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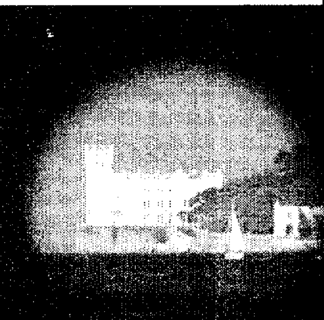

preprint

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DERIVATION OF THE POLYAKOV ACTION

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Abstract

We develop another method to get the Polyakov action that is: the solution of the conformal Ward identity on a Riemann surface Σ. We find that this action is the sum of two terms: the first one is expressed in terms of the projective connection and produces the diffeomorphism anomaly and the second one is anomaly and contains the globally defined zero modes of the Ward identity. The explicit expression of this action is given on the complex plane.

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I.Introduction

The effective action of an arbitrary conformal field theory on a Riemann surface Σ is a functional of the Beltrami differentials μ and $\overline{\mu}$ when the matter field is integrated out and is denoted by $\Gamma[\mu, \overline{\mu}]$[1,2]. This functional satisfies two postulates which have been shown in a number of examples but are expected to hold in general[3]. They are the following:

i/ Holomorphic factorization which expresses the fact that the vacuum functional $\Gamma[\mu, \overline{\mu}]$ splits as the sum of the chiral and the antichiral sectors actions:

$$\Gamma[\mu, \overline{\mu}] = \Gamma[\mu] + \Gamma[\overline{\mu}] \tag{I.1}$$

ii/ $\Gamma[\mu]$ is the unique solution of the conformal Ward identity in the infinite plane [4]:

$$(\overline{\partial} - \mu\partial - 2\partial\mu)\frac{\delta\Gamma[\mu]}{\delta\mu} = \frac{-k}{24\pi}(\partial^3 \mu + 2R_0\partial\mu + \partial R_0 \mu) \tag{I.2}$$

and

$$\text{c.c.} \tag{I.3}$$

which expresses the anomalous breakdown of the diffeomorphism symmetry [2,4]. k is the central charge of the chiral sector and R_0 is a holomorphic projective connection in the reference conformal structure that ensures the correct conformal covariance of the right-hand side of eq.(I.2). This means that the anomaly $A(c, \mu) = -\mu(L_3 c)$, where c is the ghost field and $L_3 = \partial^3 + 2R_0\partial + \partial R_0$ is the third Bol's operator [5], transforms with a jacobian under a conformal change of coordinates:

$$A(c, \mu)' = (\partial' z)(\overline{\partial' z})A(c, \mu), \tag{I.4}$$

with $\partial' z \equiv \partial z / \partial z'$.

The set of all Beltrami differentials on a Riemann surface Σ parametrizes the set of all conformal structures on this surface. Indeed, to any Beltrami differential μ there is associated a conformal structure, say C_μ on Σ whose generic coordinate Z is a local solution of the Beltrami equation

$$(\overline{\partial} - \mu\partial)Z = 0 \tag{I.5}$$

and satisfies the invertibility condition

$$|\partial Z|^2 - |\overline{\partial} Z|^2 \succ 0. \tag{I.6}$$

This , with the help of eq.(I.5), can be expressed as

$$|\mu(z, \overline{z})| \prec 1. \tag{I.7}$$

In particular, the reference conformal structure C_0 corresponds to the vanishing Beltrami differential; $C_0 \equiv C_{(\mu=0)}$ and then, the diffeomorphism

$$(z,\bar{z}) \leftrightarrow (Z(z,\bar{z}),\bar{Z}(z,\bar{z})) \tag{I.8}$$

becomes a conformal transformation; $\bar{\partial}Z = 0$.

A holomorphic projective connection R_0 on the surface Σ is an assignment to any coordinate z of C_0 of a smooth function R_0 defined in the domain of z with the following properties: on the overlapping domains of z and z' one has

$$R'_{z'z'}(z') = (\partial' z)^2 [R_{zz}(z) - S(z';z)], \tag{I.9}$$

where

$$S(z';z) = \partial^2 \ln \partial z' - \tfrac{1}{2}(\partial \ln \partial z')^2 \tag{I.10}$$

is the Schwarzian derivative.

However, a projective connection R is μ-holomorphic and verifies[1]

$$(\bar{\partial} - \mu\partial - 2\partial\mu)R = \partial^3 \mu. \tag{I.11}$$

In particular, one can deduce from eq.(I.11) that in the reference conformal structure C_0 a projective connection R becomes a holomorphic one:

$$\bar{\partial}R_0 = 0. \tag{I.12}$$

To any element R of the set of all projective connections there is canonically associated a projective structure subordinated to C_μ. This is a maximal coordinate covering contained in C_μ whose transition functions are restrictions of elements of the Mobius group $PSL(2,\mathbb{C})$ and whose generic coordinate Z satisfies, eq.(I.5) and eq.(I.6), the following one:

$$(\partial^2 + \tfrac{1}{2}R)(\partial Z)^{-\frac{1}{2}}. \tag{I.13}$$

Indeed, if we put the definition

$$R \equiv S(Z;z), \tag{I.14}$$

that expresses the one-to-one correspondence between projective structures and projective connections, in eq.(I.10) we get (I.13). Moreover, the local holomorphic coordinates $(Z,\bar{Z})$ that parametrizes the complex structure defined by μ correspond to the C^∞ change of local coordinates; eq.(I.8) which induces the variation of the differential form dZ:

$$\begin{aligned} dZ &= \partial Z(dz + \bar{\partial}Z d\bar{z}) \\ &= \partial Z(dz + \mu_{\bar{z}}^{z} d\bar{z}). \end{aligned} \tag{I.15}$$

Then, eq.(I.15) implies that the function ∂Z, besides eq.(I.13), satisfies the constraint

$$\left(\bar{\partial} - \mu\partial\right)\lambda = \lambda\partial\mu\,, \tag{I.16}$$

where $\lambda \equiv \partial Z$. Then, eq.(I.6) can be rewritten as

$$\bar{\partial}\Lambda = \mu\partial\Lambda + \partial\mu\,, \tag{I.17}$$

with $\Lambda \equiv \partial Z$.

Furthermore, by putting $\Lambda \equiv \ln(\partial Z)$ in eq.(I.10), with the help of eq.(I.14), we get

$$R = \partial^2\Lambda - \tfrac{1}{2}(\partial\Lambda)^2. \tag{I.18}$$

Then, we deduce from (I.17) and (I.18) the relation (I.11).
This shows that the μ-holomorphic projective connection is a consequence of the deformation of the reference conformal structure by the diffeomorphism (I.8).
We note here, that one can recover our results of ref.[1] concerning the explicit expression of the projective connection on the complex plane by using the relation (I.8) and the expression of the function Λ fund in [2].

II. The Polyakov action in terms of the projective connection

The physics of a conformal model on a two-dimensional Riemann surface Σ is independent of the choice of the coordinates system on which the model is constructed. Furthermore, the constructed Lagrangians of these models are related to the geometric objects defined on the surface Σ. Indeed, the field variables of the model are components of these intrinsic objects in order to have a well defined Lagrangian as a (1,1)-differential form on Σ. Then, the energy-momentum tensor related to the physics of the conformal model is expressed in terms of the geometry of the surface Σ; the reference conformal structure characterized by the holomorphic projectiv connection R_0 and the μ-holomorphic projective connection R. Indeed, by inserting eq.(I.11) in the Ward identity (I.2) and taking into acount eq.(I.2), we deduce the following relation:

$$\Delta_\mu T = \tfrac{k}{24\pi}\Delta_\mu\left(R_0 - R\right), \tag{II.1}$$

where $\Delta_\mu = \bar{\partial} - \mu\partial - 2\partial\mu$ and $T \equiv \dfrac{\delta\Gamma[\mu]}{\delta\mu}$ is the energy-momentum tensor whose external source is the Beltrami differential μ.
Then, we get the solution of eq.(II.1) as

$$T_{zz}\left(z,\bar{z}\right) = \tfrac{k}{24\pi}\left(R_{0_{zz}} - R_{zz}\right)\left(z,\bar{z}\right) + f_{zz}\left(z,\bar{z}\right) \tag{II.2}$$

such that the function f_{zz} is a kernel of the operator Δ_μ and verifies

$$\Delta_{\mu} f_{zz} = 0. \tag{II.3}$$

As is well known, the tensors $T_{zz}, R_{0_{zz}}, R_{zz}$ are (2,0)-differential forms on the Riemann surface Σ.
We can note that, the only remaining characteristic property of the matter field in this geometrical equation (II.2) is the central charge k of the model.
Then, by inserting the energy-momentum tensor expression, in terms of the functional derivation of the Polyakov action with respect to its external source, in eq.(II.2) and by integrating this latter over the Riemann surface Σ we get the following functional integral expression for the action $\Gamma[\mu]$:

$$\Gamma[\mu] = \tfrac{k}{12\pi} \int_{\Sigma} dm_0 \left[\mu(R_0 - R)\right](z,\bar{z}) + 2\int_{\Sigma} dm_0 \mu f(z,\bar{z}), \tag{II.4}$$

where

$$dm_0 \equiv \frac{dz \wedge d\bar{z}}{2i}.$$

One can notice at this level that the first term on the r.h.s. of eq.(II.4) is exactly the action that requires the holomorphic projective connection to write down the chirally split form of the diffeomorphism anomaly and then ensures the holomorphic factorization property of the vacuum functional $\Gamma[\mu, \bar{\mu}]$. Moreover, as this action contains the μ-holomorphic projective connection it was introduced in ref.[6], besides two other terms, to shift the Weyl anomaly into the diffeomorphism anomaly.
Hence, the procedure that we develop here to establish eq.(II.4) is another set up to get the solution of the Ward identity; eq.(I.2) on a Riemann surface Σ. However, in order to obtain the explicit expression of the Polyakov action we have to solve eq.(II.3). This is the subject of the next section.

III. Zero modes of the operator Δ_{μ}

The equation (II.3) written as

$$\left(\bar{\partial} - \mu\partial - 2\partial\mu\right)f = 0 \tag{III.1}$$

can be interpreted as the "non-anomalous Ward identity" corresponding to the free anomaly part of the two-dimensional conformal model. Moreover, it can be viewed as a μ-holomorphic (2,0)-differential form f equation[7] in comparison with eq.(I.11). Then, we learn from eq.(II.4) that the conformal anomaly, appearing in the Ward identity, comes only from the first term on the r.h.s. of eq.(II.4) and it is expressed only in terms of the projective connection up to the central charge coefficient.
Hence, at this level we were able to express the Polyakov action as the sum of the term that exhibits the conformal anomaly and of the other one that is anomaly free and that contains the globally defined zero modes of the operator Δ_{μ}. Due to this fact, the solution of the Ward identity on any Riemann surface Σ is not unique as it was stressed in ref.[7].

The equation (III.1) can be rewritten in the form

$$\bar{\partial} f = \mu \partial f + 2 \partial \mu f \qquad \text{(III.2)}$$

and with the help of the definition $F = \ln f$, we get the following equation

$$\bar{\partial} F = \mu \partial F + 2 \partial \mu . \qquad \text{(III.3)}$$

Then, by using the $\bar{\partial}$-Cauchy kernel on the complex plane introduced in [2] we establish the iterative solution of eq.(III.3) as a functional of the Beltrami differential μ that is; the function F is the Newmann series

$$F(z, \bar{z}) = \sum_{n=1}^{\infty} F_n(z, \bar{z}), \qquad \text{(III.4)}$$

where

$$F_n(z, \bar{z}) = -(-2)^{n+1} \int_{\mathbb{C}} \prod_{i=1}^{n} \left(\frac{dm_i \mu_i}{z_{i-1i}^2} \right) \frac{z_{01}}{z_{n-1n}} \qquad \text{(III.5)}$$

with the notation

$$dm_i \equiv \frac{dz_i \wedge d\overline{z_i}}{2i}, z_{ij} = z_i - z_j, \mu_i = \mu(z_i, \overline{z_i}).$$

Hence, the quadratic zero modes of the operator Δ_μ and the Polyakov action on the complex plane are respectively given by

$$f(z, \bar{z}) = \exp\left[-\sum_{n=1}^{\infty} (-2)^{n+1} \int_{\mathbb{C}} \prod_{i=1}^{n} \left(\frac{dm_i \mu_i}{z_{i-1i}^2} \right) \frac{z_{01}}{z_{n-1n}} \right] \qquad \text{(III.6)}$$

and

$$\Gamma[\mu] = \frac{k}{24\pi} \int_{\mathbb{C}} dm_0 \mu_0 (R_0 - R) + 2 \int_{\mathbb{C}} dm_0 \mu_0 \exp\left[-\sum_{n=1}^{\infty} (-2)^{n+1} \int_{\mathbb{C}} \prod_{i=1}^{n} \left(\frac{dm_i \mu_i}{z_{i-1i}^2} \right) \frac{z_{01}}{z_{n-1n}} \right], \text{(III.7)}$$

where

$$\mu_0 \equiv \mu(z, \bar{z}) \text{ and } dm_0 \equiv \frac{dz \wedge d\bar{z}}{2i}.$$

IV. Conclusion and open problems

We have developed another method to get the Polyakov action on a Riemann surface. In particular, we obtain an explicit expression for this action on the complex plane by determining the zero modes of the Ward operator on this particular surface.

The eq.(III.1) can be solved on any Riemann surface and then, the corresponding Polyakov action can be given on it and can be compared to that established in [7]. Furthermore, the comparison of the action (III.7) with that of ref.[2] can lead to the explicit expression of the projective connection as the solution of eq.(I.11).

The results developed here and in the references therein , can be translated to the torus and to the supersymmetric extensions of these Riemann surfaces by using the techniques introduced in [2,8,9].

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