equation (2.3), this is not fulfilled for reasonable values of $v_0$, due to the high Coloumb collision frequency in the bulk, which makes the increments large, consequently also the region $\tilde{A}$ large. We then need to randomize a large number of particles at each time step resulting in a small gain, or even loss, in computation time.

This method is general in the sense that it does not depend on the differential equation. However, it is limited by the requirement of a finite range of $\zeta$. 
4.2 "Smooth interaction" methods

The "Sharp boundary" method could be improved if it was not necessary to initialize a distribution in \( A \) at each time step. This could be achieved by calculating the Monte Carlo distribution, \( g \) in \( A \). However, by doing so we create a new boundary problem at the lower boundary of \( A \).

In the case of the Fokker Planck equation (2.3), it is expensive to calculate distribution in the entire \( A \). However, the probability that a particle in the lower regions of \( A \) should travel to \( B \) is very small. If \( g \) only deviates from a Maxwellian in these lower regions of \( A \), it will have a small effect on the flux from \( A \) to \( B \). This means that the boundary problem at the lower boundary of \( A \) could be solved with a low order approximation.

To define \( A \), a finite range of \( \zeta \) is required. But if \( \zeta \) is allowed to have infinite range, it would still be possible to define a subregion of \( A \), from which the probability of traveling to \( B \) is smaller than any number \( \chi \) \( \in (\chi_0, 1) \), where \( \chi_0 = P(v_{n+1} > v_0, v_n = 0) \).

The "Smooth interaction" method uses a boundary of finite thickness, \( \Delta \), to divide the velocity space into three regions

\[
A = \{ v | 0 < v < v_1 \} \\
\Delta = \{ v | v_1 < v < v_2 \} \\
B = \{ v | v_2 < v < \infty \}.
\] (4.2)

Where \( \Delta \) is a region containing the particles needed to create the flux from \( A \) to \( B \). The main contribution to this flux comes from the region where \( v \sim v_2 \), which requires that the Monte Carlo distribution is Maxwellian in this region. However, the contribution from the Monte Carlo particles with \( v \ll v_2 \) is small. The Monte Carlo distribution \( g \) can therefore smoothly fade to zero as \( v \sim v_1 \), without creating a large error in the flux.

In the region \( \Delta \) the distribution function, \( f \), is given as the sum of a Monte Carlo and fluid distribution

\[
f = h f_{\text{max}} + g \\
g = 0 \quad \forall v \in A \\
h = 0 \quad \forall v \in B
\] (4.3)

where \( f_{\text{max}} \) is the Maxwellian, \( g \) is the Monte Carlo distribution and \( h \) is a continuous function introduced to make a smooth transition between the fluid model in \( A \) and the Monte Carlo
4. Monte Carlo boundary methods for the Hybrid model

model in B. In A the function \( h \) describes the deviation from a Maxwellian and is to the first order approximated by unity in the hybrid model, with a Maxwellian in A, see figure 4.3. Inserting eq. (4.3) into eq. (2.3) gives

\[
\frac{\partial g}{\partial t} = C[g + f_{\text{max}}h] + Q[g + f_{\text{max}}h] - h \frac{\partial f_{\text{max}}}{\partial t} \quad (4.4)
\]

Since \( h = 0 \), \( v \in \mathcal{B} \) eq. (4.4) turns into eq. (2.3). In \( \Delta \) the distribution is given by a sum of a Monte Carlo and a Maxwellian distribution, each term including \( f_{\text{max}} \) is a source term. This yields a total source function

\[
\mathcal{S} = C[f_{\text{max}}h] + Q[f_{\text{max}}h] - h \frac{\partial f_{\text{max}}}{\partial t} \quad (4.5)
\]

If the source function is positive, particles should be added, whereas if it is negative, particles should be removed. The first term in equation (4.5) is the dominating one and is due to Coloumb collisions. If \( h \) is a decreasing function of velocity in \( \Delta \), this term will be positive. The second term is due to RF-heating and should be low at these relatively low velocities. The third term is due to the fact that heating decreases the number of particles at low velocities. It is needed to keep the distribution continuous and to conserve the number of particles, during ICRH.

When adding particles to the distribution, the number of particles that should be added \( N_{\Delta} \), is given by the integral of the source function

\[
N_{\Delta} = \int_{\Delta} \mathcal{S} \, dv.
\]

The distribution function which the added particles should follow, \( f_{\Delta} \), is the normalized source function

\[
f_{\Delta} = \frac{\mathcal{S}}{N_{\Delta}}.
\]

When removing particles the probabilities for removal of the individual particles should be determined by the source function such that it conserve energy. If not, removing particles will heat or cool the plasma. Since the dominant source term usually comes from Coloumb collisions, which is strictly positive, the removal of particles should not be of crucial importance and low order approximations can be used.

4.3 "Modified differential equation" method

The distribution of the resonant ions \( f \) is given by

\[
\begin{align*}
  f &= hf_{\text{max}} \quad \forall v \in A \cup \Delta \\
  f &= g \quad \forall v \in B \\
  g &= 0 \quad \forall v \in A
\end{align*}
\]

where \( f_{\text{max}} \) is the Maxwellian, \( g \) is the Monte Carlo distribution and \( h \) is approximated to unity in the hybrid model.

In this method we keep the region \( \Delta \), but we use it differently. Instead of using sources and sinks to create a smooth transition between the continuous and discrete distribution,
4.3. "Modified differential equation" method

one can regard the distribution in \( \Delta \) as a buffer of Monte Carlo particles with the purpose of creating a flux of particles from \( \Delta \) to \( \mathcal{B} \). If the distribution \( f \) is approximated with a Maxwell distribution, \( f_{\text{max}} \), in \( \Delta \), then \( g \) is arbitrary in \( \Delta \), provided that it gives the correct flux of particles to \( \mathcal{B} \). Since there is no direct contribution from \( g \) to \( f \) in \( \Delta \), \( g \) does not need to be a solution of the Fokker-Planck equation (2.3). In the "Modified differential equation" method the Fokker-Planck equation is modified in \( \Delta \).

To create the correct flux of particles from \( \Delta \) to \( \mathcal{B} \), the particles leaving \( \Delta \) have to come from a subregion of \( \Delta \) that is close to Maxwellian. This requires \( g \) to be close to Maxwellian for \( v \sim v_2 \). In the lower limit where \( v = v_1 \) we can put \( f(v_1) = 0 \). This closes the system so that no particles can travel to \( \mathcal{A} \).

In practise the differential equation is modified by replacing the function \( \alpha(v), \beta(v) \) and \( K \) in equation (2.3) with \( a(v)\alpha(v), b(v)\beta(v) \) and \( k(v)K \), respectively. The choice of \( a, b \) and \( k \), affects the increments, \( R\Delta t + \zeta S\sqrt{\Delta t} \) given by eq. (A.6) and (A.7), in Appendix A. When modifying the differential equation \( R \) and \( S \) will be modified to

\[
R = a\alpha v_1 + b\beta/2 + 3kK
\]
\[
S = v_1\sqrt{(b\beta + 2kK)},
\]

where \( R \) determines the drift of a Monte Carlo particle and \( S \) determines the diffusion. There are two ways to get a boundary at \( v_1 \) for which a particle cannot cross, when approaching it from higher velocities.

1. \( \lim_{v \to v_1} R(v) = \infty \) and \( S(v_1) < \infty \)
2. \( R(v_1) > 0 \) and \( S(v_1) = 0 \)

Since the infinity gives rise to numerical problems, the second approach is taken here. The condition that \( S(v_1) = 0 \) means that there is no diffusion at \( v_1 \). This trivially achieve when

\[
b(v_1) = k(v_1) = 0.
\]

Inserting this into the equation for \( R(v_1) > 0 \) results in

\[
a(v_1)\Delta v_1 > 0.
\]

Particles that come from \( \Delta \) and travel to \( \mathcal{A} \) have to come from a distribution that is close to Maxwellian. Therefore the distribution function has to be continuous at the upper boundary at \( v_2 \). To achieve this requires that the increments and their derivatives are continuous at \( v_2 \). This gives us the following conditions for \( a, b \) and \( k \)

\[
a(v_2) = b(v_2) = k(v_2) = 1
\]
\[
a'(v_2) = b'(v_2) = k'(v_2) = 0.
\]

We note that the constrains on \( b \) and \( k \) are equal, since both functions start from zero at \( v_1 \) and end at unity for \( v = v_2 \). The following analysis is considerably simplified by choosing \( k \) to be equal to \( b \). The calculations have been done for a general \( k \), but they add no information of value for this work.

An analytic solution similar to equation (2.4) can be obtained for the modified equation in \( \Delta \)

\[
f(v) = f(v_2)e^{f(v)}, \quad (4.6)
\]
where

\[ I(v) = - \int_{v}^{v_2} dv \frac{-2a\alpha + \frac{1}{\nu^2} \frac{\beta}{\nu} (b\beta v^2)}{b\beta + 2k\beta}. \]

With \( k = b \) one obtains

\[ I(v) = - \int_{v}^{v_2} dv \frac{-2a\alpha + \frac{1}{\nu^2} \frac{\beta}{\nu} (b\beta v^2)}{b(\beta + 2K)}. \]  

(4.7)

Since \( a \) and \( b \) needs to be continuous near \( v = v_2 \), the Stix solution eq. (2.5) is factorized from eq. (4.7)

\[ f(v) = f(v_2) e^{S(v)} e^{I(v)}, \]  

(4.8)

where

\[ I(v) = \int_{v}^{v_2} dv \frac{1}{1 + \frac{K}{\beta}} \times \left( \frac{\alpha}{\beta} \left( 1 - \frac{a}{b} \right) + \frac{1}{2b} \frac{\partial b}{\partial v} - \frac{2K}{\beta^2} \left( -\alpha + \frac{1}{2v^2} \frac{\partial}{\partial v} (v^2\beta) \right) \right). \]  

(4.9)

Here, \( I_S(v) \) is the Stix solution given by equation (4.7). To choose \( a \) and \( b \) such that the boundary model creates the needed flux from \( A \) to \( B \) is non-trivial. A parameter that we would like choose is the deviation from the Stix solution, which is given by a factor \( e^J \). Here \( J = J(a, b) \), but if it is possible to invert this relationship one can get \( a = a(b, J) \), or \( b = b(J, a) \). This leaves \( (b, J) \), or \( (J, a) \) as the function that should be chosen, instead of \( (a, b) \).

Let us define

\[ \phi(v) \equiv e^{J(v)}, \]

which is equivalent to

\[ J'(v) = \frac{\phi'(v)}{\phi(v)}. \]

Here, prime stands for differentiation with respect to velocity. From eq. (4.9) we obtain

\[ J' = \frac{1}{1 + \frac{K}{\beta}} \left( \frac{\alpha}{\beta} \left( 1 - \frac{a}{b} \right) + \frac{1}{2b} \frac{\partial b}{\partial v} - \frac{2K}{\beta^2} \left( -\alpha + \frac{1}{2v^2} \frac{\partial}{\partial v} (v^2\beta) \right) \right). \]

From this equation \( a \) is solvable. Straight forward algebra gives

\[ a = \left( 1 + \frac{2K}{\beta} \left( -\frac{1}{2v^2} \frac{\partial}{\partial v} (v^2\beta) - 1 \right) + \frac{\beta + 2K \phi'}{\alpha} \right) b + \frac{1}{2} \frac{\beta - b'}. \]  

(4.10)

To close the system at the lower boundary \( v = v_1 \) requires that \( \phi(v_1) = 0 \). This gives a singularity in eq. (4.10), when dividing by \( \phi \). But this singularity is removable if

\[ \lim_{v \to v_1} \frac{b(v)}{\phi(v)} < \infty. \]  

(4.11)

Now a boundary method can be obtained for any choice of \( \phi \) (or \( J \)) and \( b \), satisfying eq. (4.11). However, every choice does not give a boundary method that behaves well. The following choices of \( \phi \) and \( b \) have been tested using the code \textsc{hybrid}

\[ \phi(v) = b(v) = \sin \left( \frac{v - v_1}{v_2 - v_1} \right). \]

This choice has resulted in a boundary method that gives small errors in the distribution function (see chapter 5.2). Other problems with the boundary model are discussed in the following subsections.
4.3.1 Sub time stepping

For an ordinary Monte Carlo method using random numbers with infinite range any particle could get large negative increments that gives a negative velocity. Although the range of the random number is limited, there still exist a set of combinations of velocities and possible random numbers that gives a negative velocity. This is of course unphysical and is not possible according to the Fokker-Planck equation. However, due to the finite time steps, a particle can travel from one velocity to another without taking into account how \( \alpha \) and \( \beta \) changes between these velocities.

For any Monte Carlo methods with a boundary in phase space the finite time steps will make it possible for a particle to cross this boundary, although the time continuous equation forbids the particle to cross the boundary. This appears also in the "Modified differential equation" method. Here the boundary at \( v = v_1 \) should be the lowest possible velocity that a particle could have. However, because of the finite time steps particles could end up with a velocity less than \( v_1 \).

This problem can be solved if one for each particle that gets too large increment, divide the time step into two sub time steps. Then the path during the first half of the time step is determined by \( \alpha \) and \( \beta \) at the velocity of the particle before the time step, and the path during the second half of the time step is determined by \( \alpha \) and \( \beta \) at the velocity of the particle after one sub time step. If the particle still does not stay above \( v_2 \), we divide the two steps into four, and so on.

Large increments often appear for large random numbers \( \zeta \). But since all values of \( \zeta \) have to be treated equally, the large random numbers resulting in large increments have to be kept during all sub time steps. \( \zeta \) is then fixed during sub time stepping and the equation of motion has to be changed slightly, compared to equation (3.1)

\[
X_i(n + 1 / \gamma) = X_i(n) + \frac{R[X_i(n)] \Delta t + \zeta_n S[X_i(n)]}{} \frac{\sqrt{\Delta t}}{\gamma}.
\]

Here \( \gamma = 1, 2, 3, \ldots \), is the number of sub time steps which the ordinary time step has been divided into. The random variable \( \zeta_n \) should be kept the same during the sub time stepping (that is for all \( \gamma \)). Note that \( \Delta t \) is the ordinary time step.

4.3.2 Tail creation

When heating the plasma, the fraction of particles that are in the low velocity region \( \Delta \) decreases, since particles are dragged up to the higher velocity regime forming a high energetic tail. This gives rise to a net flux of particles from \( \Delta \) to \( \beta \), which has to come from \( \gamma \in \Delta \), since \( \gamma \in \Delta \) includes no particles. The flux is obtained by adding or removing particles from \( \gamma \in \Delta \) by a feedback method that requires that the distribution function is nearly continuous at \( v = v_2 \) and that the number of particles is conserved. When there are too few or too many particles in the tail this gives rise to a discontinuity in the distribution function. Particles are then added or removed from \( \Delta \) so that the same discontinuity appears in \( \gamma \). During the following time steps the added particles will diffuse out in the tail, or if particles are removed, particles from \( \beta \) will then diffuse into \( \Delta \). The number of particles that should be added (or removed for negative values) is \( \delta N \)

\[
\delta N = N_\Delta \frac{\bar{f}_{\max}(v_2) - g(v_2)}{g(v_2)}.
\]
where

\[ N_\Delta = \int_{v_1}^{v_2} g(v) dv. \]

Here \( g \) is the distribution of Monte Carlo particles and \( f_{\text{max}} \) is the Maxwellian. A steady state in \( \Delta \) should not be affected by the added particles. Therefore each particle that should added to \( g \) should be added according to the distribution function obtained by normalizing \( g \).

The distribution function \( g \) is made up by discreet particles. To obtain the value \( g(v_2) \), the distribution is filtered. Since the number of particles is finite, the value \( g(v_2) \) includes fluctuations, which affects \( \delta N \). The fluctuations in \( \delta N \) can be reduced by adding damping factor \( \kappa \), such that

\[ \delta N = \frac{1}{\kappa} N_\Delta \frac{f_{\text{max}}(v_2) - g(v_2)}{g(v_2)}. \]

The damping also affects the net flux from \( A \) to \( B \) by slowing down the time scale under which the tail is formed.
5. Test made on the 'Modified differential equation' method

5.1 Physical and numerical parameters

<table>
<thead>
<tr>
<th>Numerical parameters</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of Monte Carlo particles :</td>
<td>30000 – 50000</td>
</tr>
<tr>
<td>Time step :</td>
<td>0.5 [ms]</td>
</tr>
<tr>
<td>Discharge time :</td>
<td>0.5 – 1.0 [s]</td>
</tr>
<tr>
<td>Number of grids :</td>
<td>3000</td>
</tr>
<tr>
<td>Maximum velocity :</td>
<td>6.0 [Mm/s]</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Physical parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>Resonant ion species :</td>
</tr>
<tr>
<td>Background species :</td>
</tr>
<tr>
<td>Densities :</td>
</tr>
<tr>
<td>Background temperature :</td>
</tr>
<tr>
<td>Thermal velocity :</td>
</tr>
</tbody>
</table>

The number of Monte Carlo particles will affect the magnitude of the fluctuations in the distribution function.

The finite time step will give rise to an error in the distribution function, which is further discussed in section 5.2. The time steps here are short compared with the time steps used in FIDO (FIDO's time step ≈ 1.0 ms), since in FIDO the main interests are in the physics of the high energetic particles with low collision frequency, while the tests using HYBRID are made in the bulk.

The discharge time has to be long enough to allow the distribution to thermalize. This is easy to check by observing the temperature evolution.

The number of grids is the number of intervals which the velocity space has been divided into when constructing a continuous distribution function. This distribution is truncated at maximum velocity.

The physical parameters have been chosen to be similar to a JET DT discharge. The frequency for the ICRH is chosen at the fundamental cyclotron frequency of the resonant ions.

5.2 Maintaining a Maxwellian without heating

Without the ICRH, the resonant ions should have a Maxwellian distribution, with a temperature determined by the background ions. Whether this property is maintained when introducing the "Modified differential equation" method has been tested in two different ways. The first test was to observe the logarithm of $f(v^2)$ fig. 5.1, which should be a linear function. A second and more detailed verification was made from the collisional power transferred from the resonant ions to the electrons, see appendix B. These tests give a high resolution of small changes in the distribution, but are also numerically sensitive to a number of parameters in the integration in eq. (B.6), at page 32. If both the resonant ions and the electrons are Maxwell distributed with the same temperature, there should be no net power transfer. The power transfer was therefore studied as a function of $v_1$ for six fixed values of $v_2$, and is shown in fig. 5.2, where the power is normalized to the transferred power from a resonant ion distribution with a temperature that is one percent higher than the background ion temperature.
Figure 5.1: The Maxwell distribution, dotted line, is well represented by the HYBRID solution, solid line. The noise at high velocities is caused by low density of Monte Carlo particles. The upper boundary $v_2$ is at one thermal velocity.

An ordinary Monte Carlo method uses finite time steps that creates a small perturbation from the Maxwellian distribution resulting in a net collision power transfer from the resonant ions to the background plasma species. This power transfer remains even when the number of Monte Carlo particles goes to infinity. To get zero power transfer the number of Monte Carlo particles has to go to infinity and the length of the time steps has to go to zero. The net power transfer from an ordinary Monte Carlo method, is shown in the fig. 5.2 represented by the dotted-dashed lines.

It is clear from fig. 5.2, that the "Modified differential equation" method produces an error for any value of $v_1$. In the distribution function this error created by the boundary method will appear as a discontinuity in the derivative of the distribution function at $v = v_2$. Above $v_2$ the derivative quickly fades out to its Maxwellian value. Therefore does the boundary method not affect the shape of the distribution function for $v \neq v_2$, but the normalization of the distribution function. This gives too many particles above $v_2$ and too few below, see fig.5.3.

To get a Maxwellian, the flux from $\Delta$ to $\beta$ has to be statistically the same as the flux from $\Delta$ to $\beta$ using an ordinary Monte Carlo method. However, since the lower regions of $\Delta$ are not Maxwell distributed, the particles present will not create the correct flux from $\Delta$ to $\beta$, and this will cause an error in the distribution $f(v)$.

For small values of $v_1$, the error created by the boundary method is of the same order of magnitude as the error from the finite time steps. When $v_1/v_2 - 1$ the power transfer increases rapidly with $v_1/v_2$. However, the figure shows that the error created by the boundary method clearly depends on $v_2$. One reason for this is that for high values of $v_2$ there will be very few particles that are above $v_2$. Increasing the distribution function above $v_2$ and conserving the number of particles will require a small decrease below $v_2$. This will be a small perturbation in the collisional power transfer. While for $v_2 \sim v_{th}$ the number of particles above $v_2$ will be similar to the number below. The perturbation at $v > v_2$ will then give rise to a large increase above $v_2$ and a large decrease below, which will cause a large perturbation in the collisional power transfer.

5.3 Heating and tail formation

When a tail is formed, new particles are added to $\Delta$ according to eq. (4.12). This feedback method was tested for a number of choices of $v_2$ and $v_1$, see fig. 5.4.

Since the deviation from a Maxwellian during heating, is a strictly increasing function of velocity (see fig. 2.1), the approximation that the low velocity distribution is Maxwellian (the
5.3. Heating and tail formation

Figure 5.2: Collisional power transfer from the resonant ions to the electrons as a function of $v_1/v_2$, for different $v_2$. The absorption has been normalized to the power transfer from a Maxwellian distribution with temperature, $T = 1.01T_0$, where $T_0$ is the temperature of the background ion species. The dotted-dashed line is the collisional power transfer obtained using an ordinary Monte Carlo method.
5. Test made on the 'Modified differential equation' method

Figure 5.3: When \( \Delta \) gets too thin, an error is created by the boundary method. Then \( g \) will no longer be able to produce the same flux from \( \Delta \) to \( B \) as an ordinary Monte Carlo method would have produced. This gives a discontinuity in the derivative of the distribution function, giving too many particles above \( v_2 \) and too few below.

Figure 5.4: When a tail is formed the number of particles in the tail is increased and the density in the bulk is decreased. Here \( v_2 = v_{1h} \).

Figure 5.5: Here \( v_2 = 1.4v_{1h} \) is chosen to be too large, to clearly show that a hybrid approximation gives a distribution that is too cold in the bulk and hence has too few particles in the tail.
5.4. Computation time reduction

hybrid approximation) will give a distribution that always is too cold in the bulk. As could be seen in fig. 5.5, this gives too many particles in the bulk, and too few in the tail. This strongly limits the choice of \( v_2 \). Above \( v_2 \) the derivative of the distribution function is the same as that of the analytic solution, eq. (2.4).

Tests have not been made to determine whether this feedback operation affect the time scale for processes in the neighborhood of \( v_z \). However, the collision time in \( \Delta \) is very short, so the added particles will quickly thermalize. But the number of particles that should be added or removed is systematically underestimated by the damping factor \( \alpha \). This is important especially when the time scale of the tail formation is of the same order as the time steps multiplied with the damping factor.

5.4 Computation time reduction

The advantage with the "Modified differential equation" method is that it reduces the computation time compared to an ordinary Monte Carlo method, since it does not calculate the distribution in the bulk. The time saved can be measured from the number of particles used in \( g, N_{\text{hybrid}} \), compared to the number needed to fill up the entire velocity space, \( N_{\text{MC}} \). We therefore define the ratio, \( \Omega \), between the time, \( t_{\text{MC}} \), it takes to calculate the distribution using an ordinary Monte Carlo method, and the time, \( t_{\text{hybrid}} \), needed for an equivalent calculation using the "Modified differential equation" method

\[
\Omega \triangleq \frac{t_{\text{MC}}}{t_{\text{hybrid}}} \approx \frac{N_{\text{MC}}}{N_{\text{hybrid}}}. \tag{5.1}
\]

The ratio \( \Omega \) depends strongly on \( v_2 \) and has therefore an implicit dependence on the heating \( K \) (in eq. (2.3)), which reduces the region where the distribution can be assumed Maxwellian. However, \( \Omega \) depends also directly on \( K \), since \( K \) gives rise to a tail in the distribution function, resulting in a decrease in the number of particles in the bulk and an increasing of the number in the tail. The direct dependence on \( K \) therefore decreases the ratio \( \Omega \) for fixed \( v_2 \).

\( \Omega \) has been estimated for a number of values of \( v_2 \) when \( K = 0 \).

<table>
<thead>
<tr>
<th>( v_2/v_{th} )</th>
<th>( \Omega )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>1.2</td>
</tr>
<tr>
<td>1.2</td>
<td>2.5</td>
</tr>
<tr>
<td>1.4</td>
<td>7.0</td>
</tr>
<tr>
<td>1.6</td>
<td>12</td>
</tr>
<tr>
<td>1.8</td>
<td>36</td>
</tr>
<tr>
<td>2.0</td>
<td>70</td>
</tr>
</tbody>
</table>

The region of interest is for \( v_{th} < v_2 < 2v_{th} \), where \( v_{th} \) is the thermal velocity. For each \( v_2 \), the value of \( v_1 \) is chosen such that the power transferred from the resonant ions to the electrons is the same as if the resonant ions had a temperature \( \pm 1\% \) higher than that of the electrons. Because of the direct dependence on \( K \), these values of \( \Omega \) will be reduced when \( K > 0 \). Notice that two thermal velocities is four thermal energies.

When introducing the boundary method, new functions are introduced into the code, which increase the computation time. However, for the "Modified differential equation" method, these new functions do not increase the computation time significantly. The functions for computing the modification of the increments do not affect the computation time, since the algorithms for computing the function \( \alpha, \beta \) and \( K \) are much more complicated than the algorithms for computing \( a, b \) and \( k \). A small change in computation time comes from the adding
5. Test made on the 'Modified differential equation' method of new particles during tail formation, but the change is small since adding and removing only represents the net flux of particles across the boundary.
6. Conclusions and discussions

Three boundary methods are proposed and discussed in section 4.1, 4.2 and 4.3. The "Modified differential equation" method, which requires only that the net flux of particles across the boundary is added or removed, was considered most promising. A Monte Carlo code HYBRID was therefore developed to test this method. These tests have shown that it can reproduce a Maxwell distribution with errors of the same order as the errors from the finite time steps, and that it within the errors of the hybrid model can reproduce the analytic solution (eq. 2.4) during ICRH at the fundamental cyclotron frequency of the resonant ions.

The gain in speed depends on how high $v_2$ can be chosen. For high RF-power densities the deviation from a Maxwellian in the bulk becomes large requiring a rather low value of $v_2$ making the method less effective.

An interesting improvement of the hybrid model is to use higher order analytic approximations than the Maxwell distribution for the low velocities. This would make it possible to increase $v_2$ and to avoid the effects giving a too cool bulk during heating, discussed in section 5.3.

The one dimensional equation (2.3) could numerically be solved by evaluating the integral in eq. (2.5) or by a finite element or finite difference method. Any of these methods would be less time consuming and give higher accuracy than a Monte Carlo method. It is for high dimensions that the Monte Carlo method is of practical interest. The aim with this work has been to study Monte Carlo boundary methods. In one dimensions these studies become easier to overview and to implement, than in higher dimensions.

In a thermonuclear fusion experiment the power is transfered from the RF-wave to the ions at the centre of the plasma. This power is often high, creating large perturbations fairly far down in the bulk. Here $v_2$ could not take on values higher then the thermal velocity. Then a factor $< 1.2(= 1)$ should be gained in computation time. Outside the heated centre of the plasma, the situation is different. Here the particles are not directly heated by the ICRH, instead heated particles from the centre of the plasma get here by diffusion or drift. Therefore the deviation from a Maxwellian is much smaller and higher values of $v_2$ can be chosen. This region is large and it is here that the hybrid model could save computation time. Close to the wall the temperature is very low compared to the centre of the plasma, making it possible to choose high values of $v_2$. If $v_2$ could be chosen to be two thermal velocities, a gain in computation time of a factor of ~ 70 would be achieved!

From this discussion it seems reasonable to use an ordinary Monte Carlo method to calculate the distribution function at the heated centre of the plasma and to use the "Modified differential equation" method outside this region. To estimate the computation time saved, using the hybrid model, we need an approximate value of $\Omega$ in the regions that are not heated, and the ratio between the volume of the central strongly heated plasma and the total plasma volume.

For a different Monte Carlo boundary problem, all three methods should be considered. The "Sharp boundary" method is general in the sense that it does not depend on the differential equation, but it requires a finite range of the random number $\zeta$. The other two methods can use any random number, but they depend on the differential equation.
Acknowledgments

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Bibliography


A. Monte Carlo increments

The evolution of the Monte Carlo distribution is determined by eq. (3.1). The functions $R$ and $S$, which in one dimension both become scalars $R$ and $S$, are here computed for the differential equation (2.3). For a particle with velocity $v$, the distribution is

$$f(v) = \frac{\delta(v - v_i)}{4\pi v^2}. \quad \text{(A.1)}$$

$R$ and $S$ are defined by

$$R = \dot{\mathcal{E}} \left[ \frac{v^2}{2} \right] \quad \text{(A.2)}$$
$$S = \sqrt{\dot{\mathcal{V}} \left[ \frac{v^2}{2} \right]}, \quad \text{(A.3)}$$

where $\mathcal{E}$ is the expectation value and $\mathcal{V}$ is the variance. The time derivative of the variance can be expressed as

$$\dot{\mathcal{V}}[x] = \frac{\partial}{\partial t} \left( \mathcal{E}[x^2] - \mathcal{E}[x]^2 \right) = \mathcal{E}[x^2] - 2\mathcal{E}[x] \dot{\mathcal{E}}[x]. \quad \text{(A.4)}$$

The $n$:th moment is obtained by multiplying the distribution function by $v^n$ and integrating

$$\mathcal{E}[v^n] = \frac{\partial}{\partial t} \int_0^\infty v^n f(v) 4\pi v^2 dv = \int_0^\infty v^n \frac{\partial f(v)}{\partial t} - 4\pi v^2 dv$$

for any positive integer $n$. The time derivative of the distribution is given by equation (2.3)

$$\dot{\mathcal{E}}[v^n] = \int_0^\infty v^n \frac{\partial}{\partial v} \left( -\alpha v^2 f + \frac{1}{2} \frac{\partial}{\partial v} \left( v^2 \beta f \right) + K v^2 \frac{\partial f}{\partial v} \right) 4\pi dv.$$

Since the distribution is finite at zero velocity and goes to zero quicker than $1/v^2$ for large $v$, the equation can be partially integrated

$$\dot{\mathcal{E}}[v^n] = -\int_0^n n v^{n-1} \left( -\alpha v^2 f + \frac{1}{2} \frac{\partial}{\partial v} \left( v^2 \beta f \right) + K v^2 \frac{\partial f}{\partial v} \right) 4\pi dv.$$

Another partial integration assuming that $n \leq 2$ yields

$$\dot{\mathcal{E}}[v^n] = \int_0^n n v^{n-2} \left( \alpha u^3 + (n - 1)v^2 \beta/2 + (n + 1)Kv^2 \right) 4\pi dv.$$ 

By substituting the distribution according to equation (A.1) and carrying out the integration over the Dirac distribution we obtain

$$\dot{\mathcal{E}}[v^n] = n v_i^{n-2} \left( \alpha v_i + (n - 1)\beta/2 + (n + 1)K \right). \quad \text{(A.5)}$$

Now $R$ and $S$ can be calculated from eq. (A.2), (A.3), (A.4) and (A.5).

$$R = \alpha v_i + \beta/2 + 3K \quad \text{(A.6)}$$
$$S = v_i \sqrt{\beta + 2K} \quad \text{(A.7)}$$
B. Collisional power transfer to the electrons

The Coloumb operator $C[f]$ in eq. (2.3) describes the energy exchange between the resonant ions and the different background particle species $i$. It can be written as

$$C[f] = \sum_i \xi[f_i].$$

The Coloumb operator $C[f]$ is given by the functions $\alpha$ and $\beta$, which are also separable into functions of $f_i$ such that

$$\alpha = \sum_i \alpha[f_i],$$
$$\beta = \sum_i \beta[f_i],$$
$$\xi[f_i] = \frac{1}{v^2} \frac{\partial}{\partial v} \left( -\alpha[f_i] v^2 f + \frac{\partial}{\partial v} \left( v^2 \beta[f_i] f \right) \right).$$

The collisional power transfer from the resonant ions to the electrons is given by

$$P[f] = -\int_0^\infty \frac{m v^2}{2} \left( \frac{\partial f}{\partial t} \right)_{col} 4\pi v^2 dv,$$  \hspace{1cm} (B.1)

where $\text{col}$ stands for "due to collisions". The collision part of (2.3) is then inserted into eq. (B.1)

$$P[f] = -\int_0^\infty \frac{m v^2}{2} C[f] 4\pi v^2 dv$$
$$= -\sum_i \int_0^\infty \frac{m v^2}{2} \xi[f_i] 4\pi v^2 dv.$$  \hspace{1cm} (B.2)

Each term of (B.2) represents the power absorption of the individual particle species $f_i$. The power absorbed by the electrons is then given by

$$P_{e}[f] = -\int_0^\infty \frac{m v^2}{2} \xi[f_e] 4\pi v^2 dv$$
$$= -\int_0^\infty \frac{m v^2}{2} \frac{1}{v^2} \frac{\partial}{\partial v} \left( -\alpha[f_e] v^2 f + \frac{1}{2} \frac{\partial}{\partial v} \left( v^2 \beta[f_e] f \right) \right) 4\pi v^2 dv.$$  \hspace{1cm} (B.3)

Partial integration yield

$$P_{e}[f] = \int_0^\infty 4\pi m v \left( -\alpha[f_e] v^2 f + \frac{1}{2} \frac{\partial}{\partial v} \left( v^2 \beta[f_e] f \right) \right) dv.$$  \hspace{1cm} (B.4)

The Monte Carlo particles are mathematically defined as Dirac distributions. To be able to compute the derivatives inside the integral the distribution function has to be filtered from a sum of Dirac distributions to a continuous function. This is achieved by expanding $f_e$ in terms of basis functions, $\varphi_n(v), i = 1, 2, \ldots, N$,

$$f_e(v) = \sum_{n=1}^{N} f_n \varphi_n(v).$$  \hspace{1cm} (B.5)
B. Collisional power transfer to the electrons

where

$$\varphi_l(v) = \begin{cases} 
0 & \text{if } v < v_{n-1} \\
\frac{v-v_{n-1}}{v_{n-1}-v_{n+1}} & \text{if } v_{n-1} \leq v < v_n \\
1 - \frac{v-v_n}{v_{n+1}-v_n} & \text{if } v_n \leq v < v_{n+1} \\
0 & \text{if } v \geq v_{n+1}.
\end{cases}$$

The finite number of basis functions will truncate the distribution at high velocities, but the truncation error is tested to be negligible. Inserting eq. (B.5) into eq. (B.4) yields

$$P_{e[f]} = \sum_{n=1}^{N} \int_{v_{n-1}}^{v_{n+1}} 4\pi m v \left(-\bar{\alpha}[f_e]v^2\varphi_n + \frac{1}{2} \frac{\partial}{\partial v} \left(v^2 \bar{\beta}[f_e] \varphi_n\right)\right) dv. \quad (B.6)$$

These integrals are calculated in the HYBRID code. The main use of the collisional power transfer to the electrons is for testing the errors created by the boundary method, see section 5.2.
Monte Carlo boundary methods for RF-heating of fusion plasma

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Abstract

A fusion plasma can be heated by launching an electromagnetic wave into the plasma with a frequency close to the cyclotron frequency of a minority ion species. This heating process creates a non-Maxwellian distribution function, that is difficult to solve numerically in toroidal geometry. Solutions have previously been found using a Monte Carlo code FIDO. However, the computations are rather time consuming. Therefore methods to speed up the computations, using Monte Carlo boundary methods have been studied.

The ion cyclotron frequency heating mainly perturbs the high velocity distribution, while the low velocity distribution remains approximately Maxwellian. An hybrid model is therefore proposed, assuming a Maxwellian at low velocities and calculating the high velocity distribution with a Monte Carlo method. Three different methods to treat the boundary between the low and the high velocity regime are presented. A Monte Carlo code HYBRID has been developed to test the most promising method, the "Modified differential equation" method, for a one dimensional problem. The results show good agreement with analytical solutions.

Key words: Monte Carlo simulations, variance reduction methods, ICRH, quasilinear Fokker-Planck equation, high energy tail.