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Abstract

In this paper, we construct and study the representation theory of a Hopf C^*-algebra with approximate unit, which constitutes quantum analogue of a compact group C^*-algebra. The construction is done by first introducing a convolution-product on an arbitrary Hopf algebra H with integral, and then constructing the L_2 and C^*-envelopes of H (with the new convolution-product) when H is a compact Hopf $*$-algebra.

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1. INTRODUCTION

Compact quantum groups were introduced by Woronowicz [10] and studied by several other authors [1, 3, 9], as non-commutative analogues of the function algebras on compact groups. The aim of this work is to construct some quantum analogues of the compact group C^*-algebras, which turn out to be non-cocommutative Hopf C^*-algebras with approximate unit.

More precisely, starting from a compact Hopf $*$-algebra, we construct a Hopf C^*-algebra with approximate unit in such a way that when the original compact Hopf $*$-algebra is the function algebra on certain compact group G, the resulting Hopf C^*-algebra reduces to the group C^*-algebra $C^*(G)$.

Recall that for a compact group G, the group C^*-algebra $C^*(G)$ is the completion by operator norm of the algebra $L^1(G)$ of absolutely integrable complex-valued functions on G. The product on $L^1(G)$ is the convolution-product, given by

$$(g * f)(x) = \int_G f(y)g(y^{-1}x)dy.$$

With respect to this product, $C^*(G)$ is a non-unital C^*-algebra. In that case, there is an important notion of δ-type sequences in $C^*(G)$, that approximate the unity. Such a sequence is called an approximate unit in $C^*(G)$. Moreover, the action of the group G on any representation can be recovered from the corresponding action of the algebra $C^*(G)$, by using δ-type sequences. In our case, however, we do not know what the quantum group defining our C^*-algebra is, hence our defining a Hopf algebra structure to exhibit the "group property" of our algebra.

Our construction is done in several steps. First we construct a convolution product on an arbitrary Hopf algebra with integrals, being motivated by the classical convolution product for $L_1(G)$, the completion of which (under the operator norm) defines $C^*(G)$ (see 2.2). Denoting by $\check{H}$ the vector space H with the new convolution product, we show that $\check{H}$ is a non-unital algebra that is an ideal of H^* (the dual of H); that the category of completely reducible H-comodules is equivalent to the category of completely reducible $\check{H}$-comodules (2.4, 2.5) and that when H is co-semi-simple then $\check{H}$ is isomorphic to a direct sum of full endomorphism rings of simple H-comodules (2.6).

In section 4, we now focus on the case when H is a compact Hopf $*$-algebra. Through the works of Woronowicz, Koorwinder and Djikhuizen, [10, 2], compact Hopf $*$-algebra are known to play a role analogous to that of the algebra of functions on compact groups in the classical theory. Since by definition, H has a complex scalar product, we complete it to a Hilbert space H_{L^2}, which is the quantum analogue of the algebra $L^2(G)$. We show in 3.2 that the convolution product on $\check{H}$ extends to H_{L^2} and that H_{L^2} possesses a (topological) coproduct and an approximate antipode and hence is a Hopf algebra with approximate unit (3.4). Since the antipode on H is not involutive, the $*$-structure defined on H by $h^* := S(h^*)$ cannot be extended to H_{L^2} because it is not continuous with respect to the given norm. To overcome this problem, we pass to the C^*-enveloping of H_{L^2} to obtain H_{C^*} that is the required Hopf C^*-algebra with approximate unit (see 4.5). We show that there is a one-one correspondence between the irreducible unitary representations of H_{C^*} and those of $\check{H}$ (see 4.4).

2. THE CONVOLUTION PRODUCT AND CO-SEMI-SIMPLICITY

We work over an algebraically closed field $\mathbb{K}$ of characteristic zero. Let (H, m, η) be an algebra over a field $\mathbb{K}$, where m denotes the product, η denotes the map $\mathbb{K} \longrightarrow H$, $1_{\mathbb{K}} \longrightarrow 1_H$. A bialgebra structure on H is a pair of linear maps $\Delta : H \longrightarrow H \otimes H$, $\varepsilon : H \longrightarrow \mathbb{K}$, satisfying

- Δ and ε are homomorphism of algebras.
- $(\varepsilon \otimes \mathrm{id}_H)\Delta = (\mathrm{id}_H \otimes \varepsilon)\Delta = \mathrm{id}_H$.
- $(\Delta \otimes \mathrm{id}_H)\Delta = (\mathrm{id}_H \otimes \Delta)\Delta$.

An antipode on H is a linear map $H \longrightarrow H$, satisfying

- $m \circ (S \otimes \mathrm{id}_H)\Delta = m \circ (\mathrm{id}_H \otimes S)\Delta = \eta\varepsilon$.

A bialgebra equipped with an antipode is called Hopf algebra. The antipode is then uniquely determined.

Let $(H, m, \eta, \Delta, \varepsilon)$ be a Hopf algebra. A right coaction of H on a vector space V is a linear map $\delta : V \longrightarrow V \otimes H$, satisfying

$$(1) \qquad \begin{aligned} (\mathrm{id}_V \otimes \varepsilon)\delta &= \mathrm{id}_V, \\ (\delta \otimes \mathrm{id}_H)\delta &= (\mathrm{id}_V \otimes \Delta)\delta, \end{aligned}$$

in the first identity we identify V with $V \otimes \mathbb{C}$. V is then called a right H-co-module.

We shall frequently use Sweedler's notation, in particular $\Delta(x) := \sum_{(x)} x_{(1)} \otimes x_{(2)} = x_1 \otimes x_2, \delta(v) = \sum_{(v)} v_{(0)} \otimes v_{(1)} = v_0 \otimes v_1$.

The elements v_1's in the presentation $\delta(v) = v_0 \otimes v_1$ are called coefficients of the coaction δ, and the space they span is called coefficient space. This space is a subcoalgebra of H. To see this, fix a basis $x_1, x_2, \dots, x_d$. Then, the coaction is given by $\delta(x_i) = x_j \otimes a_i^j$ and the $\{a_i^j\}$ span the coefficient space. On the other hand, from (1) it follows $\Delta(a_i^j) = a_k^j \otimes a_i^k$, $\varepsilon(a_i^k) = \delta_i^k$. The comodule V is simple iff $\{a_i^j\}$ is a basis for the coefficient space, see [4].

A left (right) integral on a Hopf algebra is a (non-trivial) linear functional $\int : H \longrightarrow \mathbb{K}$, which is a left (right) H-comodule homomorphism, where H is a left (right) H-comodule by means of the coproduct Δ and $\mathbb{K}$ is a left (right) H-comodule by means of the unity map η. Explicitly, a left (resp. right) integral $\int_l \left(\text{resp.}, \int_r\right)$ on H satisfies

$$(2) \qquad \int_l (x) = x_1 \int_l (x_2) \quad \left(\text{resp.} \int_r (x) = \int_r (x_1) \cdot x_2\right).$$

It was shown by Sullivan [7] that the integral on a Hopf algebra, if it exists, is defined uniquely up to a constant. Further, we have

Lemma 2.1. [8, 5, 6] *Let $\int$ be a left integral on H. Then the bilinear form $b(g,h) = \int(gS(h))$ is non-degenerate on H, that is*

$$(3) \qquad \begin{aligned} \int(gS(h)) = 0, \forall h \quad &\Longrightarrow \quad g = 0. \\ \int(gS(h)) = 0, \forall g \quad &\Longrightarrow \quad h = 0. \end{aligned}$$

In the rest of this work, we fix a left integral $\int$ on H.

We mention an identity, due originally to Sweedler [7], which plays a crucial role in our computations:

$$(4) \qquad \int(gS(h_1)) \cdot h_2 = g_1 \int(g_2 S(h)),$$

if $\Delta(g) = g_1 \otimes g_2$ and $\Delta(h) = h_1 \otimes h_2$ in the notations of Sweedler.

The coalgebra structure on H induces an algebra structure on its dual H^* – the space of linear forms on H. The product of ϕ, ψ in H^* is given by

$$\phi * \psi(h) = \phi(h_1)\psi(h_2), \forall h \in H.$$

The unit element in H^* is the counit ε of H.

Convolution product on H. We first recall the classical structure. Let G be a compact group. Then there exists a unique normalized Haar measure on G which induces the Haar integral on $L^1(G)$. The $*$-product is defined on $L^1(G)$ as follows:

$$(g * f)(x) := \int_G f(y)g(y^{-1}x)dy. \tag{5}$$

Now, let H be a Hopf algebra with an integral. Being motivated by (5) we define the convolution product on H by:

$$g * f := \int (fS(g_1)) \cdot g_2. \tag{6}$$

According to (4), we also have $g * f = f_1 \int (f_2 S(g))$.

Lemma 2.2. *H equipped with the $*$-product defined above is a (non-unital) algebra.*

Proof. We only have to check the associativity:

$$\begin{aligned}
(h * g) * f &= \int (fS((h * g)_1)) \cdot (h * g)_2 \\
&= \int \left(fS\left(\int (gS(h_1) \cdot h_2)\right)\right) \cdot h_2 \\
&= \int \left(fS(h_2) \cdot \int (gS(h_1)\right) \cdot h_3 \\
&= \int \left(f \int (gS(h_1)) \cdot S(h_2)\right) \cdot h_3 (\text{using } (4)) \\
&= \int \left(fS(g_1) \int (g_2 S(h_1))\right) \cdot h_2 \\
&= \int \left(fS(g_1) \cdot \int (g_2 S(h_1))\right) \cdot h_2 \\
&= \int \left(\int (fS(g_1)) \cdot g_2 S(h_1)\right) \cdot h_2 \\
&= \int ((g * f)S(h_1)) \cdot h_2 \\
&= h * (g * f).
\end{aligned}$$

Lemma 2.2 is proved. ∎

We denote by $\check{H}$ the vector space H, equipped with the convolution product $*$.

H-comodules and $\check{H}$-modules. We now study the correspondence between H-comodules and $\check{H}$-modules. Let V be an H-comodule. We define an action of $\check{H}$ on V as follows:

$$h * v := v_0 \int (v_1 S(h)) \tag{7}$$

We check the associativity:

$$\begin{aligned}
g * (h * v) &= v_0 \int (v_1 S(g)) \cdot \int (v_2 S(h)) \\
(\text{using } (4)) &= v_0 \int \left(v_1 S(h_1) \int (h_2 S(g))\right) \\
&= v_0 \int (v_1 S(g * h)) \\
&= (g * h) * v.
\end{aligned}$$

Let $\phi : V \longrightarrow W$ be a morphism of H-comodules, i.e., $\phi(v)_0 \otimes \phi(v)_1 = \phi(v_0) \otimes v_1$. Then, for $h \in H, v \in v$,

$$\begin{aligned} \phi(h * v) &= \phi(v_0 \int (v_1 S(h)) \\ &= \phi(v_0) \int (v_1 S(h)) \\ &= \phi(v)_0 \int (\phi(v)_1 S(h)) \\ &= h * \phi(v). \end{aligned}$$

Thus, ϕ is a morphism of $\check{H}$ modules. We therefore have a functor $\mathcal{F}$ from the category of H-comodules into the category of $\check{H}$-modules, which is the identity functor on the underlying category of vector spaces.

Proposition 2.3. *The functor $\mathcal{F}$ defined above is full, faithful and exact.*

Proof. From the definition of $\mathcal{F}$, we see that, as vector spaces, $\mathcal{F}(V) = V$ and $\mathcal{F}(\phi) = \phi$. Thus, the functor $\mathcal{F}$ is faithful and exact. It remains to show that $\mathcal{F}$ is full, which amounts to showing that if $\phi : V \longrightarrow W$ is a morphism of $\check{H}$-module then it is a morphism of H-comodule. By assumption, ϕ satisfies

$$h * (\phi(v)) = \phi(h * v), \quad \forall h \in \check{H}$$

or, equivalently,

$$\phi(v)_0 \int (\phi(v_1) S(h)) = \phi(v_0) \int ((v_1 S(h)), \quad \forall h \in \check{H}.$$

Since $\int$ is faithful (see (3)), we conclude that

$$\phi(v_0) \otimes \phi(v_1) = \phi(v_0) \otimes v_1.$$

In other words, ϕ is a homomorphism of H-comodules.∎

Let V now be a cyclic $\check{H}$-module, i.e., there exists an element $\bar{v} \in V$, such that all $v \in V$ is obtained from $\bar{v}$ by the action of some $f \in H$. We then define the coaction of H on V by

$$\delta(v) := f_1 * \bar{v} \otimes S(f_2), \tag{8}$$

where f is such that $f * \bar{v} = v$.

First, we have to show that this coaction is well defined, which means that it does not depend on the choice of the representative elements $\bar{v}$ and f. To show that the definition does not depend on $\bar{v}$ we let $\bar{v} = g * \tilde{v}$ and show that the definition does not change when $\bar{v}$ is replaced by $\tilde{v}$, which means

$$f_1 * \bar{v} \otimes S(f_2) = (f * g)_1 * \tilde{v} \otimes S((f * g)_2).$$

Replacing $\bar{v}$ on the left-hand side of this equation by $g * \tilde{v}$ and canceling $\tilde{v}$ in both sides, one is led to the following equation

$$f_1 * g \otimes f_2 = (f * g)_1 \otimes (f * g)_2,$$

which follows immediately from the definition.

To show the independence on the choice of f we assume $f * \bar{v} = 0$ and show that $f_1 * \bar{v} \otimes S(f_2) = 0$. Indeed, we have

$$\int (f_2 S(g)) \cdot f_1 * \bar{v} = (g * f) * \bar{v} = g * (f * v) = 0,$$

for all g. Hence, according to Lemma 2.1 $f_1 * \bar{v} \otimes f_2 = 0$.

We now proceed to check the co-associativity and co-unitary:

$$\begin{aligned}(\mathrm{id}_H \otimes \delta)\delta(v) &= \delta(f_1 * \bar{v}) \otimes S(f_2) \\ &= f_1 * \bar{v} \otimes f_2 \otimes f_3) \\ &= f_1 * \bar{v} \otimes \Delta(f_2) \\ &= (\mathrm{id} \otimes \Delta_H)\delta(v).\end{aligned}$$

$$\begin{aligned}(\mathrm{id}_V \otimes \varepsilon)\delta f(v) &= f_1 * \bar{v} \otimes \varepsilon(f_2) \\ &= f * \bar{v} = v.\end{aligned}$$

Since simple modules are cyclic, we obtain a functor $\mathcal{G}$ from the category of completely reducible $\check{H}$-modules to the category of H-comodules.

Theorem 2.4. *The category of completely reducible H-comodules is equivalent to the category of completely reducible $\check{H}$-modules.*

Proof. We show that the functors $\mathcal{F} \circ \mathcal{G}$ and $\mathcal{G} \circ \mathcal{F}$ are identity-functors on the categories of completely reducible H-comodules and $\check{H}$-comodules, respectively.

Let V be a simple H-comodule. Then $\check{V} = \mathcal{F}(V)$ is a simple $\check{H}$-module. Let us fix $\bar{v} \in \check{V}$ and for $v \in \check{V}$, let $f \in \check{H}$ be such that $f * \bar{v} = v$. By definition, we have

$$v = f * \bar{v} = \bar{v}_0 \int (\bar{v}_1 S(f)).$$

The coaction of H on $\mathcal{G} \circ \mathcal{F}(V)$ is

$$\begin{aligned}\delta_{\mathcal{G}\circ\mathcal{F}(V)}(v) &= f_1 * \bar{v} \otimes f_2 \\ &= \bar{v}_0 \int (\bar{v}_1 S(f_1)) \otimes f_2 \\ &= \bar{v}_0 \otimes \bar{v}_1 \int (v_2 S(f)) \\ &= \delta_V(v).\end{aligned}$$

Thus, $\mathcal{G} \circ \mathcal{F}$ is the identity functor. The assertion for $\mathcal{F} \circ \mathcal{G}$ is proved analogously. ∎

$\check{H}$ is an ideal of H^*. In the previous section we have seen that there exists a correspondence between H-comodules and $\check{H}$-modules. On the other hand, there exists a one-to-one correspondence between H-comodules and rational H^*-modules. It is then natural to ask about the relationship between H^* and $\check{H}$. We now show that $\check{H}$ is isomorphic to the left ideal generated by the integrals in H^*.

It is well known, that the rational submodule $H^\square$ of H^*, considered as a left module on itself, i.e. the sum of all left ideals of H^*, which are finite dimensional (over $\mathbb{K}$), is an H-Hopf module (with an appropriate H-action) and hence isomorphic to the tensor product of H with the space spanned by the integrals [8, Thm 5.1.3]. Since the space of integrals is one-dimensional, we have an isomorphism between the two vector spaces H and $H^\square$. This isomorphism can be given explicitly as follows.

By means of the integral, every element of H can by considered as a linear functional on H itself: $H \ni h \longmapsto \int^h \in H^* : \int^h (g) := \int (gS(h))$.

Proposition 2.5. *The map $H \ni h \longmapsto \int^h \in H^*$ defined above is an isomorphism of algebras $\check{H} \longrightarrow H^\square \subset H^*$.*

Proof. We have

$$\begin{aligned}\int^{f*h}(g) &= \int\Big(gS(h_1\int(h_2Sf))\Big)\\ &= \int(gS(h_1))\int(h_2S(f))\\ &= \int(g_1S(f))\int(g_2S(h))\\ &= (\int^f*\int^h)(g). \blacksquare\end{aligned}$$

In the strictly algebraic sense, $\check{H}$ is not a Hopf algebra, unless H is finite-dimensional. In the next section we will show that for compact Hopf $*$-algebra, there exists a natural topology in H such that the completion of H with respect to this topology is a topological Hopf algebra with approximate unit.

Co-semi-simple Hopf algebras. A Hopf algebra H is called co-semisimple if any finite dimensional H-comodule decomposes into a direct sum of simple comodules. A Hopf algebra is co-semisimple if and only if it possesses an integral whose value at its unit element is nonzero. In this case, left and right integrals are equal [8].

Let H be a co-semi-simple Hopf algebra. Then it decomposes into a direct sum of simple sub-coalgebras, each of which is the coefficient space of a simple H-comodule [8]

$$H \cong \bigoplus_{\lambda\in\Lambda} H_\lambda. \tag{9}$$

The set Λ contains an element 0 for which $H_0 \cong \mathbb{K}$. The integral computed on H_λ is zero for $\lambda \neq 0$.

We consider the discrete topology on Λ. Let $C_0(\Lambda)$ denote the set of all compact subsets in Λ containing 0. For any compact K, let

$$H_K := \bigoplus_{\lambda\in K} H_\lambda. \tag{10}$$

Then H_K are subcoalgebra of H. For any $f \in H$, $f = \sum_{\lambda\in K} f_\lambda, f_\lambda \in H_\lambda$, for some compact K.

For each $\lambda \in \Lambda$, let V_λ be the corresponding simple H-comodule. Note that the V_λ are finite dimensional for all $\lambda \in \Lambda$.

The isomorphism (9) becomes now an isomorphism of algebras between $\check{H}$ and the direct sum of endomorphism ring of V_λ.

$$\check{H} \cong \bigoplus_{\lambda\in\Lambda} \check{H}_\lambda \cong \bigoplus_{\lambda\in\Lambda} \mathrm{End}_{\mathbb{K}}(V_\lambda). \tag{11}$$

Thus we have proved

Theorem 2.6. *Let H be a co-semisimple Hopf algebra. Then $\check{H}$ is isomorphic to the direct sum of full endomorphism rings of simple H-comodules.*

The algebra $\check{H}$ does not have a unit element. Adding a unit to this algebra is problematic when we are dealing with the norm – the unit element is something like " the Dirac delta function " which never has finite norm. Instead we have a notion of δ-type sequences, which approximate the unit.

Definition 2.7. 1) Let A be an algebra without unit. A system $\{e_i, i \in I\}$ of idempotents in A is an *approximate unit* if

(i) I is a partially ordered set.

(ii) For any $a \in A$, there exists $i = i(a)$, such that $e_ia = ae_j = a$, for all $j \geq i$.

For an approximate unit in a bialgebra we require further that

(iii) $\varepsilon(e_i) = 1, \forall i \in I$.

For an algebra with involution we require that

(iv) There exist f_i, such that $e_i = f_i f_i^*$, for all $i \in I$.

2) For a topological algebra, the condition (ii) above is replaced by

(ii') the nets $\{e_i a | i \in I\}$ and $\{ae_i | i \in I\}$ converge to a.

3) A Hopf algebra with approximate unit is a bialgebra with approximate unit together with a system of endomorphism $\{S_i | i \in I\}$, called an *approximate antipode*, satisfying

$$m(S_i \otimes \mathrm{id})\Delta = m(\mathrm{id} \otimes S_i)\Delta = e_i \varepsilon.$$

The existence of such a sequence in our $\check{H}$ is obvious. Indeed, let e_K be the unit element in $\check{H}_K$. Then $\{e_K, K \in C_0(\Lambda)\}$ is an approximate unit in $\check{H}$. Thus $\check{H}$ is an algebra with approximate unit.

3. Compact Hopf $*$-algebras

In this section we construct from a compact Hopf $*$-algebra a bialgebra which is a Hilbert space. Thus, $\mathbb{K} = \mathbb{C}$. This is the first step toward the construction of our Hopf C^*-algebra. A good reference on compact Hopf $*$-algebra, where the algebra is referred to as CQG-algebra, is Dijkhuizen and Koornwinder [2].

By definition, a compact Hopf $*$-algebra over $\mathbb{C}$ is a co-semisimple Hopf algebra with an involutive $\mathbb{C}$-anti-linear anti-homomorphism * such that every simple comodule is unitarizable, i.e. we can define a scalar product on this comodule such that

$$< v_0, w > S(v_1) =< v, w_0 > w_1^*, \text{ for } v, w \in H. \tag{12}$$

For any orthonormal basis of this comodule, the corresponding coefficient matrix satisfies the orthogonality condition. More precisely, let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_d$ be an orthonormal basis of the comodule and $\mathbf{U} = (\mathbf{u}_i^j)$ be the corresponding coefficient matrix, i.e., $\delta(\mathbf{x}_i) = \mathbf{x}_k \otimes \mathbf{u}_i^k$. Then $\mathbf{U}$ satisfies $\mathbf{U}\mathbf{U}^* = \mathbf{U}^*\mathbf{U} = I$, where $\mathbf{U}^{*}{}_i^j := \mathbf{u}_j^{i*}$.

Lemma 3.1. *Let H be a co-semisimple Hopf algebra. Then the square of the antipode on H is co-inner, i.e., it can be given in terms of an invertible element q of H^* (i.e. a linear form of H):*

$$S^2(h) = q(h_1)h_2 q^{-1}(h_2),$$

where the linear form q is given by $q(h) = \int (S^2(h_1)S(h_2))$.

Proof. First we show that $S^2(h_1)q(h_2) = q(h_1)h_2$, or equivalently

$$S^2(h_1) \int (S^2(h_2)S(h_3)) = \int (S^2(h_1)S(h_2)) \cdot h_3. \tag{13}$$

We have

$$\begin{aligned} \int (S^2(h_2)S(h_3)) \cdot h_4 S(h_1) &= S^2(h_2) \int (S^2(h_3)S(h_4)) \cdot S(h_1) (\text{by } (4)) \\ &= \int (S^2(h_1)S(h_2)). \end{aligned}$$

Thus, the lemma will be proved if we can show that q is invertible as an element of H^*.

Let V be a finite dimensional H-comodule and V^{**} be its double-dual. As vector space, V^{**} is isomorphic to V and the coaction of H on V^{**} is given by $\delta_{V^{**}}(v) = v_0 \otimes S^2(v_1)$. Equation (13) shows that the map $v \longmapsto v_0 \otimes q(v_1) : V \longrightarrow V^{**}$ is a morphism of H-comodules. If V is simple then V^*

and hence V^{**} are simple. Therefore, the map above should be zero or invertible. To see that it cannot be zero we fix a basis $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_d$ of V and let $\mathbf{U} = (\mathbf{u}_i^j)$ be the coefficient matrix. Since we have $\Delta(\mathbf{u}_i^j) = \sum_k \mathbf{u}_k^j \otimes \mathbf{u}_i^k, \varepsilon(\mathbf{u}_i^j) = \delta_i^j$,

$$q(\sum_i \mathbf{u}_i^i) = \int(\mathbf{u}_i^k S(\mathbf{u}_k^i)) = d.$$

Thus, the map $v \longmapsto v_0 \otimes q(v_1) : V \longrightarrow V^{**}$ is an isomorphism of H-comodules. Therefore the form q is invertible. ∎

Set $Q_i^j = q(\mathbf{u}_i^j)$. Then, according to Lemma 3.1 the matrix Q is the matrix of the isomorphism $V \longrightarrow V^{**}$ with respect to the basis $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_d$ of V and we have

$$S^2(\mathbf{u}_i^j) = Q_k^j \mathbf{u}_l^k Q^{-1}{}_j^l. \tag{14}$$

The matrix Q is very important in the study of H and will be called *reflection matrix.* If V is irreducible then the integral on $\mathbf{u}_i^j S(\mathbf{u}_k^l)$ can be given in terms of Q. In fact, from the left-invariance of $\int$ we have

$$\int(\mathbf{u}_i^j S(\mathbf{u}_k^l)) = \mathbf{u}_m^j S(\mathbf{u}_k^n) \int(\mathbf{u}_i^m S(\mathbf{u}_n^l)) \tag{15}$$

Or equivalently

$$\int(\mathbf{u}_i^j S(\mathbf{u}_k^l)) \cdot \mathbf{u}_n^k = \mathbf{u}_m^j \int(\mathbf{u}_i^m S(\mathbf{u}_n^l)).$$

Since $\{\mathbf{u}_i^j\}$ are linearly independent, $\int(\mathbf{u}_i^j S(\mathbf{u}_k^l)) = \delta_k^j C_i^l$ for some matrix $C = (C_i^l)$.

On the other hand, according to (14), we have $\mathbf{u}_i^m Q_m^n S(\mathbf{u}_n^j) = Q_i^j$. Substituting this into (15) we get $\delta_n^m C_i^j Q_m^n = Q_j^i$. Thus $\mathrm{tr}(Q) \neq 0$ and $C_i^j = Q_i^j/\mathrm{tr}(Q)$, and we get

$$\int(\mathbf{u}_i^j S(\mathbf{u}_k^l)) = \delta_k^j Q_i^l/\mathrm{tr}(Q). \tag{16}$$

Analogously we have

$$\int(S(\mathbf{u}_i^j)\mathbf{u}_k^l) = \delta_k^j (Q^{-1})_i^l/\mathrm{tr}(Q^{-1}). \tag{17}$$

From Equations (16) and (17) we get the rule for the convolution product for coefficients of simple comodules. Let V_λ and V_μ be simple comodules and $\mathbf{U}_\lambda, \mathbf{U}_\mu$ be the coefficient matrix with respect to some (orthogonal) bases of V_λ and V_μ, and $Q_\lambda := q(\mathbf{U}_\lambda), Q_\mu := q(\mathbf{U}_\mu)$. An element f of H_λ can be represented by a matrix F_λ: $f = F_\lambda{}_j^i \mathbf{u}_\lambda{}_i^j = \mathrm{trace}(F_\lambda \mathbf{U}_\lambda)$. Let $c_\lambda = \mathrm{trace}(C_\lambda \mathbf{U}_\lambda), d_\mu = \mathrm{trace}(D_\mu \mathbf{U}_\mu)$. The convolution product has the form

$$\begin{aligned} c_\lambda * d_\lambda &= C_\lambda{}_j^i D_\lambda{}_l^k (\mathbf{u}_\lambda)_i^j * (\mathbf{u}_\mu)_k^l \\ &= C_\lambda{}_j^i D_\lambda{}_l^k \delta_\mu^\lambda \frac{Q_\lambda{}_k^j}{\mathrm{tr}(Q_\lambda)} (\mathbf{u}_\lambda)_i^l \quad \text{(from (6) and (16))} \\ &= \frac{C_\lambda Q_\lambda D_\lambda}{\mathrm{tr}(Q)}, \end{aligned} \tag{18}$$

where in the left-hand side is the matrix product: $(CD)_i^j = C_k^i D_j^k$. Thus, with the $*$-product, H_λ becomes a unital algebra, denoted by $\check{H}_\lambda$, and $\check{H}_\lambda \cong \mathrm{End}_{\mathbb{C}}(V_\lambda)$. Moreover, for any compact $K \subset \Lambda$, $\check{H}_K := \bigoplus_{\lambda \in K} \check{H}_\lambda$ is also a unital algebra.

Let V_λ be a simple H-comodule, $\check{V}_\lambda$ be the corresponding $\check{H}$-module. The representation of $\check{H}$ on $\check{V}_\lambda$ will be denoted by π_λ. Then according to (7), the action of $f = \mathrm{trace}(F_\lambda \mathbf{U}_\lambda)$ on $\mathbf{x}_i$ is given by

$$f * \mathbf{x}_i = \mathbf{x}_j \int(\mathbf{u}_i^j S(f)) = \mathbf{x}_j F_\lambda{}_l^j Q_\lambda{}_i^l. \tag{19}$$

In other words, the matrix of $\pi_\lambda(f)$ with respect to the basis $\{\mathbf{x}_i\}$ is $F_\lambda Q_\lambda$.

For compact Hopf $*$-algebra, we can show that the matrix Q, with respect to an orthonormal basis, is positive definite [2, 10]. Let V^* be the dual to the comodule V. Let $\xi_1, \xi_2, \ldots, \xi_d$ be the basis of V^*, dual to the basis $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_d$. The corresponding coefficient matrix is then $S(\mathbf{U}^t)$. Since V^* is unitarizable, there exists a basis $\eta_1, \eta_2, \ldots, \eta_d$ of V^* which is orthonormal with respect to a scalar product, such that the corresponding coefficient matrix $\mathbf{W} = (\mathbf{w}^j_i)$ satisfies $S(\mathbf{W}) = \mathbf{W}^*$. Let T be a matrix such that $\xi^i = \eta_j T^i_j$ then

$$\mathbf{w}^i_j = {T^{-1}}^i_k S(\mathbf{u}^k_l) T^l_j = {T^{-1}}^i_k \mathbf{u}^{l*}_k T^l_j, \tag{20}$$

since $(\mathbf{u}^i_j)$ is also unitary. Therefore, by the involutivity of $*$

$$S(\mathbf{w}^j_i) = \mathbf{w}^{i*}_j = \bar{T}^{-1i}_k \mathbf{u}^l_k \bar{T}^l_j.$$

Substituting $S(w^j_i)$ into the preceding equation and using (14) we, get $Q = \text{const} \cdot T^*T$. A direct consequence of this fact is that $\int$ is positive definite on H.

For any $\lambda \in \Lambda$, let λ^* be such that $(V_\lambda)^* = V_{\lambda^*}$. According to Lemma 3.1, $*$ is an involutive map on Λ. From above we see that the involution $*$ on H maps H_λ onto H_{λ^*}. Since the integral is zero on H_λ except for $\lambda = 0$, we see that $\int(a) = \overline{\int(a^*)}$. Consequently $\int$ defines a scalar product $<,>$ on H, $< a, b >:= \int(ab^*)$ giving rise to a norm on H called L^2-norm. Let H_{L^2} denote the completion of H with respect to the L^2-norm. Then H_{L^2} is a Hilbert space.

The Hopf Algebra H_{L^2}. Our aim is to define a Hopf algebra structure on H_{L^2}, where the product is an extension of the convolution product on $\check{H}$. First we have to extend the convolution product to H_{L^2}. This can be done provided that we showed that this product is continuous with respect to the L^2-norm.

Lemma 3.2. *The convolution product on $\check{H}$ satisfies*

$$||f * g||_{L^2} \leq ||f||_{L^2} \cdot ||g||_{L^2}.$$

Proof. It is sufficient to show this inequality for f and g belonging to the same coefficient space of a simple comodule. Then, in that case, f and g can be represented in the form $f = \text{tr}(CU)$, $g = \text{tr}(DU)$, for a unitary multiplicative matrix U and some matrices C, D with complex scalar entries. Using (16), (18) the indicated inequality is equivalent to the following

$$\frac{\text{tr}(CQC^*)\text{tr}(DQD^*)}{tr(Q)^2} \geq \frac{\text{tr}(DQCQ(DQC)^*)}{\text{tr}(Q)^3}.$$

Let T be such that $Q = TT^*$. Set $C_1 = CT$, $D_1 = DT$ the above inequality has the form

$$\text{tr}(C_1C_1^*)\text{tr}(D_1D_1^*)\text{tr}(TT^*) \geq \text{tr}(C_1T^*D_1D_1^*TC_1^*).$$

The last inequality follows immediately from the Minkowski inequality since

$$\text{tr}(CC^*) = \sum |c^j_i|^2. \blacksquare$$

Therefore, the convolution product on $\check{H}$ can be extended to H_{L^2}.

Lemma 3.3. *The family $\{e_K | K \in C_0(\Lambda)\}$, where e_K is the unit element of $\check{H}_K$, is an approximate unit in H_{L^2}.*

Proof. What we need to show is that for any $h \in H_{L^2}$, there exists a composition sequence $K_1 \subset K_2 \subset \ldots$ such that

$$\lim_{n\to\infty} ||e_{K_n} * h - h|| = \lim_{n\to\infty} ||h * e_{K_n} - h|| = 0$$

Now, for any $h \in H_{L^2}$, there exists at most a countable set of λ, $\lambda \in \Lambda$ such that $h_\lambda = h * e_\lambda \neq 0$ and $h = \sum_\lambda h_\lambda$. The last series converges absolutely whence the assertion follows. ∎

Then we have to define the coproduct. Notice that, the coproduct on H_{L^2}, if it exists, should be dual to the original product on H by means of the integral. Thus, we consider, for an element $h \in H$, a linear functional $\phi_h : H \otimes H \longrightarrow \mathbb{C}$, $\phi_h(g \otimes f) = \int(hgf)$. This is obviously continuous hence is extendable on $H_{L^2}\hat{\otimes}H_{L^2}$ ($\hat{\otimes}$ denotes the tensor product of Hilbert spaces). Hence, by Riesz theorem, there exists an element $\Delta_*(f)$ of $H_{L^2}\hat{\otimes}H_{L^2}$, such that $\phi_h(g \otimes f) = \int\int(\Delta_*(h), g \otimes f)$. Thus, we get a map $H \longrightarrow H_{L^2}$, $h \longmapsto \Delta_*(h)$. Again this is continuous and hence induces a map $H_{L^2} \longrightarrow H_{L^2}\hat{\otimes}H_{L^2}$ which is the coproduct on H_{L^2}.

The counit is given by $\varepsilon_*(f) = \int(f)$.

We check the axioms for the coproduct and counit.

Theorem 3.4. *$(H_{L^2}, *, e_K, \Delta_*, \varepsilon_*)$ is a Hopf algebra with approximate unit.*

Proof. The proof consists of some lemmas. For convenience we shall use the notation

$$\Delta_*(f) = \sum_{(f)} f^{(1)} \otimes f^{(2)} = f^1 \otimes f^2,$$

where the sum on the right-hand side is an absolute convergent series in $H_{L^2}\hat{\otimes}H_{L^2}$. ∎

Lemma 3.5. *The coproduct and the counit satisfy*

$$(\Delta_* \otimes id_{H_{L^2}})\Delta_* = (id_{H_{L^2}} \otimes \Delta_*)\Delta_*,$$

$$(\varepsilon_* \otimes id_{H_{L^2}})\Delta_* = (id_{H_{L^2}} \otimes \varepsilon_*)\Delta_* = id_{H_{L^2}},$$

in the second equation we identify $\mathbb{C}\hat{\otimes}H_{L^2}$ and $H_{L^2}\hat{\otimes}\mathbb{C}$ with H_{L^2}.

Proof. By the faithfulness of $\int$, the first equation is equivalent to

$$\int\int\int(\Delta_* \otimes id)\Delta_*(f) \cdot g \otimes h \otimes k = \int\int\int((id \otimes \Delta_*)\Delta_* f \cdot g \otimes h \cdot k), \forall g, h, k \in H, f \in H_{L^2}$$

By definition, the left-hand side is equal to

$$\int\int\int(f^{11} \otimes f^{12} \otimes f^2 \cdot g \otimes h \otimes k) = \int(f^1 \cdot gh)\int(f^2 k) = \int fghk\cdot$$

So is also the right-hand side, whence we obtain the first equation of the lemma.

For the second equation of the lemma, we have, for all $g \in H$,

$$\int(\varepsilon(f^1)f^2 \cdot g = \int(f^1)\int(f^2 g) = \int fg\cdot$$

The lemma 3.5 is therefore proved. ∎

Lemma 3.6. *The map $\Delta_* : H_{L^2} \to H_{L^2}\hat{\otimes}H_{L^2}$ is a homomorphism of algebras.*

Proof. This is equivalent to the fact that for any $h \cdot k \in H$, $f, g \in H_{L^2}$,

$$\int(f * g)hk = \int((f^1 * g^1)h) \cdot \int((f^2 * g^2)k).$$

The left-hand side is equal to

$$\int(fh_1k_1) \cdot \int(gh_2k_2) = \int(f^1h_1)\int(f^2k_1)\int(g^1h_2)\int(g^2k_2) = \int((f^1 * g^1)h) \cdot \int((f^2 * g^2)k).$$

The Lemma 3.6 is proved. ∎

The above two lemmas imply that H_{L^2} is a bialgebra with approximate unity. It remains to find an approximate antipode. By definition, we have to find a system $\{S_{*K}|K \in C_0(\Lambda)\}$ of linear endomorphism on H_{L^2}, such that $m_*(S_{*K} \otimes \mathrm{id})\Delta_*$ and $m_*(\mathrm{id} \otimes S_{*K})\Delta_*$ are approximate units.

For any $h \in H_\lambda$, we define $S_*(h) \in H_{\lambda *}$ to be such that $\forall g \in H_\lambda$,

$$\int (hS(g)) = \int (S_*(h)g).$$

In fact, $S_*(h)$ can be computed explicitly

$$S_*(h) = S(h_1)p(h_2)q(h_3),$$

where q is the linear functional defined in Lemma 3.1, p is given by

$$p(h) = \int S(h_2)h_1, \forall h \in H.$$

Set $S_{*\lambda}|_{H_\mu} = 0, \forall \mu \neq \lambda$. For any $K \subset C_0(\Lambda)$ set $S_{*K} = \bigoplus_{\lambda \in K} S_{*\lambda}$.

Lemma 3.7. *$\{S_{*K}|K \in C_0(\Lambda)\}$ is an approximate antipode.*

Proof. We show that

$$S_{*K}(f^1) * f^2 = \varepsilon_*(f)e_K = \int (f)e_K, \forall f \in H, K \in C_0(\Lambda).$$

Since $\int S_{*K}(f)g = \int fS(g_K)$, we have

$$\begin{aligned} \int ((S_{*K}(f^1) * f^2)g) &= \int (S_{*K}(f^1)g_1) \cdot \int (f^2 g_2) \\ &= \int (f^1 S(g_{K1})) \cdot \int (f^2 g_2) \\ &= \int (f) \int (e_K g), \end{aligned}$$

whence the assertion follows. The lemma is proved. ∎

We have therefore finished the proof of Theorem 3.4. ∎

Remark. Since the antipode on H is not involutive, the $*$-structure on $\check{H}$ defined by $h^\star = S(h^*)$ cannot be extended to H_{L^2}, because it is not continuous with respect to the given norm. To overcome this obstruction we have to pass to the C^* envelope of H_{L^2} which is the subject of our next section.

4. The C^*-algebra H_{C^*}

Let H be a compact Hopf $*$-algebra. Let $\star$ be the involutive map on $\check{H}$ defined in the previous section $f^\star = S(f^*)$.

Lemma 4.1. *All simple $\check{H}$-modules have the structure of $\star$-modules.*

Proof. According to Theorem 2.4, a simple $\check{H}$-comodule is equivalent to some module $\check{V}_\lambda$ induced from a simple unitary H-comodule V_λ. We check that, with respect to the given scalar product on V_λ,

$$< h * v, w > = < v, h^\star * w >, \forall h \in \check{H}, v, w \in \check{V}_\lambda.$$

Indeed,

$$\begin{aligned} < h * v, w > &= < v_0, w > \overline{\int (hS(v_1))} \\ &= < v, w_0 > \overline{\int (hw_1 *)} \quad \text{according to (12)} \\ &= < v, w_0 > \int (w_1 h^*) \\ &= < v, w_0 > \int (S(h^*) S(w_1)) \\ &= < v, h^* * w > . \blacksquare \end{aligned}$$

We introduce the following semi-norm on $\check{H}$

(21) $$||f||_{C^*} := \sup_{\lambda \in \Lambda} ||\pi_\lambda(f)||.$$

The lemma below will show that this is a norm, i.e., it is bounded.

Let $\mathbf{U} = \mathbf{U}_\lambda$ be a unitary coefficient matrix corresponding to the simple unitary comodule V_λ and $Q = Q_\lambda$ be the corresponding reflection matrix. If $0 \neq f \in \check{H}_\lambda$, then $f = F^i_j \mathbf{u}^j_i$ for a complex matrix F. Since f act on $\check{V}_\mu$ as zero if $\mu \neq \lambda$, then according to (19),

(22) $$||f||_{C^*} = ||\pi_\lambda(f)|| = ||FQ|| > 0.$$

Lemma 4.2. *The L^2- and C^*-norms on $\check{H}$ satisfy*

$$||f||_{L^2} \geq ||f||_{C^*}.$$

Proof. Since $||f||^2_{L^2} = \sum_\lambda ||f_\lambda||^2_{L^2}$ and $||f||_{C^*} = \sup_\lambda ||f_\lambda||_{C^*}$, it is sufficient to check the above equation for $f \in \check{H}_\lambda$. Thus $f = F^i_j \mathbf{u}^j_i$, for $(\mathbf{u}^j_i) = (\mathbf{u}_\lambda)^j_i$ a unitary multiplicative matrix.

The desired inequality is then equivalent to

$$\frac{\mathrm{tr}(FQF^*)}{\mathrm{tr}(Q)} \geq \frac{||FQ||^2}{\mathrm{tr}(Q)^2}.$$

Since Q is positive definite, $Q = T^*T$. Hence

$$||FQ||^2 = ||FTT^*|| \leq ||FT||^2 ||T^*||^2 \leq \mathrm{tr}(FQF^*)\mathrm{tr}(Q).$$

Here we use the inequality

$$||A||^2 \leq \mathrm{tr}(AA^*). \blacksquare$$

We define the algebra H_{C^*} to be the completion of $\check{H}$ with respect to the norm $||\cdot||_{C^*}$. Since $||\cdot||_{C^*}$ is an operator norm, H_{C^*} is a C^*-algebra. By virtue of Lemma 4.2 we have

(23) $$H_{L^2} \subset H_{C^*}.$$

From the decomposition (11) of $\check{H}$ and the definition of the norm $||\cdot||$, we can easily obtain an isomorphism of the two C^*-algebras

(24) $$H_{C^*} \cong \prod_{\lambda \in \Lambda}{}' \mathrm{End}_{\mathbb{C}}(V_\lambda),$$

where the product on the right-hand side of the equation is over all families $\{x_\lambda \in \mathrm{End}_{\mathbb{C}}(V_\lambda)\}$ with $||x_\lambda|| \longrightarrow 0$ as $\lambda \longrightarrow \infty$.

Lemma 4.3. *All irreducible unitary representations of H_{C^*} are finite-dimensional and irreducible over $\check{H}$.*

Proof. Let $\pi : H_{C^*} \longrightarrow \mathcal{B}(\mathcal{H})$ be an irreducible unitary representation of H_{C^*} in a Hilbert space $\mathcal{H}$. Then $\mathcal{H}$ is a $\check{H}$-module. Let $M \subset \mathcal{H}$ be a simple $\check{H}$-submodule. Then it has a structure of a simple H-comodule, hence is finite dimensional over $\mathbb{C}$. Thus M is closed in $\mathcal{H}$.

Since $\check{H}$ is dense in H_{C^*} and M is closed in $\mathcal{H}$, M is a representation of H_{C^*}. By the irreduciblity of π, we conclude that $M = \mathcal{H}$.∎

As a corollary of Lemmas 4.1 and 4.3, we have

Proposition 4.4. *There exists a 1-1 correspondence between irreducible unitary representations of H_{C^*} and those of $\check{H}$.*

Thus H_{C^*} is a C^*-algebra of type I.

From the construction of H_{C^*} it is easy to see, that if G is a compact group and $H = \mathbb{C}[G]$ - the Hopf algebra of representative functions on G, which is a dense subalgebra of the algebra $C^\infty(G)$ of all continuous complex valued functions on G, then $H_{C^*} \cong C^*(G)$ - the group C^*-algebra of G.

In the case of a compact group G, the action of G on any representation can be recovered by the action of $C^*(G)$, by using δ-type sequences. In our case, we do not know, what the quantum group defining H is. Therefore we have to introduce a Hopf algebra structure on H_{C^*} to exhibit the "group" property of this algebra.

Theorem 4.5. *H_{C^*} is a Hopf C^*-algebra with approximate unit.*

Proof. Recall that the topological tensor product $H_{C^*}\hat{\otimes}H_{C^*}$ is the completion of $\check{H} \otimes \check{H}$ with respect to the norm

$$||f \otimes g||_{C^*} := \sup_{\lambda,\mu\in\Lambda} ||\pi_\lambda(f) \otimes \pi_\mu(g)||.$$

For any $f \in \check{H} \subset H_{L^2}$, $\Delta(f) \in H_{C^*}\hat{\otimes}H_{C^*}$. Thus, it is sufficient to show that

$$||\Delta(f)||_{C^*} \leq ||f||_{C^*}.$$

But this inequality is obvious, as for any $\lambda, \mu \in \Lambda$, $V_\lambda \otimes V_\mu$ is a representation of H_{L^2} by means of the map $(\pi_\lambda \otimes \pi_\mu)\Delta$, hence decomposes into a direct sum of irreducible representations, namely $\pi_\lambda \otimes \pi_\mu \cong \oplus_\gamma c^\gamma_{\lambda\mu}\pi_\gamma$, consequently

$$||(\pi_\lambda \otimes \pi_\mu)\Delta(f)|| \leq \sup_{\gamma\in\Lambda} ||\pi_\gamma(f)|| = ||f||_{C^*}.$$

To show that ε_* extends onto H_{C^*} we remark that $\varepsilon_* : \check{H} \longrightarrow \mathbb{C}$ is multiplicative, i.e., induces a representation, for

$$\varepsilon_*(f * g) = \int (f_1 \int (f_2 S(g))) = \int (f) \int (g) = \varepsilon_*(f)\varepsilon_*(g).$$

Hence

$$|\varepsilon_*(f) \leq ||f||_{C^*}$$

thus ε_* extends onto H_{C^*}.

According to Lemma 4.1, for $f \in \check{H}$,

$$\pi_\lambda(f^*) = \pi_\lambda(f)^*, \forall \lambda \in \Lambda.$$

Therefore the involutive map $\star$ extends on H_{C^*}.

Using (14) and (18), we can show that the unit element e_λ in H_λ satisfies $e^\star_\lambda = e_\lambda$. Since $\{e_K | K \in C_0(\Lambda)\}$ is an approximate unit of $\check{H}$ and since $\check{H}$ is dense in H_{C^*}, $\{e_K | K \in C_0(\Lambda)\}$ is an approximate unit on H_{C^*}. Consequently, $\{S_K | K \in C_0(\Lambda)\}$ is an approximate antipode on H_{C^*}. The proof of Theorem 4.5 is completed. ∎

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