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R. Kumar

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KAPL ATOMIC POWER LABORATORY

SCHENECTADY, NEW YORK 12301

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An Algebraic Stress/Flux Model for Two-Phase Turbulent Flow

**Ranganathan Kumar
Lockheed Martin Corporation
Schenectady, NY 12301**

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Abstract

An algebraic stress model (ASM) for turbulent Reynolds stress and a flux model for turbulent heat flux are proposed for two-phase bubbly and slug flows. These mathematical models are derived from the two-phase transport equations for Reynolds stress and turbulent heat flux, and provide C_μ , a turbulent constant which defines the level of eddy viscosity, as a function of the interfacial terms. These models also include the effect of heat transfer. When the interfacial drag terms and the interfacial momentum transfer terms are absent, the model reduces to a single-phase model used in the literature.

Nomenclature

B_0	buoyancy parameter in the turbulence model
C	model constants
C_D	drag coefficient
C_μ	constant in the turbulence model
D_b	bubble diameter
D^{kl}	drag term coupling phases k and l
d^{kl}	fluctuation of drag coefficient coupling phases k and l
\mathcal{D}_{ij}	diffusion of $\overline{u_i u_j}$
$\mathcal{D}_{j\theta}$	diffusion of $\overline{u_j \theta}$
F_w	wall friction force
g	acceleration due to gravity
G	generation rate of turbulent energy due to buoyancy effects
G_{ij}	kinematic production rate of $\overline{u_i u_j}$ by buoyancy forces
$G_{j\theta}$	kinematic production rate of $\overline{u_j \theta}$ due to buoyancy effects
k	turbulent kinetic energy
m'''	interfield mass term
M_i	momentum exchange
p	pressure
p'	pressure fluctuation
P	generation rate of turbulence energy due to mean velocity gradients
P_{ij}	kinematic production rate of $\overline{u_i u_j}$ by mean velocity gradients
$P_{j\theta}$	kinematic production rate of $\overline{u_j \theta}$ by mean velocity gradients
R	time scale ratio of thermal and hydrodynamic fields
t	time
T	temperature
u	velocity fluctuation

v	velocity fluctuation in the transverse direction
U	mean velocity
U^r	relative velocity
x	flow direction
y	thickness direction

Greek Symbols

α	vapor fraction
β	coefficient of thermal expansion
Γ	volumetric phase change rate
ε	dissipation rate of turbulent kinetic energy
ε_{ij}	dissipation rate of $\overline{u_i u_j}$
$\varepsilon_{j\theta}$	dissipation rate of $\overline{u_j \theta}$
θ	temperature fluctuation
$\overline{\theta^2}$	variance of temperature fluctuations
λ	thermal conductivity
μ	viscosity
ω	model parameter
π_{ij}	pressure-strain correlation
$\pi_{ij,1}$	first part of π_{ij} associated with turbulence interactions
$\pi_{ij,2}$	second part of π_{ij} associated with mean strain
$\pi_{ij,3}$	third part of π_{ij} associated with buoyancy
$\pi_{j\theta}$	pressure-temperature gradient correlation
$\pi_{j\theta,1}$	first part of $\pi_{j\theta}$ associated with turbulence interactions
$\pi_{j\theta,2}$	second part of $\pi_{j\theta}$ associated with mean strain
$\pi_{j\theta,3}$	third part of $\pi_{j\theta}$ associated with buoyancy
ρ	density
ρ'	density fluctuation (included in buoyancy term only)

$\sigma_t, \sigma_\varepsilon$ turbulence parameters

τ shear stress

τ' shear stress fluctuation

τ_t Reynolds' stress

ν kinematic viscosity

ν_t turbulent viscosity

Subscripts

i, j, k directional indices

int interface

m repeated index

w wall

Superscripts

k phase

kl from phase k to l

lk from phase l to k

r relative

Introduction

For accurate predictions of two-phase flow behavior, microscopic structures of two-phase flow such as turbulence need to be modeled. There appears to be a consensus in the literature that the random fluctuations due to the flow around the bubbles can be called pseudo-turbulence and that there is a secondary contribution to kinetic energy fluctuations from the bubble wakes. This bubble-induced turbulence has a second time scale, called the residence time of the turbulence eddy in the vicinity of the bubble, given by the bubble diameter and relative velocity as $t_d \propto \frac{D_b}{U'}$. This time scale was then used to obtain the bubble contribution to the Reynolds stress and kinetic energy described by the irrotational motion, which were then superimposed with the shear-induced turbulent counterparts. Researchers have used the concept of superposition in air-water flows with low void fraction and with or without grid turbulence generation [Buyevich (1972), Sato, et al (1981), Theofanous and Sullivan (1982), Bertadano et al (1990), Lance and Bataille (1991) among others]. The bubble induced expression for v_t was given by

$$v_t = C_\mu \frac{k^2}{\varepsilon} + C_{\mu b} D_b |U'| \alpha \quad (1)$$

In eq. (1), the asymptotic bubble-induced turbulent kinetic energy of the liquid is obtained from an inviscid analysis of the relative motion between the dispersed phase and the continuous phase. The last term is the bubble induced viscosity model due to Sato et al (1981).

Even in flows with low void fraction where the bubble-to-bubble interactions may be considered negligible, superposition of stresses inherently assumes that the bubble frequency may be isolated from the frequency of turbulent eddies. In other words, it is assumed that the two time scales discussed above are widely separated. Such an assumption is not necessarily valid, especially for high void fractions. Secondly, turbulence suppression has been observed in the experimental results of Serizawa et al (1975), Ohba and Yuhara (1982), Lee, et al (1989) and Kataoka, et al (1993). Therefore, a simple sum of bubble-induced turbulence and single phase flow turbulence is not appropriate to describe two-phase turbulent flow.

The proposed model departs from the linear superposition approximation. The extended pro-

cedure involves developing a Reynolds stress transport equation by Reynolds averaging the two-fluid momentum equation. Since the equations are too complex to involve all interfacial terms, only the interfacial drag and interphase mass transfer terms are included. Since the drag term consists of $\frac{D_b}{U^r C_D}$, which is proportional to the bubble time constant, this time scale describing the bubble motion naturally occurs in the derivation.

Next, a simple model for temperature variance, $\overline{\theta^2}$, appearing in the buoyant production term in the turbulent heat flux equation is postulated. Starting from these equations, equivalent algebraic stress models for $\overline{u_i u_j^k}$ and $\overline{u_j \theta^k}$ are developed. Finally, these algebraic models are brought down to the form of an eddy viscosity/diffusivity relation for 2-D flows. An expression for C_μ will be derived for the 2-D, bubbly flow using the algebraic stress model.

It should be emphasized that the eddy viscosity concept is a deficiency in both the k-ε model and the mixing length models. The constant, C_μ , which defines the level of eddy viscosity has a value of 0.09 in equilibrium shear layers. Inaccurate prediction of differences in normal Reynolds stresses makes eddy viscosity models incapable of describing secondary flows in non-circular ducts. However, by obtaining an expression for C_μ including the effects of shear-induced production and dissipation, buoyancy and bubble motion, it is hoped that the turbulence in two-phase flow will be adequately described without having to solve the transport equations for Reynolds stress and heat flux.

Model Development

In order to develop a two-phase transport equation for Reynolds stress, first, the ensemble-averaged continuity and momentum equations are considered.

$$\frac{\partial}{\partial t} \alpha^k \rho^k + \frac{\partial}{\partial x_j} \alpha^k \rho^k \overline{U_j^k} = \Gamma^{lk} - \Gamma^{kl} \quad (2)$$

$$\frac{\partial}{\partial t} (\alpha^k \rho^k \overline{U_i^k}) + \frac{\partial}{\partial x_j} (\alpha^k \rho^k \overline{U_i^k} \overline{U_j^k}) = -\alpha^k \frac{\partial \overline{p}}{\partial x_i} + \frac{\partial}{\partial x_j} \alpha^k (\overline{\tau_{ji}^k} + \overline{\tau_{ji}^k}) + \quad (3)$$

$$(\overline{p_{int}^k} - \overline{p}^k) \frac{\partial \alpha^k}{\partial x_j} + F_w^k + \alpha^k \rho^k g_i + M_i^k + m_i^{m^k} \overline{U_i^k} + \overline{U_i^l} \Gamma^{lk} - \overline{U_i^k} \Gamma^{kl}$$

The superscript, l , describes the second phase in the above equations. The turbulent shear stress $\overline{\tau_{ji}^k} = -\rho^k \overline{u_i u_j^k}$ is the term that needs to be modeled. Momentum due to interfield mass transfer will not be included in the two-field model. The interfacial momentum exchange term, M_i^k , is given as the sum of drag and non-drag components. The non-drag components along with the $(\overline{p_{int}^k} - \overline{p^k}) \frac{\partial \alpha^k}{\partial x_j}$ term, and wall friction term are ignored in the development of the two-phase Reynolds stress equation, which follows below.

In contrast to Kataoka and Serizawa's(1989) local instantaneous formulation of two-phase flow equations, the formulation used here starts from the already averaged equations. Since this formulation accounts for averages over a cell which contains many bubbles or particles, it can only be accurate for bubble sizes smaller than this cell. This means that the formulation is good only when the turbulent length scales are larger than the cell. Equation (3) can now be considered to be an instantaneous momentum equation, and averaged terms such as $\overline{U_i^k}$ and $\overline{D^{kl}}$ are cast as instantaneous quantities \tilde{U}_i^k and \tilde{D}^{kl} . The drag term, $M_i^k = \tilde{D}^{kl} \tilde{U}_i^r$, and momentum from the interphase mass transfer, $\Gamma^{lk} \tilde{U}_i^l - \Gamma^{kl} \tilde{U}_i^k$ are the only two-phase terms considered for the derivation of the Reynolds transport equation. Notice that \tilde{D}^{kl} , the instantaneous interface friction coefficient, is given by

$$\tilde{D}^{kl} = \frac{3}{4} \tilde{\alpha}^k \frac{C_D}{D_b} \rho^k |\tilde{U}_i^r| \quad (4)$$

and that the average values of interphase mass have been used. As is standard practice in single phase turbulence modeling, density fluctuations are neglected in the development of the Reynolds stress equations, except for contribution to the buoyancy term.

The instantaneous relative velocity and the instantaneous drag coefficient are decomposed into their respective mean value and fluctuation due to 'true' and pseudo turbulence as given below.

$$\overline{\tilde{D}^{kl} \tilde{U}_i^r} = \overline{(D^{kl} + d^{kl}) (U_i^r + u_i^r)} = D^{kl} U_i^r + \overline{d^{kl} u_i^r} \quad (5)$$

The first term is the well-known drag term in the momentum equation and the second term may be called a turbulent dispersion force. The dispersion term has been modeled as proportional to $\nabla(\ln\alpha)$ by de Bertadano et al (1990), and may be retained in the momentum equation. The remaining bubbly forces are neglected in the Reynolds stress equation, however, it is important to retain these forces in the momentum equation, to predict turbulence suppression, as will be explained later.

Reynolds Stress Transport Equation

Consistent with the discussion in the previous section, equations (2) and (3) are decomposed and averaged. The continuity equation given in (2) is now considered to be an instantaneous equation and can be written as

$$\frac{\partial}{\partial t}\alpha^k\rho^k + \frac{\partial}{\partial x_j}\alpha^k\rho^k(U_j + u_j) = \Gamma^{lk} - \Gamma^{kl} \quad (6)$$

The fluctuation continuity equation can be written as:

$$\frac{\partial}{\partial x_j}\alpha^k\rho^k u_j = 0 \quad (7)$$

The transport equation for fluctuating momentum component, u_i , is obtained by subtracting the instantaneous equation from its averaged equation. Note that this averaged equation is different from the original ensemble-averaged equation given in (3). Using equations (4) and (5), the fluctuation i-momentum equation is written as follows.

Fluctuation i-momentum equation:

$$\begin{aligned} \frac{\partial}{\partial t}(\alpha^k\rho^k u_i^k) + \frac{\partial}{\partial x_j}(\alpha^k\rho^k [u_i^k U_j^k + u_j^k U_i^k + u_i^k u_j^k - \overline{u_i^k u_j^k}]) = \alpha^k \frac{\partial p'}{\partial x_i} + \frac{\partial}{\partial x_j} \alpha^k \tau_{ji}^k + \\ \alpha^k \rho^k g_i + D^{kl} u_i^r + d^{kl} U_i^r + d^{kl} u_i^r - \overline{d^{kl} u_i^r} + \Gamma^{lk} u_i^l - \Gamma^{kl} u_i^k \end{aligned} \quad (8)$$

Multiplying by u_k^k ,

$$u_k^k \frac{\partial}{\partial t} (\alpha^k \rho^k u_i^k) + u_k^k \frac{\partial}{\partial x_j} \alpha^k \rho^k [u_i^k U_j^k + u_j^k U_i^k + u_i^k u_j^k - \overline{u_i u_j^k}] = u_k^k \alpha^k \frac{\partial p'}{\partial x_i} + u_k^k \frac{\partial \alpha^k \tau_{ji}^k}{\partial x_j} + \quad (9)$$

$$u_k^k \alpha^k \rho^k g_i + u_k^k D^{kl} u_i^r + u_k^k d^{kl} U_i^r + u_k^k d^{kl} u_i^r - u_k^k \overline{d^{kl} u_i^r} + u_k^k \Gamma^{lk} u_i^l - u_k^k \Gamma^{kl} u_i^k$$

The procedure for obtaining the final equation for $\overline{u_i u_j^k}$ involves the following steps. First, an equation similar to (9) is written by interchanging the subscripts i and k. The resulting equation is added to (9) and averaged. Now, the chain rule is applied and the continuity equations given in (6) and (7) are invoked, and an equation for $\overline{u_i u_j^k}$ is obtained. Interchanging the subscripts, j and k:

$$\frac{\partial}{\partial t} \alpha^k \rho^k \overline{u_i u_j^k} + \frac{\partial}{\partial x_k} \{ \alpha^k \rho^k U_k \overline{u_i u_j^k} \} = \mathcal{D}_{ij} + P_{ij} + G_{ij} + \alpha^k \rho^k (\pi_{ij,1} + \pi_{ij,2} + \pi_{ij,3})$$

$$- \alpha^k \rho^k \epsilon_{ij}$$

$$+ D^{kl} \{ \overline{u_j^k u_i^r} + \overline{u_i^k u_j^r} \} \quad I$$

$$+ U_i^r \overline{d^{kl} u_j^k} + U_j^r \overline{d^{kl} u_i^k} \quad II$$

$$+ \Gamma^{lk} \{ \overline{u_i^l u_j^k} + \overline{u_i^k u_j^l} \} - (\Gamma^{kl} + \Gamma^{lk}) \overline{u_i^k u_j^k} \quad III$$

Eq.(10) is the Reynolds stress transport equation for phase-k. The terms, I, II, and III indicate the new two-phase terms. The viscous dissipation of $\overline{u_i u_j^k}$ is given by

$$- \alpha^k \rho^k \epsilon_{ij} = -2\mu^k \alpha^k \frac{\partial u_i^k}{\partial x_m} \frac{\partial u_j^k}{\partial x_m} \quad (11)$$

It should be noted that the exact derivation of the $\overline{u_i u_j}^k$ equations includes an extra viscous term involving the gradient of vapor fraction and this term is ignored in the current analysis. At high Reynolds numbers, the turbulence is locally isotropic so that the same amount of energy is dissipated in each energy component. When this local isotropy prevails, the dissipation correlation given in (11) is zero when $i \neq j$. Therefore, the dissipation term in (11) can be labelled as

$$-\alpha^k \rho^k \epsilon_{ij} = -\frac{2}{3} \alpha^k \rho^k \epsilon \delta_{ij} \quad (12)$$

The remaining terms appearing on the right hand side of equation (10) are those terms modeled for single phase flows by Launder, Reese and Rodi (1975). Straight forward extension to multiple fields yields:

$$\mathcal{D}_{ij} = C_s \frac{\partial}{\partial x_k} \alpha^k \rho^k \left\{ \frac{k}{\epsilon} \overline{u_k u_l}^k \frac{\partial}{\partial x_l} \overline{u_i u_j}^k \right\} \quad (13)$$

$$P_{ij} = \rho^k \alpha^k \left\{ -\overline{u_i u_k}^k \frac{\partial \overline{U_j}^k}{\partial x_k} - \overline{u_j u_k}^k \frac{\partial \overline{U_i}^k}{\partial x_k} \right\} \quad (14)$$

$$\begin{aligned} G_{ij} &= (\overline{\rho' u_i}^k g_j + \overline{\rho' u_j}^k g_i) \alpha^k \\ &= -\beta^k \rho^k \alpha^k (\overline{u_i \theta}^k g_j + \overline{u_j \theta}^k g_i) \end{aligned} \quad (15)$$

where β^k is the volume coefficient of expansion for phase k ,

$$\pi_{ij,1} = -C_1 \frac{\epsilon}{k} (\overline{u_i u_j}^k - \frac{2}{3} \delta_{ij} k) \quad (16)$$

$$\pi_{ij,2} = -C_2 (P_{ij} - \frac{2}{3} \delta_{ij} P) \quad (17)$$

$$\pi_{ij,3} = -C_3 \left(G_{ij} - \frac{2}{3} \delta_{ij} G \right) \quad (18)$$

The diffusion term, \mathcal{D}_{ij} in eq. (13) has been approximated by the generalized diffusion model of Daly and Harlow (1970). Eq. (14) represents the production rate of $\overline{u_i u_j}$ by mean velocity gradients. The direct influence of buoyancy in turbulence is given by eq. (15) where the cross correlation of density and velocity fluctuations is written in terms of turbulent heat flux, $\overline{u_j \theta^k}$, using Boussinesq approximation. In the literature, eddy diffusivities have been found to be sensitive to turbulent heat fluxes, and the contribution to turbulence from eq. (15) is expected to be significant. Equations (16), (17), and (18) represent the models for pressure-strain correlation: return-to-isotropy model of Rotta (1951), mean flow model of Launder, Reese and Rodi (1975), and buoyancy model of Launder (1975) and Ljuboja and Rodi (1980) respectively.

Considering the term I in eq.(10),

$$I = \overline{D^{kl} u_i u_j^l \left(1 - \frac{u_j^r}{u_j^l} \right)} - \overline{D^{kl} u_i u_j^k \left(1 - \frac{u_j^r}{u_j^k} \right)} \quad (19)$$

which can then be written as

$$I = D^{kl} (C_{k1} \overline{u_i u_j^l} - C_{k2} \overline{u_i u_j^k}) \quad (20)$$

This term can then be simplified to

$$I = -2D^{kl} C_k \overline{u_i u_j^k} \quad (21)$$

The second term II involves turbulent dispersion, and although it is an important term in the momentum equation, its importance in the Reynolds stress and kinetic energy equations is unknown and as a first approximation, this term will be dropped for the current analysis.

Considering the interfacial term III,

$$III = \Gamma^{lk} \left[\overline{u_i^l u_j^l \left(1 - \frac{u_j^r}{u_j^l} \right)} + \overline{u_i^k u_j^k \left(1 + \frac{u_j^r}{u_j^k} \right)} \right] - \overline{u_i u_j^k} (\Gamma^{kl} + \Gamma^{lk}) \quad (22)$$

which can then be written as

$$III = \Gamma^{lk} [C_{k3} \overline{u_i u_j^l} + C_{k4} \overline{u_i u_j^k}] - \overline{u_i u_j^k} (\Gamma^{kl} + \Gamma^{lk}) \quad (23)$$

and can further be approximated as

$$III = \Gamma^{lk} C_{\Gamma} \overline{u_i u_j^k} - \Gamma^{kl} \overline{u_i u_j^k} \quad (24)$$

where C_{Γ} is a constant that could vary between 0 and 0.2.

The current approach used in the derivation of the Reynolds stress equation is somewhat similar to the procedure adopted by Besnard and Harlow (1988). They chose momentum and volume fractions as the primary variables, and decomposed them into mean and fluctuating parts. The drag coefficient which they called the coupling coefficient was also decomposed; however, in their analysis, they neglected the fluctuating part of the coupling coefficient. In their derivation, the effects of coupling between the two fields were taken into account in a term that was very similar to equation (21).

Exact comparison of the final equations of Besnard and Harlow (1988) and the current work is not possible because of the different starting points. The common feature is that both start with ensemble averaged momentum equations, and have therefore not accounted for a part of the turbulent kinetic energy produced at scales which are of the order of the bubble sizes. Since these formulations make an inherent assumption that there are no interactions between the bubbles, the volume fraction must be small. Besnard, Kataoka and Serizawa (1991) developed Reynolds stress equations in terms of interfacial area concentration and showed that small scale interfacial structure has the same importance as the large scale interfacial structure.

Different interpretations for large and small scale interfaces and their effect on turbulence were given by the same group of researchers [Kataoka, Besnard and Serizawa (1991) and Besnard, Kataoka and Serizawa (1991)]. Although they assumed a sparsely dispersed field, they allowed for interaction between the particles or bubbles through coupling to the surrounding fluid.

Thus, the additional turbulent kinetic energy produced at scales of the order of bubble size is taken into account through an inter-penetrational energy source. This term is the bubble-induced turbulence generation term that the current formulation could not explicitly account for, and hence will be introduced as a source term in the two-phase Reynolds stress transport equation. Accordingly, this bubble-induced generation term is modeled here as:

$$IV = -2D^{kl}C_{ks}U_m^rU_m^r \quad (25)$$

Here, summation is performed on repeated m-index. In eq. (25), D^{kl} is positive and the entire right side of the equation becomes positive representing generation of turbulence.

Turbulence suppression is a phenomenon that was seen in bubbly flow data of Serizawa et al (1975), Obha and Yuhara (1982), Lee et al (1986), and Kataoka et al (1993) in air-water flows in circular tubes. The local kinetic energy in two-phase flow becomes smaller than that in single phase for the same liquid flux. It should be noted that suppression does not mean that the total turbulent kinetic energy in two phase is lower than that in single-phase flow. Turbulence enhancement is usually seen close to the wall. Kataoka et al (1992) modeled turbulence suppression by considering small scale interfaces. But, turbulence suppression is expected to naturally occur when the two-phase equations are modeled correctly with all the bubbly forces in the momentum equation. This approach will be used in the current work and turbulence suppression will not be modeled separately. Equations (21), (24) and (25) can be put together as the two-phase terms in the final form of the phasic Reynolds transport equation, as given below.

$$\begin{aligned} \frac{\partial}{\partial t} \alpha^k \rho^k \overline{u_i u_j^k} + \frac{\partial}{\partial x_k} \{ \alpha^k \rho^k U_k \overline{u_i u_j^k} \} = & \mathcal{D}_{ij} + P_{ij} + G_{ij} + \alpha^k \rho^k (\pi_{ij,1} + \pi_{ij,2} + \pi_{ij,3}) \\ & - \frac{2}{3} \alpha^k \rho^k \delta_{ij} \varepsilon \\ & - 2D^{kl}C_k \overline{u_i u_j^k} + 2D^{kl}C_{ks}U_k^rU_k^r + \Gamma^{lk}C_{\Gamma} \overline{u_i u_j^k} - \Gamma^{kl} \overline{u_i u_j^k} \end{aligned} \quad (26)$$

In summary, the phasic Reynolds transport equation was developed starting from the ensemble-averaged momentum equation including only the drag and the interphase mass transfer terms. This transport equation given in equation (26) involves six components of Reynolds stress, $\overline{u_i u_j}$, and three components of scalar flux, $\overline{u_j \theta}$, and a scalar function, $\overline{\theta^2}$. The last two components appear in transport equations and algebraic equations and will be discussed later. Thus, there are ten dependent variables that need to be solved for each phase and this is no trivial matter even with the most modern numerical schemes and the fastest computers. Therefore, an algebraic stress model proposed by Rodi [1976] will be used in this work in conjunction with transport equations for two-phase kinetic energy and dissipation. First, the transport equations are derived in the following section.

Transport Equations for Turbulent Kinetic Energy and Dissipation

The sum of the diagonal components of the Reynolds stress gives twice the turbulent kinetic energy. Therefore, taking half of the trace of the Reynolds stress equation given in (26) yields the following identities.

$$\frac{1}{2} \left[\frac{\partial}{\partial t} \alpha^k \rho^k \overline{u_i u_j}^k + \frac{\partial}{\partial x_k} (\alpha^k \rho^k U_k \overline{u_i u_j}^k) \right] \longrightarrow \frac{\partial}{\partial t} \alpha^k \rho^k k + \frac{\partial}{\partial x_k} (\alpha^k \rho^k U_k k) \quad (27)$$

$$P_{ij} \longrightarrow \frac{1}{2} P_{ii} = P = -\alpha^k \rho^k \overline{u_i u_k}^k \frac{\partial U_i}{\partial x_k} \quad (28)$$

$$\begin{aligned} G_{ij} \longrightarrow \frac{1}{2} G_{ii} = G &= \overline{\rho' u_i}^k g_i \\ &= -\beta^k \alpha^k \overline{u_i \theta}^k g_i \end{aligned} \quad (29)$$

where β^k is volumetric coefficient of expansion.

$$-\frac{2}{3} \alpha^k \rho^k \delta_{ij} \epsilon \longrightarrow -\alpha^k \rho^k \epsilon \quad (30)$$

$$\mathcal{D}_{ij} \longrightarrow \mathcal{D}_k \quad (31)$$

All the π_{ij} terms vanish, and using equations (27) through (31), and following Launder and Spalding's (1972, 1974) work, $k - \epsilon$ equations may be written as follows.

k - equation:

$$\begin{aligned} \frac{\partial}{\partial t} \alpha^k \rho^k k + \frac{\partial}{\partial x_k} \{ \alpha^k \rho^k U_k k \} &= \frac{\partial}{\partial x_j} \alpha^k \left(\mu^k + \frac{\mu_t}{\sigma_k} \right) \frac{\partial k}{\partial x_j} + (P + G) - \alpha^k \rho^k \epsilon \\ &- D^{kl} C_k k + D^{kl} C_{ks} U_k^r U_k^r + \frac{1}{2} (\Gamma^{lk} C_\Gamma - \Gamma^{kl}) k \end{aligned} \quad (32)$$

ϵ - equation:

$$\begin{aligned} \frac{\partial}{\partial t} \alpha^k \rho^k \epsilon + \frac{\partial}{\partial x_k} \{ \alpha^k \rho^k U_k \epsilon \} &= \frac{\partial}{\partial x_j} \alpha^k \left(\mu + \frac{\mu_t}{\sigma_\epsilon} \right) \frac{\partial \epsilon}{\partial x_j} + \frac{\epsilon}{k} (C_{\epsilon 1} P + C_{\epsilon 3} G) \\ &- C_{\epsilon 2} \alpha^k \rho^k \frac{\epsilon^2}{k} - D^{kl} C_\epsilon \epsilon + D^{kl} C_4 C_{ks} \frac{\epsilon}{k} U_k^r U_k^r + (\Gamma^{lk} C_\Gamma - \Gamma^{kl}) C_5 \epsilon \end{aligned} \quad (33)$$

D^{kl} in the above equations is given by

$$D^{kl} = \frac{3}{4} \alpha^k \frac{C_D}{D_b} \rho^k |U_k^r| \quad (34)$$

Transport Equations for Turbulent Heat Flux

As explained in the previous section, turbulent heat flux, $\overline{u_j \theta}$, appears in the Reynolds stress and kinetic energy equations and needs to be modeled. To obtain an algebraic expression for $\overline{u_j \theta}$, first a transport equation for $\overline{u_j \theta}$ is written. As done previously to derive an equation for $\overline{u_i u_j}$, the

fluctuation momentum equation is written and correlated with temperature fluctuation, θ . Additionally, the fluctuation energy equation is correlated with the velocity fluctuation, u_j . These two equations are then summed and simplified yielding:

Turbulent Heat Flux Equation:

$$\frac{D}{Dt} \overline{u_j \theta^k} = \overline{-u_j u_k} \frac{\partial T^k}{\partial x_k} - \overline{u_k \theta^k} \frac{\partial U_j^k}{\partial x_k}$$

$P_{j\theta}$, Production

$$-\overline{\beta \theta^2} g_j + \{ -C_{1\theta} \overline{\frac{\epsilon}{k}} u_j \theta^k \}$$

$G_{j\theta}$: Buoyant production turbulent interaction part of pressure-temperature gradient correlation, $\pi_{j\theta, 1}$

$$\{ -C_{2\theta} P_{j\theta} \} + \{ -C_{3\theta} G_{j\theta} \}$$

mean strain part, buoyancy part,
 $\pi_{j\theta, 2}$ $\pi_{j\theta, 3}$

$$+ \frac{1}{\rho^k} \frac{\partial}{\partial x_k} \left[-\rho^k \overline{u_k u_j \theta^k} + \frac{\lambda^k}{C_p^k} \overline{u_j \frac{\partial \theta^k}{\partial x_k}} + \mu^k \theta^k \left(\frac{\partial u_j}{\partial x} + \frac{\partial u_k}{\partial x_j} \right) - \overline{p' \theta^k} \delta_{jk} \right]$$

diffusion terms, $\mathcal{D}_{j\theta}$

$$+ \nu^k \frac{\partial \theta}{\partial x_k} \left(\frac{\partial u_j^k}{\partial x_k} + \frac{\partial u_k^k}{\partial x_j} \right) + \frac{\lambda^k}{\rho^k C_p^k} \frac{\partial \theta}{\partial x_k} \frac{\partial u_j^k}{\partial x_k} \quad (35)$$

dissipation terms, $\epsilon_{j\theta}$

where λ^k is thermal conductivity.

The physical interpretation of the individual terms is provided along with the equation. The turbulent heat flux is governed by similar processes as in Reynolds stress. It is assumed that the velocity fluctuation only due to the presence of the bubble and the temperature fluctuation are not

strongly correlated to contribute to the transport of turbulent heat flux. However, since the bubble-induced turbulence is already accounted for in the $\overline{u_j u_k}$ term, the bubble contribution is indirectly incorporated in the turbulent heat flux equation. As in the case of the π_{ij} term in the Reynolds stress equation, the $\pi_{j\theta}$ term here can be divided into a turbulence part, $\pi_{j\theta,1}$, a mean-strain part, $\pi_{j\theta,2}$, and a buoyancy part, $\pi_{j\theta,3}$. The most widely used model for $\pi_{j\theta,1}$ is that of Monin (1965) which is a counterpart of Rotta's return-to-isotropy approximation in $\pi_{ij,1}$. The other two terms, $\pi_{j\theta,2}$ and $\pi_{j\theta,3}$ were proposed by Launder (1975) and used by Sha and Launder (1979) and Ljuboja and Rodi (1981). Lumley and Khajeh-Nouri (1974) and Lumley et al (1978) assumed that $\varepsilon_{j\theta}$ is negligible when local isotropy also prevails in the thermal field, i.e.,

$$\overline{\left(\frac{\partial\theta}{\partial x}\right)^2} = \overline{\left(\frac{\partial\theta}{\partial y}\right)^2} = \overline{\left(\frac{\partial\theta}{\partial z}\right)^2} \quad (36)$$

This is the assumption made in the current work, however, as de Lemos and Sesonske (1985) have shown, the anisotropy can be modeled easily through the $\pi_{ij,1}$ term later, if necessary.

The next step is to model the $\overline{\theta^2}$ term in the $\overline{u_j \theta^k}$ equation. The dissipation of the temperature variance, $\overline{\theta^2}$, is related to the dissipation of turbulent kinetic energy, ε , by assuming a constant ratio, R , of the time scales for thermal and hydrodynamic fields as suggested by Launder (1975) and used by Ljuboja and Rodi (1981).

$$R = \frac{\overline{\left(\frac{\theta^2}{2\varepsilon_\theta}\right)}}{k/\varepsilon} \quad (37)$$

Experiments indicate that R varies from 0.5 to 1. A value of 0.8 was used by Gibson and Launder (1978). Lumley and Khajeh-Nouri (1974) give such a variation in R as the reason to use a transport equation for ε_θ . Another transport equation will introduce additional complexity and will not be considered here. However, in local equilibrium, the transport equation for $\overline{\theta^2}$ reduces to

$$-\overline{u_j \theta}^k \frac{\partial T^k}{\partial x_j} = \varepsilon_\theta = \lambda^k \left(\frac{\partial \theta}{\partial x_j} \right) \quad (38)$$

Using the above expressions given in equations (36) and (37), $\overline{\theta^2}$ may be written as

$$\overline{\theta^2} = -2R \frac{k}{\varepsilon} \overline{u_j \theta}^k \frac{\partial T}{\partial x_j} \quad (39)$$

Now, all the transport equations are appropriately derived and modeled as explained in this section and the next step is to obtain algebraic expressions for $\overline{u_i u_j}^k$ and $\overline{u_j \theta}^k$.

Algebraic Stress/Flux Model

The algebraic models arise when approximations for the net transport terms are introduced. The advantage of these algebraic models over transport equation models is computational simplicity. Gradients of velocity and temperature appear in the transport equations in the local acceleration, convection and diffusion terms. Hence, when these gradients can be eliminated by model approximations, the differential equations can be converted into algebraic expressions [(Rodi, 1976) and Naot and Rodi (1982)]. The basic assumption in the algebraic stress model is that the convective and diffusive transport of individual Reynolds stress components locally are proportional to the transport of kinetic energy.

Mathematically,

$$\begin{aligned} \frac{D}{Dt} \alpha^k \rho^k \overline{u_i u_j}^k - \mathcal{D}_{ij} = \alpha^k \rho^k \frac{\overline{u_i u_j}^k}{k} \left(\frac{Dk}{Dt} - \mathcal{D}_k \right) = \frac{\overline{u_i u_j}^k}{k} (P + G - \alpha^k \rho^k \varepsilon \\ - 2D^{kl} C_k k + 2D^{kl} C_{ks} U_m^r U_m^r + \Gamma^{lk} C_\Gamma k - \Gamma^{kl} k) \end{aligned} \quad (40)$$

Again, summation is performed on repeated k-index. For the thermal field, local equilibrium is assumed. In other words, the rate of production locally balances the rate of dissipation.

$$\frac{D}{Dt} \overline{u_j \theta} - \mathcal{D}_{j\theta} = 0 \quad (41)$$

Now apply the above models in the respective transport equations to get algebraic stress and flux expressions. Equation (40) reduces to

$$\begin{aligned} \frac{\overline{u_i u_j^k}}{k} - \frac{2}{3} \delta_{ij} = & \frac{1}{\left\{ \frac{K_0}{\epsilon} + C_1 + 2 \frac{k}{\epsilon} (D^{kl} C_k + \frac{1}{2} (\Gamma^{kl} - \Gamma^{lk} C_\Gamma)) \right\}} \times \\ & \left[\left\{ (1 - C_2) \left(\frac{P_{ij}}{\epsilon} - \frac{2}{3} \delta_{ij} \frac{P}{\epsilon} \right) + (1 - C_3) \left(\frac{G_{ij}}{\epsilon} - \frac{2}{3} \delta_{ij} \frac{G}{\epsilon} \right) \right\} + 2 \frac{D^{kl}}{\epsilon} (C_{ks} (U_m^r)^2) \right. \\ & \left. - \frac{2}{3} \delta_{ij} \left(\frac{k}{\epsilon} C_k D^{kl} + \frac{1}{2} (\Gamma^{kl} - \Gamma^{lk} C_\Gamma) \frac{k}{\epsilon} + C_{ks} \frac{D^{kl}}{\epsilon} (U_m^r)^2 \right) \right] \end{aligned} \quad (42)$$

where

$$\begin{aligned} K_0 &= P + G + B + C - \epsilon \\ B &= -C_k D^{kl} k + \frac{1}{2} k (C_\Gamma \Gamma^{lk} - \Gamma^{kl}) \\ C &= D^{kl} C_{ks} (U_m^r)^2 \end{aligned} \quad (43)$$

Using eq. (39), an implicit expression for $\overline{u_j \theta}$ may be obtained from eq. (35) as follows:

$$-\overline{u_j u_k} \frac{\partial T}{\partial x_k} + P_{j\theta} + G_{j\theta} + \pi_{j\theta,1} + \pi_{j\theta,2} + \pi_{j\theta,3} - \epsilon_{j\theta} = 0 \quad (44)$$

Equations (42), (43) and (44) in conjunction with the kinetic energy and dissipation equations given in (32) and (33) form the complete algebraic stress/flux model for a three-dimensional two-phase flow in a heated duct. These equations provide implicit expressions for Reynolds stress and turbulent heat flux which may cause convergence problems even in single-phase flows. To circumvent this situation, one idea may be to evaluate the stresses and flux at each point in the flow by iteratively solving equations (42) and (44). This, however, is uneconomical and such a solution procedure is dropped in favor of a simpler two-dimensional approach.

Two-Dimensional Model

If the primary flow direction is considered to be x, the thickness direction as y and the width direction as z, and the corresponding velocities in these directions as u, v and w, an easier and more useful expression may be obtained for \overline{uv} , $\overline{v^2}$, $\overline{u\theta}$ and $\overline{v\theta}$. Knowing $\overline{u\theta}$ and $\overline{v\theta}$, θ^2 can be obtained from eq. (39). Boussinesq's eddy viscosity concept assumes that the turbulent stresses are proportional to the mean velocity gradients. This can be expressed as

$$-\overline{u_i u_j} = \nu_t \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) - \frac{2}{3} k \delta_{ij} \quad (45)$$

where ν_t is the turbulent or eddy viscosity which is not a fluid property but depends on the state of turbulence. In two dimensions, \overline{uv} becomes

$$\overline{uv} = -\nu_t \frac{\partial U}{\partial y} \quad (46)$$

This relation is usually written for single-phase flow turbulence based on the consideration of mixing length. According to Kataoka and Serizawa (1989), in two-phase flows, there exist two mixing lengths, one related to the size of the eddy and the other related to the size of the interfacial configuration. In this work, both are considered directly or indirectly and have been incorporated in C_μ .

An equivalent expression can be written for the turbulent heat flux in direct analogy to turbulent momentum transport, representing the eddy diffusivity.

$$\overline{v\theta} = \frac{\nu_t \partial T}{\sigma_t \partial y} \quad (47)$$

$$\overline{u\theta} = \frac{k}{\epsilon C_{1\theta}} \left\{ \overline{uv} \frac{\partial T}{\partial y} + (1 - C_{2\theta}) \overline{v\theta} \frac{\partial U}{\partial y} - \beta g \overline{\theta^2} (1 - C_{3\theta}) \right\} \quad (48)$$

$$\overline{\theta^2} = -2R \frac{k}{\epsilon} \overline{v\theta} \frac{\partial T}{\partial y} \quad (49)$$

$$\overline{\theta^2} = -2R \frac{k}{\varepsilon} \frac{\partial T}{\partial y} \quad (49)$$

where

$$v_t = C_\mu \frac{k^2}{\varepsilon} \quad (50)$$

$$\sigma_t = \omega C_{1\theta} \quad (51)$$

$$C_\mu = \frac{\overline{v^2}}{k} \omega \quad (52)$$

where

$$\omega = (1 - C_2) + \frac{(1 - C_3)}{C_{1\theta}} B_0 + \frac{2D^{kl} C_{ks} (U^r)^2}{\varepsilon \frac{\overline{v^2} \partial U}{\partial y}} \quad (53)$$

$$B_0 = \beta g \frac{k}{\varepsilon} \frac{\partial T / \partial y}{\partial U / \partial y} \quad (54)$$

$$\frac{\overline{v^2}}{k} = \frac{1}{Const} \times \frac{2}{3} \left[C_2 \frac{P}{\varepsilon} + C_3 \frac{G}{\varepsilon} + C_1 - 1 + C_{ks} \frac{D^{kl}}{\varepsilon} (U_m^r)^2 \right] \quad (55)$$

where

$$Const = \frac{P}{\varepsilon} + \frac{G}{\varepsilon} + C_1 - 1 + \frac{k}{\varepsilon} D^{kl} C_k + \frac{k}{2\varepsilon} (\Gamma^{kl} - \Gamma^{lk} C_\Gamma) - \frac{D^{kl}}{\varepsilon} C_{ks} (U_k^r)^2 \quad (56)$$

It can be seen from the above expressions that C_μ is now a function of $\frac{P}{\varepsilon}$, $\frac{G}{\varepsilon}$ and the interfacial terms involving D^{kl} , Γ^{kl} and Γ^{lk} .

and Rodi (1982). The two-phase constants will have to be adjusted based on experimental data. Some suggestions for model constants are given below.

$$C_1 = 1.8 - 2.2; C_2 = 0.55 - 0.6; C_3 = 0.55 - 0.6$$

$$C_{1\theta} = 3.0; C_{2\theta} = 0.5; C_{3\theta} = 0.5$$

$$C_{ks} = 0.02; R = 0.8; C_\Gamma \leq 0.2; C_k \leq 0.5$$

$$\sigma_k = 1; \sigma_\epsilon = 1.33$$

$$C_\epsilon = 1.44; C_{\epsilon 2} = 1.92; C_{\epsilon 3} = 1.44; C_4 = 1; C_5 = 0.5$$

Equations (32), (33), (46) through (56) form the two-dimensional, two-phase, turbulent flow-model. The constants given above may need to be varied. P and G in these equations are given in equations (28) and (29) and D^{kl} in eq. (4).

It should be noted that all the calculations can take place in a three-dimensional field while the model itself is two-dimensional. This means that the turbulence model does not account for gradients in the width direction. But since the y-derivatives of U and T are expected to change in the z-direction, a new value of C_μ will be calculated at every point in the three-dimensional field.

Summary

This report presents a turbulence model that is free from some of the deficiencies pointed out in the introduction. It includes the turbulence effect of the dispersed phase naturally from the derivation of a phasic Reynolds stress transport equation. It also includes the effect of heat transfer which is not available in the literature for two-phase flows. The model provides a function for C_μ and accounts for the different production or destruction processes acting on the different stress and flux components. Therefore, this model is more general than the standard k- ϵ model which employs an isotropic eddy viscosity. The model is algebraic and therefore excessive coding normally required for transport equations is not necessary here. When the interfacial drag terms and the interfacial momentum transfer terms are absent, the model reduces to a single-phase model used successfully in the literature. Single-phase model constants proposed in the literature for various flows will be used first with minor adjustments for any heat transfer effect. The current model generates additional constants however, and sensitivity analyses will have to be done and the

model tested extensively against data. As with all turbulence models, the proposed model is not deemed to be universal, but since the model is theoretically based with different time scales for the dispersed phase, it certainly bears hope. The model is expected to be valid for high void fractions up until the slug flow regime since dispersed phase interactions are taken into account. Appropriate drag models should be used in various flow regimes to test the validity of the model in slug flows.

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