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IN M-THEORY

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CALIBRATED GEOMETRIES AND NON PERTURBATIVE SUPERPOTENTIALS IN M-THEORY

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Abstract

We consider non perturbative effects in M-theory compactifications on a seven-manifold of G_2 holonomy arising from membranes wrapped on supersymmetric three-cycles. When membranes are wrapped on associative submanifolds they induce a superpotential that can be calculated using calibrated geometry. This superpotential is also derived from compactification on a seven-manifold, to four dimensional Anti-de Sitter spacetime, of eleven dimensional supergravity with non vanishing expectation value of the four-form field strength.

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1 Introduction

The theory of minimal surfaces has been for a long time an area of active research in mathematics. A deep novelty on it was the theory of calibrations, introduced by Harvey and Lawson [1] in 1982 to characterize geometries with the property that all submanifolds in the geometry are volume minimizing. With this defining property, minimal p-surfaces, or calibrated submanifolds, can be naturally employed to construct supersymmetric configurations of Dirichlet $p$branes in string theory, as in references [2]-[17], an approach which is also probably able to shed some light [18, 19] on the study of the moduli space of calibrated submanifolds [20, 21], whose global structure seems difficult to understand [22, 23].

Motivated by reference [15], where BPS solitons in the effective field theory arising from compactification of type IIA string theory, with non vanishing Ramond-Ramond fluxes, on a Calabi-Yau fourfold, were identified by Gukov with D-branes wrapped over calibrated submanifolds in the internal Calabi-Yau space to provide a simple and geometrical derivation of the superpotential in the two dimensional field theory [24, 25], in this paper we will consider compactification of M-theory on a seven-manifold of G_2 holonomy, which leads to a four dimensional field theory with $N = 1$ supersymmetry, and apply ideas, as in [15], of the theory of calibrations to understand the superpotential.

Superpotentials induced by membrane instantons in M-theory on a seven-manifold have been previously considered from the point of view of geometric engineering of field theories by Acharya in [26], where fractional membrane instantons, arising from compactification on Joyce orbifolds, are argued to generate a superpotential and, more recently, by Harvey and Moore in [14], where membranes wrapped on rigid supersymmetric three-cycles are shown to induce non zero corrections to the superpotential, that can be expressed in terms of topological invariants of the three-cycle. In this paper we will argue how the allowed calibrations in G_2 holonomy seven-manifolds imply that the only contributions to the superpotential in the four dimensional field theory will come from membranes whose worldvolume has been wrapped on supersymmetric three-cycles of the internal manifold, and exclude posible corrections from eleven dimensional fivebranes, in the same way as in [15] various D-branes in type IIA string theory are shown to contribute to the superpotential in the two dimensional $N = (2, 2)$ field theory, when wrapping calibrated submanifolds in the Calabi-Yau fourfold. However, we will not consider compactifications to four dimensional Minkowski spacetime, as in [14], but to four dimensional Anti-de Sitter space, to avoid the constraint that the fourth rank tensor field strength must be vanishing [27]. This will allow a nice geometrical picture for the superpotential.

Relevant notions on calibrated geometries, and its realizations in supersymmetric compactifications, are described in section 2. Section 3 is devoted to the construction of the superpotential induced by membrane instantons, identifying instantons in the four dimensional field theory with membranes whose worldvolume is wrapped over associative submanifolds in the G_2 holonomy

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seven-manifold. The expression for the superpotential is justified in section 4 by compactification of eleven dimensional supergravity, with non trivial fourth rank tensor field strength, on a manifold with G_2 holonomy to a four dimensional Anti-de Sitter spacetime. In section 5 we present some concluding remarks and possible implications of the work presented here.

2 Supersymmetry and the Calibration Bound

In this section we will review some generalities about calibrated geometry[2], and its relation to supersymmetric compactifications, and introduce some results and definitions relevant to the rest of the paper.

Given a Riemannian manifold M, with metric g, an oriented tangent k-plane V on M is a vector subspace V of the tangent space T_pM to M, with dim $(V) = k$, equipped with an orientation. The restriction $g|_V$ is the Euclidean metric on V, and allows to define, together with the orientation on V, a natural volume form Vol_V on the tangent space V.

A closed k-form ψ is said to be a *calibration on* M if for every oriented k-plane V on M it is satisfied

$$\psi|_V \le \mathrm{Vol}_V \tag{2.1}$$

where, by $\psi|_V$, we mean

$$\psi|_V = \alpha \, . \, \mathrm{Vol}_V \tag{2.2}$$

for some $\alpha \in \mathbf{R}$, so that condition (2.1) holds only if $\alpha \le 1$.

Now let N be an oriented submanifold of M, with dimension k, so that each tangent space T_pN, for $p \in N$, is an oriented tangent k-plane. The submanifold N is called a *calibrated submanifold* with respect to the calibration ψ if

$$\psi|_{T_pN} = \mathrm{Vol}_{T_pN} \tag{2.3}$$

for all $p \in N$. It is easy to show that calibrated manifolds have minimal area in their homology class [1]; to see this, let us denote by $[\psi]$ the de Rham cohomology class, $[\psi] \in H^k(M; \mathbf{R})$, and by $[N]$ the homology class, $[N] \in H_k(M; \mathbf{R})$. Then,

$$[\psi] \, . \, [N] = \int_{p \in N} \psi|_{T_pN} . \tag{2.4}$$

But condition (2.1) implies

$$\int_{p \in N} \psi|_{T_pN} \le \int_{p \in N} \mathrm{Vol}_{T_pN} = \mathrm{Vol}(N), \tag{2.5}$$

so that $\mathrm{Vol}(N) \ge [\psi] \, . \, [N]$. Equality holds for calibrated submanifolds [1].

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[2]For a more detailed and formal treatment on calibrated geometry we refer the reader to the original reference [1], or [28] for a collection of results.

In this paper we will be interested in calibrated geometries in seven-dimensional Joyce manifolds with G_2 holonomy so that M, in what follows, will be a compact seven-manifold, $M = X_7$, with a torsion free structure, that we will denote by $\Psi^{(3)}$ (the holonomy group of the metric associated to $\Psi^{(3)}$ is G_2 if and only if $\pi_1(X_7)$ is finite [29]).

The covariantly constant three-form $\Psi^{(3)}$ constitutes the *associative calibration*, with respect to the set of all associative three-planes, which are the canonically oriented imaginary part of any quaternion subalgebra of $\mathbf{O}$. Calibrated submanifolds with respect to $\Psi^{(3)}$ will, in what follows, be denoted by S_Ψ, and referred to as *associative submanifolds*. The Hodge dual to the associative calibration is a four-form $^*\Psi^{(4)}$, known as the *coassociative calibration*, responsible for *coassociative submanifolds*.

The existence of associative submanifolds, volume minimizing, can be directly understood from the point of view of compactifications of M-theory on manifolds with G_2 holonomy [3]. Let us see how this comes about. The low energy limit of M-theory is expected to be eleven dimensional supergravity, which is a theory that contains membrane solutions, described by the action [30, 31] [3]

$$S_{\text{Membrane}} = \int d^3\sigma \sqrt{h} [\frac{1}{2} h^{\alpha\beta} \partial_\alpha X^M \partial_\beta X^N g_{MN} - \frac{1}{2} - i\bar{\theta}\Gamma^\alpha \nabla_\alpha \theta$$

$$+ \frac{i}{3!} \epsilon^{\alpha\beta\gamma} C_{MNP} \partial_\alpha X^M \partial_\beta X^N \partial_\gamma X^P + \cdots], \tag{2.6}$$

where $X^M(\sigma^i)$ represents the membrane configuration, θ is an eleven dimensional Dirac spinor and $h_{\alpha\beta}$ is the induced metric on the worldvolume of the membrane (upper case latin indices are defined in eleven dimensions, while greek indices α, β, γ label coordinates on the worldvolume).

On the membrane fields, global fermionic symmetries act as

$$\delta_\eta \theta = \eta,$$
$$\delta_\eta X^M = i\bar{\eta}\Gamma^M \theta, \tag{2.7}$$

where η is an eleven dimensional constant anticommuting spinor. But the theory is also invariant under local fermionic transformations,

$$\delta_\kappa \theta = 2P_+ \kappa(\sigma),$$
$$\delta_\kappa X^M = 2i\bar{\theta}\Gamma^M P_+ \kappa(\sigma), \tag{2.8}$$

with κ some eleven dimensional spinor, and P_+ a projection operator [32]; the operators $P_\pm$ are defined as

$$P_\pm = \frac{1}{2}(1 \pm \frac{i}{3!} \epsilon^{\alpha\beta\gamma} \partial_\alpha X^M \partial_\beta X^N \partial_\gamma X^P \Gamma_{MNP}), \tag{2.9}$$

and can be easily shown to satisfy $P_\pm^2 = P_\pm$, $P_+ P_- = 0$ and $P_+ + P_- = 1$.

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[3]We will chose units such that the eleven dimensional Planck length is set to one.

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As bosonic membrane configurations break all the global supersymmetries generated by η, unbroken supersymmetry remains only if there is a spinor $\kappa(\sigma)$ such that

$$\delta_\kappa \theta + \delta_\eta \theta = 2P_+\kappa(\sigma) + \eta = 0 \tag{2.10}$$

or, equivalently,

$$P_-[2P_+\kappa(\sigma) + \eta] = P_-\eta = 0. \tag{2.11}$$

If the eleven dimensional theory is compactified on a seven-manifold of G_2 holonomy, finding a covariantly constant spinor η satisfying condition (2.11) is equivalent to the existence of a supersymmetric three-cycle in the seven-manifold, which is precisely the associative submanifold [3]. To prove this, we will formally write the covariantly constant three-form (the associative calibration) as

$$\Psi^{(3)} = \frac{1}{3!}\psi_{mnp}dX^m \wedge dX^n \wedge dX^p, \tag{2.12}$$

and decompose the eleven dimensional spinor η in terms of a four dimensional spinor, ϵ, and the covariantly constant spinor ξ on the manifold with G_2 holonomy (this spinor is unique up to scale),

$$\eta = \epsilon \otimes \xi. \tag{2.13}$$

Now, if we chose a normalization such that the action of the seven dimensional gamma matrices on the covariantly constant spinor of the internal manifold is

$$\gamma_{mnp}\, \xi = \psi_{mnp}\, \xi, \tag{2.14}$$

the condition (2.11) for unbroken supersymmetry becomes

$$\frac{1}{2}\left(1 - \frac{i}{3!}\epsilon^{\alpha\beta\gamma}\partial_\alpha X^m \partial_\beta X^n \partial_\gamma X^p \psi_{mnp}\right)\xi = 0, \tag{2.15}$$

which is equivalent to [3],

$$\frac{1}{3!}\partial_{[\alpha}X^m \partial_\beta X^n \partial_{\gamma]}X^p \psi_{mnp} = \epsilon_{\alpha\beta\gamma}, \tag{2.16}$$

so that the pull back of the three-form is proportional to the volume element [4].

Condition (2.16) implies that the membrane worldvolume, wrapped on the three-cycle calibrated by $\Psi^{(3)}$, has been minimized, as can be easily seen from the inequality

$$\int d^3\sigma\sqrt{h}(P_-\xi)^\dagger(P_-\xi) \geq 0, \tag{2.17}$$

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[4]An identical condition can be similarly derived for the coassociative calibration [3],

$$\frac{1}{4!}\partial_{[\alpha}X^m \partial_\beta X^n \partial_\gamma X^p \partial_{\delta]}X^q \psi_{mnpq} = \epsilon_{\alpha\beta\gamma\delta}.$$

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as in [2]. Using the projector (2.9), and the fact that $P_-^\dagger P_- = P_-$, (2.17) becomes

$$\int d^3\sigma \sqrt{h}\, \bar{\xi}\xi \geq \int d^3\sigma\, \bar{\xi}\frac{i}{3!}\epsilon^{\alpha\beta\gamma}\partial_\alpha X^m \partial_\beta X^n \partial_\gamma X^p \psi_{mnp}\xi, \tag{2.18}$$

or $V_3 \geq \int_{S_\Psi} \Psi^{(3)}$. The bound is saturated if and only if $P_-\xi = 0$, which is precisely the condition for unbroken supersymmetry.

3 Associative Calibrations and Superpotential for M-Theory Compactifications

M-theory compactification on a seven-manifold with G_2 holonomy produces a four dimensional field theory with $N = 1$ supersymmetry. At low energy, trusting eleven dimensional supergravity as an aproximation to M-theory, the effective four dimensional supergravity theory describing the massless modes is $N = 1$ supergravity coupled to b_2 vector multiplets, and b_3 chiral multiplets, where b_2 and b_3 are Betti numbers of X_7 [33]. However, in this paper we will not be interested in this relatively poor, from the point of view of physics, spectrum, or posible increases of interest when singularities are allowed in the seven-manifold, as in [26]. What we will wonder about is the generation of a non perturbative superpotential, arising in the effective four dimensional field theory from M-theory effects, applying the ideas in [15].

In reference [15], the generation of a superpotential in the two dimensional theories obtained when compactifying type IIA string theory on Calabi-Yau fourfold with background Ramond-Ramond fluxes was considered, identifying BPS solitons in the field theory with D-branes wrapped over calibrated submanifolds in the internal manifold. In this section, following the same reasoning, we will wrap the M-theory membrane worldvolume over some associative submanifold in the seven-manifold X_7, $S_\Psi \in H_3(X_7, \mathbf{Z})$, which is a supersymmetric three-cycle, defined through condition (2.16). This state, from the point of view of the four dimensional field theory, is an instanton. The flux of the fourth-rank antisymmetric field strength, $F = dC$, associated to the membrane, over the "complement" $(S_\Psi)^\perp$ in X_7, which will be some four dimensional manifold, Y_4, jumps by one when crossing the membrane. To understand this, two points P and Q can be chosen to ly at each side of the three-cycle S_Ψ, with the membrane worldvolume wrapped on it. If T is a path joining these two points, and it is chosen to intersect the membrane at a single point, the five-manifold $Y_4 \times T$ will also intersect the membrane at a single point. Now, as $\frac{dF}{2\pi}$ is a delta function for the membrane,

$$\int_{Y_4 \times T} \frac{dF}{2\pi} = 1 \tag{3.1}$$

or, once Stoke's theorem is employed,

$$\int_{Y_4 \times P} \frac{F}{2\pi} - \int_{Y_4 \times Q} \frac{F}{2\pi} = 1. \tag{3.2}$$

Hence, the field theory vacua connected by the instanton solution will correspond to different four-form fluxes, F_1 and F_2; the variation $\frac{\Delta F}{2\pi}$ is then Poincaré dual to the homology class of the supersymmetric three-cycle, $[S_\Psi]$. But the amplitude for the configuration of the membrane is proportional to the volume of the associative submanifold, S_Ψ,

$$\int_{S_\Psi} \Psi^{(3)}, \tag{3.3}$$

the constant of proportionality being simply the membrane tension. If we denote the superpotential in the field theory by W, the amplitude of the instanton connecting two vacua will be given by the absolute value of ΔW; then, when (3.2) is taken into account, (3.3) becomes

$$\Delta W = \int_{S_\Psi} \Psi^{(3)} = \frac{1}{2\pi} \int_{X_7 = S_\Psi \times Y_4} \Psi^{(3)} \wedge \Delta F^{(4)}, \tag{3.4}$$

so that we expect a superpotential

$$W = \frac{1}{2\pi} \int_{X_7} \Psi^{(3)} \wedge F^{(4)}, \tag{3.5}$$

for compactification of M-theory on a seven-manifold of G_2 holonomy. Expression (3.5) is the analogous of the untwisted and twisted chiral superpotentials obtained in [15] for compactification of type IIA string theory in a Calabi-Yau fourfold with non-vanishing Ramond-Ramond fluxes.

We could now also wonder about the possible contribution of M-theory fivebranes to the superpotential. In fact, the result that $\pi_1(X_7)$ is finite if and only if the holonomy is in G_2 [29], implies that $H_6(X_7; \mathbf{Z})$ is also finite, so that we might expect instanton effects coming from the fivebrane. However, there is no six dimensional calibrated submanifold in X_7, because the only calibrations are $\Psi^{(3)}$ and $*\Psi^{(4)}$, so that there will be no BPS fivebrane instanton contribution to the superpotential.

4   Eleven Dimensional Supergravity on Seven-Manifolds

In this section we will explicitly obtain the superpotential (3.5) from supersymmetric compactification of M-theory on $AdS_4 \times X_7$. Compactification of eleven dimensional supergravity on a seven-manifold is very restrictive, because the four dimensional non compact spacetime is a supersymmetric Minkowski space only if all components of the fourth rank antisymmetric field strength vanish, $F = 0$, and the internal manifold is Ricci-flat [27]. However, if some of these conditions are relaxed, more general compactifications can be performed. In [34, 35], compactification of eleven dimensional supergravity to four dimensional Anti-de Sitter spacetime was shown to allow a four-form field strength proportional to the cosmological constant of the external space, and which therefore vanishes for compactifications to Minkowski spacetime. The requirements on the four-form F, assuming that supersymmetry remains unbroken, with zero

cosmological constant, for compactification of M-theory with a warp factor on a Calabi-Yau fourfold have been obtained in [36], and extended to a three dimensional Anti-de Sitter external space in [25]. The analysis of this section, where compactification of M-theory to a four dimensional spacetime, with non vanishing cosmological constant, on an internal manifold with G_2 holonomy will be considered, will hence follow closely those in references [36, 25, 15].

The bosonic form of the eleven dimensional effective action looks like

$$S = \frac{1}{2} \int d^{11}x \sqrt{g}[R - \frac{1}{2}F^{(4)} \wedge *F^{(4)} - \frac{1}{6}C^{(3)} \wedge F^{(4)} \wedge F^{(4)} - C^{(3)} \wedge I^{(8)}], \qquad (4.1)$$

where the gravitational Chern-Simons correction, associated to the sigma-model anomaly of the six dimensional fivebrane worldvolume [37], can be expressed in terms of the Riemann tensor [38],

$$I_8 = -\frac{1}{768}(\text{tr } R^2)^2 + \frac{1}{192}\text{tr } R^4. \qquad (4.2)$$

The complete action is invariant under local supersymmetry transformations [39]

$$\delta_\eta e_M^A = i\bar{\eta}\Gamma^A\psi_M,$$

$$\delta_\eta C_{MNP} = 3i\bar{\eta}\Gamma_{[MN}\psi_{P]},$$

$$\delta_\eta \psi_M = \nabla_M \eta - \frac{1}{288}(\Gamma_M^{PQRS} - 8\delta_M^P\Gamma^{QRS})F_{PQRS}\eta, \qquad (4.3)$$

where e_M^A is the elfbein, ψ_M the gravitino, η an eleven dimensional anticonmuting Majorana spinor, and ∇_M the covariant derivative, involving the Christoffel connection.

A supersymmetric configuration exists if and only if there is a spinor η satisfying the conditions

$$\delta_\eta e_M^A = 0,$$

$$\delta_\eta C_{MNP} = 0,$$

$$\delta_\eta \psi_M = 0. \qquad (4.4)$$

If the background is Lorentz covariant in the four dimensional spacetime, the background spinor ψ_M must vanish, so that e_M^A and C_{MNP} are unchanged by the supersymmetry transformation. Hence, the only constraint that remains to be impossed is

$$\nabla_M \eta - \frac{1}{288}(\Gamma_M^{PQRS} - 8\delta_M^P\Gamma^{QRS})F_{PQRS}\eta = 0. \qquad (4.5)$$

The most general metric, maximally symmetric, for compactification on X_7,

$$ds_{11}^2 = \Delta(x^m)^{-1}(ds_4^2(x^\mu) + ds_7^2(x^m)). \qquad (4.6)$$

includes a scalar function called the warp factor, $\Delta(x^m)$, depending on the internal dimensions. We are now choosing a notation such that the eleven dimensional upper case indices split into

four dimensional greek indices, while latin indices labell the set of coordinates tangent to the internal manifold. With the choice (4.6) for the metric, the eleven dimensional Dirac gamma matrices should also be rescaled,

$$\Gamma_M \to \Delta^{-1/2}\Gamma_M, \quad \Gamma^M \to \Delta^{1/2}\Gamma^M. \tag{4.7}$$

When the transformation rule of the covariant derivative for conformally related metrics, as in (4.6),

$$\nabla_M \eta \to \nabla_M \eta - \frac{1}{4}\Delta^{-1}\Gamma_M^N \nabla_N \Delta \eta, \tag{4.8}$$

is taken into account, condition (4.5) becomes, with the help of (4.7),

$$\nabla_M \eta - \frac{1}{4}\Gamma_M^N \partial_N(\log \Delta)\eta - \frac{1}{288}(\Gamma_M^{PQRS} - 8\delta_M^P\Gamma^{QRS})F_{PQRS}\eta = 0. \tag{4.9}$$

Now, let us decompose the gamma matrices in the convenient $11 = 7 + 4$ split,

$$\Gamma^\mu = \gamma^\mu \otimes \gamma^8,$$
$$\Gamma^m = \gamma^m \otimes \mathbb{1}, \tag{4.10}$$

where γ^8 is the seven dimensional chirality operator, defined in terms of the gamma matrices in X_7,

$$\gamma^8 = \frac{i}{7!}\epsilon_{m_1...m_7}\gamma^{m_1}\ldots\gamma^{m_7}, \tag{4.11}$$

so that $(\gamma^8)^2 = +\mathbb{1}$, and γ^m are chosen such that

$$\{\gamma^m, \gamma^n\} = 2\delta^{mn}. \tag{4.12}$$

We will also choose, as in [35], the most general covariant form for the fourth rank antisymmetric tensor field strength[5]

$$F_{\mu\nu\rho\sigma} = m\epsilon_{\mu\nu\rho\sigma},$$
$$F_{\mu\nu\rho s} = F_{\mu\nu rs} = F_{\mu nrs} = 0, \tag{4.13}$$

while

$$F_{mnpq} \text{ arbitrary}, \tag{4.14}$$

where m can depend upon the extra (internal) dimensions, and the decomposition (2.13) for the supersymmetry parameter, $\eta = \epsilon \otimes \xi$.

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[5] In compactifications of string theory or M-theory on a Calabi-Yau fourfold a global anomaly, given by the Euler number of the Calabi-Yau fourfold, arises [40, 41]. This anomaly can be cancelled if non zero background fluxes are allowed, and/or strings or membranes are introduced filling, respectively, two or three dimensional spacetime, for compactification of string theory or M-theory. These strings, or membranes, are represented by maximally symmetric tensor fields. However, for seven-manifolds there is no such interpretation for the ansatz of the first equation in (4.13).

With the split (4.10), and the ansatz (4.13), the μ-component of the supersymmetry condition (4.9) becomes

$$\nabla_\mu \eta - \frac{1}{4}\gamma_\mu \otimes \gamma_8 \gamma^n \partial_n(\log \Delta)\eta - \frac{\Delta^{3/2}}{288}\gamma_\mu \otimes \gamma_8 \gamma^{mnpq} F_{mnpq}\eta$$

$$+\frac{1}{6}\Delta^{3/2} im\gamma_\mu \eta = 0. \tag{4.15}$$

When ϵ is a four dimensional anti-conmuting Killing spinor satisfying

$$\nabla_\mu \epsilon = \frac{\Lambda}{2}\gamma_\mu \epsilon \tag{4.16}$$

and, without loss of generality,

$$\gamma_8 \xi = \xi, \tag{4.17}$$

equation (4.15), with the decomposition (2.13), leads to the solutions

$$144\Lambda\xi = \Delta^{3/2} F_{mnpq}\gamma^{mnpq} \xi \tag{4.18}$$

and

$$m = i\gamma^n \partial_n \Delta^{-3/2}. \tag{4.19}$$

Relation (4.18), obtained in [25], extends the condition $\gamma^{mnpq}F_{mnpq}\xi = 0$, for M-theory compactification on Calabi-Yau fourfolds to an external Minkowski space [36], to compactifications to Anti-de Sitter spacetime.

Similarly, the m-component of condition (4.9) becomes, after some gamma matrices algebra,

$$\nabla_m \xi - \frac{\Lambda}{2}\gamma_m \xi - \frac{4}{21}im\Delta^{3/2}\gamma_m \xi + \frac{\Delta^{3/2}}{24} F_{mnpq}\gamma^{pqr} \xi = 0, \tag{4.20}$$

once (4.18) and (4.19) are taken into account. Then, if we use the fact that on a seven-manifold of G_2 holonomy the calibration $\Psi^{(3)}$ satisfies relation (2.14), $\gamma^{mnp}\xi = \psi^{mnp}\xi$, and the fact that ξ has been chosen to be the covariantly constant spinor on the seven-manifold, relation (4.20) reproduces, when integrated over the volume of X_7, the superpotential proposed in (3.5), with a coefficient related to the warp factor, depending hence on the internal dimensions, as a consequence of the anstaz (4.13), and a non vanishing cosmological constant. The limit of no conformal factor in (4.6) relates directly the cosmological constant to the superpotential.

5   Conclusions

In this paper we have investigated the generation of a non perturbative superpotential in the four dimensional field theories obtained from compactification of M-theory on a seven-manifold

with G_2 holonomy. The allowed calibrations in these manifolds imply that the only BPS instantons are those obtained wrapping the worldvolume of the M-theory membrane over associative submanifolds, which are the analogous of special Lagrangian cycles in Calabi-Yau manifolds, while instantons coming from fivebranes wrapping six-cycles in the seven-manifold are not BPS states. Fivebranes wrapped over (complex) codimension one divisors D are however known to contribute to the superpotential in the three dimensional field theory arising upon compactification of M-theory on a Calabi-Yau fourfold if the divisor satisfies the topological requirement that $\chi(\mathcal{O}_D, D) = 1$ [42]. This condition can probably be related to the fact that fourfolds with $SU(4)$ holonomy admit, besides from Cayley calibrations, what are known as Kähler calibrations, which are obtained when considering powers of the complexified Kähler form,

$$\Psi = \frac{1}{p!}\mathcal{K}^p, \tag{5.1}$$

because the fact that $\mathcal{K}$ is covariantly constant ensures that Ψ, as defined in (5.1), is also covariantly constant. Submanifolds calibrated by Ψ are complex submanifolds, of dimension p, so that fivebranes v ill become BPS instantons once they are wrapped over some (complex) three dimensional manifold, calibrated by $\frac{1}{3!}\mathcal{K}^3$. However, a precise stament of the possible relation between the topological restriction and the analysis in [42], and calibrated geometries, is still lacking.

In reference [23], singularities of special Lagrangian three-cycles, and their compactness properties, were studied in detail, and a topological invariant, counting special Lagrangian homology three-spheres, was proposed. If an equivalent analysis can be repeated on seven-manifolds with G_2 holonomy, then an analogous topological invariant can probably be constructed, counting associative submanifolds or, which is more interesting physically, counting the number of membranes, or instantons, in the homology class $[S_\Psi]$, an idea closely related to the Gromov-Witten invariants, counting pseudo-holomorphic curves in symplectic manifolds. We hope to address this study in a subsequent paper.

Acknowledgements

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