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On Estimating Perturbative Coefficients in Quantum Field Theory and Statistical Physics

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ABSTRACT

We present a method for estimating perturbative coefficients in quantum field theory and Statistical Physics. We are able to obtain reliable error-bars for each estimate. The results, in all cases, are excellent.

It has long been a hope in perturbative quantum field theory (PQFT), first expressed by Richard Feynman, to be able to estimate, in a given order, the result for the coefficient, without the brute force evaluation of all the Feynman diagrams contributing in this order. As one goes to higher and higher order the number of diagrams, and the complexity of each, increases very rapidly. Feynman suggested that even a way of determining the sign of the contribution would be useful.

The Standard Model (SM) of particle physics seems to work extremely well. This includes Quantum Chromodynamics (QCD), the Electroweak Theory as manifested in the Weinberg-Glashow-Salam Model and Quantum Electrodynamics (QED). In each case, however, we must use perturbation theory and compute large numbers of Feynman diagrams. In most of these calculations, however, we have no idea of the size or sign of the result until the computation is completed.

Recently we proposed [1, 2, 3, 4, 5] a method to estimate coefficients in a given order of PQFT, without actually evaluating all of the Feynman diagrams in this order. In this paper, we would like to present a method for obtaining reliable error-bars for each estimate. We believe this makes our estimation method much more important and much more useful!

Our method makes use of Padé Approximants (PA) and gives us a Padé Approximant Prediction (PAP). There are many good references for P.A. See, for example refs. [6-10]. We begin by defining the PA (Type I).

$$(N, M) = \frac{a_0 + a_1 x + \dots + a_N x^N}{1 + b_1 x + \dots + b_M x^M} \tag{1}$$

to the series

$$S = S_0 + S_1 x + \ldots . + S_{N+M} x^{N+M} \tag{2}$$

where we set

$$(N, M) = S + 0(x^{N+M+1}) \tag{3}$$

We have written a computer program which solves eq. (3) and then predicts the coefficient of the next term S_{N+M+1}. It works for arbitrary N and M. Furthermore we have derived algebraic formulas for the (N,1), (N,2), (N,3) and (N,4) PA's, where N is arbitrary.

To illustrate the method, consider the simple example

$$\frac{\ln(1+x)}{x} = 1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{c} \tag{4}$$

We write the (1,1) Padé as follows:

$$(1, 1) = \frac{a_0 + a_1 x}{1 + b_1 x} \tag{5}$$

It is easy to show that

$$a_o = 1, b_1 = 2/3, a_1 = 1/6 \text{ and } c = 9/2$$

We can see that the prediction for c is close to the correct value c = 4. For x = 1, we get (1,1) = 7/10, close to the correct result, ln 2 = .6931. This is much better than the partial sum

$$1 - \frac{1}{2} + \frac{1}{3} = \frac{5}{6} = .8333 \tag{6}$$

If we now take the series

$$\frac{\ln(1+x)}{x} = 1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} \qquad (7)$$

we have $S_0 = 1,\ S_1 = -1/2,\ S_2 = 1/3,\ S_3 = -1/4$ then

$$\begin{aligned}(1,2) &= \frac{a_0 + a_1 x}{1 + b_1 x + b_2 x^2} \\ &= \frac{1 + x/2}{1 + x + x^2/6}\end{aligned} \qquad (8)$$

and for x = 1 we obtain

$$(1, 2) = 9/13 = .6923$$

very close to the correct value, ln 2 = .6931. (The partial sum is .58.) The PAP is

$$S_4\,(1,2) = 7/36 = .1944\,, \qquad (9)$$

very close to the correct value of 1/5.

The error bars are obtained by considering the magnitude of the coefficients $|S_n|$ (We use both S_n and $|S_n|$ and take the larger error.) First we consider the reciprocals

$$r_n = 1/s_n \qquad (10)$$

find the PAP for r_{n+1}, and then take the reciprocal. This gives us an upper bound (UB). Then we consider the differences

$$t_n = r_{n+1} - r_n \tag{11}$$

and find the PAP for t_n. We then have

$$r_{n+1} = r_n + t_n \tag{12}$$

and then take the reciprocal

$$S_{n+1} = 1/r_{n+1} \tag{13}$$

This gives us a lower bound (LB). For the example above where $S_n = (n + 1)$ we find for the $r_n = 1/S_n$ method the(n-1, 2) PAP for r_{n+2} has

$$\%\ error = \frac{-4}{(n+1)^2(n+2)^2} \tag{14}$$

and for the t_n method for r_{n+2} the

$$\%\ error = \frac{+12}{n(n+1)^2(n+2)^2} \tag{15}$$

Thus the first method provides an UB for S_n and the second provides a LB. For the above example for $S_n = (n+1)$ the UB's are

$$S_4 = 5.144 \ \ and \ \ S_5 = 6.0606 \tag{16}$$

while the LB's are

$$S_4 = 4.69 \quad and \quad S_5 = 5.9418 \tag{17}$$

We take as our error here Δ where Δ is the magnitude of the difference between eqs (16) and (17). So our estimates for S_4 and S_5 are

$$\begin{aligned} S_4 &= 5.00 \pm .45 \\ S_5 &= 6.00 \pm 0.12 \end{aligned} \tag{18}$$

The estimates are exact in this case. We now generalize this procedure and take Δ as our error-bars.

We now apply this method to several examples from QED, QCD, Statistical Physics and Mathematics. For odd N + M we use the (N, N+1) and (N+1, N) PAP's calculating an estimate and an error bar for each. For even N + M we use (N, N), (N-1, N+1) and (N+1, N-1). We then combine the estimates for a given coefficient statistically.

In Table I we present the results for $a_\mu - a_e$ where $a = \left(\frac{g-2}{2}\right)$ and a_e and a_μ are the anomalous magnetic moments of the muon and electron respectively. Our result for tenth-order is consistent with the known result and we give our prediction for twelfth-order.

$$a_\mu^{(12)} - a_e^{(12)} = 2499 \pm 482 \tag{19}$$

In Table II we present the estimates for a_e in eighth-order and tenth-order.[12] The result in eighth-order

$$a_e^{(8)} = -1.55\,(46) \tag{20a}$$

is excellent and our estimate for tenth-order is

$$a_e^{(10)} = 1.75 \pm .56 \quad \textbf{(20b)}$$

In Table III we present the results for the τ lepton,[13] $a_\tau - a_e$. The results for tenth-order and twelfth-order are excellent and our estimate for fourteenth-order is

$$a_\tau^{(14)} - a_e^{(14)} = 27,427 \pm 3615 \quad \textbf{(21)}$$

The conservative approach would be to double all the error-bars, using 2Δ instead of Δ for the error. However, these error-bars are conservative and one can safely take Δ/2 as the error-bar in most cases. These errors should be considered as one standard deviation, σ.

In Table IV we present the results for the 5-loop β function in $_g\phi^4$ theory.[14] The results for the 4-loop and 5-loop coefficients are very good and the estimate for the 6-loop (unknown) coefficient is

$$\beta^{(6)} = -15{,}934 \pm 4588 \quad \textbf{(22)}$$

In Table V we present the results for the cumulative partitions of n into 4 non-zero integers while Tables VII and VIII are for 3 and 2 integers respectively. The results can be seen to be very good. Tables VI, IX, X and XI are results from Statistical Physics.[15, 16, 17] All of these results are very good.

In Table XII we present the results for the number of partitions of n into non-zero positive integers. The results can be seen to be very good. Table XIII given the PAP estimates for the R ratio in the $\overline{MS}$ Scheme, in Perturbative Quantum Chromodynamics (PQCD). The 4-loop estimate is R(4) = -10.20 ± 1.53, in agreement with the known result -12.805. Our estimate for the 5-loop result is R(5) = -87.5 ± 10.8. The results for the MS Scheme are given in Table XIV. Here the estimate for the 4-loop result is extremely accurate and the error estimate is overly conservative. The 5-loop estimate is 69.7 ± 48.9. Here too we expect that the error bound is overly pessimistic.

The corresponding results for the R_τ ratio in PQCD are given in Table XV and XVI. The $\overline{MS}$ results in Table XV and the MS results in Table XVI for the 4-loop coefficient are excellent, but here again our error bound is very conservative. The estimates for the 5-loop coefficients in the $\overline{MS}$ and MS Schemes are $R_\tau^{(5)}$ = 109.2 ±12.9 and 1026.8 ± 502.0 respectively.

In conclusion we have presented a way of estimating perturbative coefficients with reliable error-bars. We believe that this method will prove to be very useful in a wide variety of areas, especially in Quantum Electrodynamics (QED) and Quantum Chromodynamics (QCD) where calculations of the next-order terms are very difficult.

After this work was completed, we received a very interesting paper by Kataev and Starshenko, in which they estimate the 5-loop coefficients for R and R_τ by a completely independent method. These results in the $\overline{MS}$ scheme $R^{(5)}$ = -96.8 and $R_\tau^{(5)}$ = 105.5 are amazingly close to our results $R^{(5)}$ = -87.5 ± 10.8 and $R_\tau^{(5)}$ = 109.2 ± 12.9 respectively!

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Table Captions

I. PAP Estimates for the difference $a_\mu - a_e$ where a_μ and a_e are the anomalous magnetic moments of the muon and electron respectively. $a = (g - 2)/2$

II. PAP Estimates for a_e

III. PAP Estimates for $a_\tau - a_e$ where a_τ is the anomalous magnetic moment of the τ lepton.

IV. PAP Estimates for the β-Function in $g\phi^4$ Theory.

V. PAP Estimates for Partitions into 4 Integers.

VI. PAP Estimates for the Spontaneous Magnetic Coefficients in the Honey-Combed Lattice.

VII. Same as Table V for Partitions into 3 Integers.

VIII. Same as Table V for Partitions into 2 Integers.

IX. Same as Table VI for the Square Lattice.

X. Same as Table VI for the Diamond Lattice.

XI. PAP Estimates for the PAD5 Spontaneous Magnetization Coefficients for The Simple Cubic Lattice in the Ising Model.

XII. PAP Estimates for the number of Partitions of n into non-zero, positive integers.

XIII. PAP Estimates for the R ratio in the $\overline{MS}$ Scheme, in perturbative QCD (PQCD) for $N_f = 5$, where N_f is the number of fermion flavors (quarks).

XIV. PAP Estimates for the R ratio in the MS Scheme, in PQCD, for $N_f = 5$.

XV. PAP Estimates for the R_τ ratio in the $\overline{MS}$ Scheme, in PQCD, for $N_f = 3$.

XVI. PAP Estimates for the R_τ ratio in the MS Scheme, in PQCD, for $N_f = 3$.

TABLE I ESTIMATE	$a_\mu - a_e$ ERROR	ERROR 2/3 EXACT	| ESTIMATE − EXACT |
705	275	570 ± 140	135
2499	482	———	———

TABLE II	a_e	ERROR 24	
-1.55	.46	-1.434 ± .138	0.116
1.75	.56	———	———

TABLE III	$a_\tau - a_e$	ERROR 4/5	
1997	795	1779	218
9697	1601	8125	1572
27,427	3615	———	———

TABLE IV	$g\phi^4$ β Fcn.	ERROR 10/11	
-94	42	-135.8	42
1146	389	1424.3	278
-15,575	3660	———	———

TABLE V ESTIMATE	PARTITIONS (4) ERROR	ERROR 18/19 EXACT	| ESTIMATE − EXACT |
45.0	11.3	35	10
73.3	8.9	70	3.3
125.9	5.6	126	0.1
209.0	3.4	210	1.0
329.7	1.7	330	0.3
495.2	0.9	495	0.2
715.03	.78	——	——

TABLE VI ESTIMATE	PAD 4 ERROR	ERROR 41 EXACT	| ESTIMATE – EXACT |
246.2	17	268	21.8
848.3	150	944	95.7
3353	265	3476	123
13,221	212	13,072	149
49,915	347	49,672	243
189,467	6406	——	——

TABLE VII	PARTS (3)	ERROR 14/15	PARTS
25	5.4	20	5
36.6	3.5	35	1.6
56.0	1.8	56	0
83.7	.94	84	0.3
119.9	.41	120	0.1
165.0135	.185	165	.0135
220.0037	.072	220	.0037
286.	.0306	286	0
364.	.0109	364	0
455	.00445	——	——

TABLE VIII ESTIMATE	PARTS(2) ERROR	ERROR 16/17 EXACT	| ESTIMATE – EXACT |
15.6	1.1	15	0.6
21.1	.43	21	0.1
27.95	.183	28	0.05
35.989	.067	36	.01096
45	.0258	45	0
55	.0088	55	0
66	.00329	——	——

TABLE IX	PAD[3]	ERROR 31	
679.5	105	714	34.5
3449	325	3472	23
17,256	612	17,318	60
87,903	123	88,048	150
454,080	350	454,380	300
2,373,100	1800	2,373,000	100
12,515,000	800	12,516,000	1000
66,549,000	2200	——	——

TABLE X ESTIMATE	PAD 1 ERROR	ERROR 21 EXACT	| ESTIMATE − EXACT |
522.7	40	534	11.3
1709	39	1732	23
5710	36	5706	4
19,028	54	19,038	10
64,157	101	64,176	19
218,200	63	218,190	10
747,052	51	747,180	128
2,574,496	100	——	——

TABLE XI

PAD 5 ESTIMATE	ERROR	EXACT	| ESTIMATE − EXACT |
-2127	657	-2148	21
7528	817	7716	188
-22,882	181	-23,262	380
80,684	1078	——	——

TABLE XII

PARTS ($R_0 = 0$)			
4	2	3	1
4.4	0.8	5	0.6
8.5	1.6	7.0	1.5
12.3	2.1	11	1.3
15.4	1.3	15	0.4
40.2	5.4	——	——

TABLE XIII

R(t = 0) ESTIMATE	$\overline{MS}$ ERROR	EXACT	| ESTIMATE − EXACT |
-10.20	1.53	-12.805	2.61
-87.5	10.8	——	——

TABLE XIV

R(t = 1.95)	MS		
14.5	6.5	16.5	2.0
69.7	24.5	——	——

TABLE XV

$R_\tau(t = 0)$	$\overline{MS}$		
27.06	6.77	26.37	0.69
109.2	12.9	——	——

TABLE XVI

$R_\tau(t = 1.95)$	MS		
92.11	23.1	99.25	7.13
1026.8	251.0	——	——