Quantum Groups in Hadron Phenomenology

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Abstract

We show that application of quantum unitary groups, in place of ordinary flavor $SU(n_f)$, to such static aspects of hadron phenomenology as hadron masses and mass formulas is indeed fruitful. So-called q-deformed mass formulas are given for octet baryons $\frac{1}{2}^+$ and decuplet baryons $\frac{3}{2}^+$, as well as for the case of vector mesons 1^- involving heavy flavors. For deformation parameter q, rigid fixation of values is used. New mass sum rules of remarkable accuracy are presented. As shown in decuplet case, the approach accounts for effects highly nonlinear in $SU(3)$-breaking. Topological implication (possible connection with knots) for singlet vector mesons and the relation $q \leftrightarrow \theta_c$ (Cabibbo angle) in case of baryons are considered.

1. Introduction

During last decade, quantum groups and quantum (or q-) algebras [1] have shown their apparent effectiveness in diverse problems of theoretical physics, see overviews [2]. In particular, one gets essential improvement of phenomenological description of superdeformed nuclear bands and spectra of diatomic molecules by replacing [3] usual $su(2)$-symmetry (underlying rigid-rotator based description of rotational spectra) with its q-deformed analogue $su(2)_q$.

More recently, in the context of hadron phenomenology the use of quantum groups/algebras have been proposed [4-8]. Here we discuss some results and implications of such an application. Basic idea of [4-7] consists in adopting the *q-deformed version of flavor symmetries* (staying formally within the first-order flavor symmetry breaking) in order to get better agreement with empirical data for hadron masses. That is, we start by replacing the usual (isospin and higher) unitary symmetries with their quantum counterparts: $SU(n_f) \to SU_q(n_f)$, $n_f \geq 2$. Main motivation of such replacement comes from the following two facts:

- Successful application of the q-algebra $su(2)_q$ for phenomenological description of rotational bands of (super)deformed nuclei and diatomic molecules [3];
- The fact known from representation theory of quantum groups and q-algebras that each finite-dimensional irreducible representation (irrep) of $SU(n)$ smoothly q-deformes [9], at q not a root of unity, to respective irrep of $SU_q(n)$ of the same dimension.

More precisely, we exploit the q-algebras $U_q(su_n)$ corresponding to $SU_q(n)$, along with their irreps, as flavor symmetries of hadrons - vector mesons 1^- and baryons $\frac{1}{2}^+$, $\frac{3}{2}^+$. To calculate hadron masses, within our model we utilized [4-7] simple and natural but sufficiently effective method of performing necessary (flavor) symmetry breaking. This method directly extends to a q-deformed case the approach, based on the concept of unitary/pseudounitary dynamical groups, earlier used in order to treat hadron masses and mass formulas described with conventional groups $SU(n)$ of flavor symmetries. The virtue of method is that it allows to bypass difficulties related with q-CGC's and q-Casimirs which for higher rank quantum groups appear rather nontrivial.

With the help of appropriate q-algebras $U_q(u_{n+1})$ or $U_q(u_{n,1})$ of 'dynamical' symmetry, one realizes necessary breaking of n-flavor symmetries up to exact (for strong interactions alone) isospin symmetry $su_q(2)_I$ and obtains the q-analogues of mass relations (MRs) [4-7]. From these q-analogues of hadron mass formulas, in the non-deformed ('classical') limit $q \to 1$, one recovers the familiar hadronic (Gell-Mann–Okubo, or GMO, and equal spacing) mass sum rules [10]. In this point our approach principally differs from that of Ref.[8], wherein classical limit implies complete degeneracy of masses both within octet and within decuplet.

At definite values of deformation parameter q, $q \neq 1$, the q-deformed baryon mass formulas produce new mass sum rules, both octet and decuplet ones, which hold with better accuracy than classical GMO sum rule and equal-spacing rule respectively.

In the case of vector mesons it turns out that all the q-dependence in expressions for masses and in resulting q-deformed MRs is expressible [4,7] in terms of certain Lorant-type polynomials (of q) related with invariants of respective torus knots. As a consequence, concrete torus knots can be associated with concrete vector quarkonia. Moreover, a possibility appears to *distinguish different flavors*, through their vector quarkonia masses, *by topological means* - that is, using 'braid overcrossing number' or 'torus winding number', see section 3.

Already in [4] it was clear that deformation parameter q 'regulates' the issue of singlet meson mixing. Say, instead of manifest introducing the ϕ–ω mixingle angle, usual for ordinary $SU(3)$-based approach, one gets agreement with data of meson mass sum rule (MSR) which involves the *physical* ϕ-meson, not mixed state, merely due to adequate value of q. This correlation: deformation parameter $\leftrightarrow$ mixing angle goes even further in the case of fermions (baryons).

In the framework of fundamental problem of fermion masses and mixings it is well-known that the Cabibbo angle θ_C can be directly related with fermion (quark) masses: the original formula [11] $\tan^2\theta_C = m_d/m_s$ was subsequently reobtained within different approaches including thoses base on supersymmetry or noncommutative geometry. As will be demonstrated in sections 4-6, the two values of deformation parameter whose fixation in the q-deformed mass formulas for octet and decuplet baryons provides us with two new mass sum rules of remarkable precision, are connectible in a simplest possible way with the Cabibbo angle.

2. q-Deformed mass formulas for vector mesons

Details of derivation are contained in [4,7]. Here we only recall the basic points (say, for 3 flavors): (i) Assign vector meson states from octet (isotriplet, two isodoublets, isosinglet) with the corresponding vectors of orthonormal Gelfand-Tsetlin basis, e.g., $|\rho\rangle = |\{8\}_3;\{3\}_2;\alpha_\rho\rangle$, $|\omega_8\rangle = |\{8\}_3;\{1\}_2;\alpha_8\rangle$, where α_ρ labels the charge states within isotriplet; (ii) Embed the octet of $U_q(u_3)$ into adjoint 15-plet representation of dynamical $U_q(u_4)$; (iii) Take mass operator for $U_q(u_3)$ symmetry breaking in terms of appropriate generators of $U_q(u_4)$, namely $\hat{M}_3 = M_0 + \gamma_3(A_{34}A_{43} + A_{43}A_{34})$; (iv) Calculate matrix elements $\langle\rho|\hat{M}_3|\rho\rangle$, $\langle\omega_8|\hat{M}_3|\omega_8\rangle$, etc. .

In more general case, $3 \le n \le 6$, we use Gelfand-Tsetlin basis for meson states from (n^2-1)-plet of 'flavor' $U_q(u_n)$ embedded into $\{(n+1)^2-1\}$-plet of 'dynamical' $U_q(u_{n+1})$; mass operator, commuting with the 'isospin and hypercharge' q-algebra $U_q(u_2)$, is constructed (bilinearly) from relevant generators of 'dynamical' algebra $U_q(u_{n+1})$, and has the form [4,7] which agrees with the concept of symmetry breaking due to quark mass differences.

The expressions for masses, obtained from calculations, depend on the symmetry breaking parameters like γ_3 and on the deformation parameter q. For example,

$$m_\rho = M_o, \qquad m_{K^*} = M_o - \gamma_3, \qquad m_{\omega_8} = M_o - 2\frac{[2]_q}{[3]_q}\gamma_3, \tag{1}$$

$$m_{D^*} = M_o + \gamma_4, \qquad m_{F^*} = M_o - \gamma_3 + \gamma_4,$$

$$m_{\omega_{15}} = M_o + 2\Big(\frac{4}{[2]_q} - \frac{[3]_q}{[4]_q} - \frac{[4]_q}{[3]_q}\Big)\gamma_3 + 2\,\frac{[3]_q}{[4]_q}\,\gamma_4, \tag{2}$$

where $[x]_q \equiv [x] \equiv (q^x - q^{-x})/(q - q^{-1})$ is the q-number $[x]$ corresponding to usual number x, and the requirement that (isodoublet) particles and their anti's have equal masses was taken in account. An important fact is that *q-dependence appears only in masses of* $\omega_8, \omega_{15}, \omega_{24}, \omega_{35}$ (isosinglet within octet and singlets within (n^2-1)-plets of $U_q(u_n)$, $n = 4, 5, 6$).

Exclusion of unknown parameters results in the q-deformed mass relations [4,7]

$$[n]_{(q)}\; m_{\omega_{n^2-1}} + (b_{n;q} + 2n - 4)\; m_\rho = 2\; m_{D_n^*} + (c_{n;q} + 2)\sum_{r=3}^{n-1} m_{D_r^*} \tag{3}$$

where the denotion $[n]_q/[n-1]_q \equiv [n]_{(q)}$ is used and

$$b_{n;q} \equiv n\; c_{n;q} - 6\;[n]^2_{(q)} + \Big(\frac{24}{[2]_q} - 1\Big)[n]_{(q)} \qquad c_{n;q} \equiv 2\;[n]^2_{(q)} - \frac{8}{[2]_q}[n]_{(q)}.$$

These q-analogues show that coefficients at masses are obtained from their 'classical' prototypes in a more complex way than merely by replacing $a \to [a]_q$.

At $n = 3$ the Eq.(3) contains the q-analogue of GMO relation

$$m_{\omega_8} + \Big(2\frac{[2]_q}{[3]_q} - 1\Big)m_\rho = 2\frac{[2]_q}{[3]_q} m_{K^*}. \tag{4}$$

This obviously yields usual GMO formula [10] $3m_{\omega_8} + m_\rho = 4m_{K^*}$ (known to require manifest introducing of singlet mixing) at $q = 1$ (in this case $[2]_q/[3]_q = 2/3$), but also produces the formula

$$m_{\omega_8} + m_\rho = 2m_{K^*} \qquad \text{at} \qquad [2]_q = [3]_q \tag{5}$$

(the latter holds if $q = q_3 = e^{i\pi/5}$). With $m_{\omega_8} \equiv m_\phi$, Eq.(5) coincides with nonet mass formula of Okubo [12] which shows ideal agreement with data [13] (up to errors of experiment and of averaging over isoplets). What are higher analogues of Okubo's sum rule? With natural choice $[n]_q = [n-1]_q$, $n = 4, 5, 6$, we get them:

$$m_{\omega_{15}} + (5 - 8/[2]_{q_4})m_\rho = 2\; m_{D^*} + (4 - 8/[2]_{q_4})m_{K^*} \tag{6}$$

$$m_{\omega_{24}} + (9 - 16/[2]_{q_5})m_\rho = 2\; m_{D_s^*} + (4 - 8/[2]_{q_5})(m_{D^*} + m_{K^*}) \tag{7}$$

$$m_{\omega_{35}} + (13 - 24/[2]_{q_6})m_\rho = 2\; m_{D_t^*} + (4 - 8/[2]_{q_6})(m_{D_s^*} + m_{D^*} + m_{K^*}). \tag{8}$$

Here q_n denote the values that solve eqns. $[n]_q - [n-1]_q = 0$, namely,

$$q_n = e^{i\pi/(2n-1)}. \tag{9}$$

Like in the case with $m_{\omega_8} \equiv m_\phi$, it is meant in (6) that J/ψ is put in place of ω_{15}, etc.

The roots (9) and their 'master' polynomials (in q) have a topological implication.

3. Knot structures associated with vector quarkonia

The quantities $[n]_q - [n-1]_q$, by their roots, reduce the q-analogues (3) to realistic mass sum rules (5), (6)-(8). At the same time, being such polynomials $P_n(q)$ that satisfy (see [14])

$$(i)\;\; P_n(q) = P_n(q^{-1}), \qquad\qquad (ii)\;\; P_n(1) = 1,$$

they do coincide with topological invariants - Alexander polynomials $\Delta(q)\{(2n-1)_1\}$ of torus $(2n-1)_1$-knots. For instance,

$$[3]_q - [2]_q = q^2 + q^{-2} - q - q^{-1} + 1 \equiv \Delta(q)\{5_1\}, \tag{10}$$

$$[4]_q - [3]_q = q^3 + q^{-3} - q^2 - q^{-2} + q + q^{-1} - 1 \equiv \Delta(q)\{7_1\}, \tag{11}$$

correspond to the 5_1- and 7_1- knots, see fig. 1. The 'extra' q-deuce in (3) may be related with the trefoil (or 3_1-) knot, since $[2]_q - 1 = q + q^{-1} - 1 \equiv \Delta(q)\{3_1\}$. Hence, *all the q-dependence* in masses of ω_{n^2-1} and in coefficients of Eq.(3) is expressible through various Alexander polynomials:

$$\frac{[3]_q}{[2]_q} = 1 + \frac{\Delta\{5_1\}}{[2]_q} = 1 + \frac{\Delta\{5_1\}}{\Delta\{3_1\}+1}, \tag{12}$$

$$\frac{[n]_q}{[n-1]_q} = 1 + \frac{\Delta\{(2n-1)_1\}}{[n-1]_q} = 1 + \frac{\Delta\{(2n-1)_1\}}{1+\sum_{r=2}^{n-1}\Delta\{(2r-1)_1\}}, \qquad n = 4,5,6. \tag{13}$$

Thus, the values (9) are roots of respective Alexander polynomials. For every fixed n, it is the 'senior' polynomial in Eqs.(12),(13) (the one in numerator) which is distinguished: it serves to extract, through its root, the corresponding MSR from q-deformed analogue. At $n = 3$ the q-GMO formula (4) generates simple and successful Okubo's relation (5); at $n = 4,5,6$, general formula (3) reduces to the higher analogues (6)-(8) of Okubo's relation.

Let us emphasize that, for particular n, the corresponding value q_n is fixed in a rigid way as a root of $\Delta\{(2n-1)_1\}$, contrary to a choice of q by fitting in other phenomenological applications [3]. For instance, rigid fixation $q = e^{i\pi/5}$ turns $\Delta\{5_1\}$ into zero as well as the ratio $[3]_q/[2]_q$ into unity, thus extracting (5) from the q-analogue (4). This extends to higher $n = 4,5,6$.

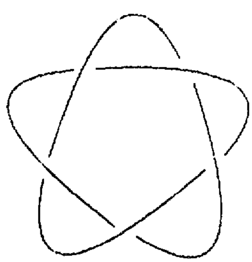

Fig. 1. Torus 5_1-knot corresponding to $[3]_q - [2]_q$

Fig. 2. Braid with 5 overcrossings

Using flavor q-algebras along with 'dynamical' q-algebras (through embedding $U_q(u_n) \subset U_q(u_{n+1})$) we get as result that the collection of torus knots 5_1, 7_1, 9_1, 11_1 is put into correspondence [7] with vector quarkonia $s\bar{s}$, $c\bar{c}$, $b\bar{b}$, and $t\bar{t}$ respectively. Thus, application of the q-algebras suggests a possibility of *topological characterization of flavors*: fixed number n just corresponds to $2n-1$ overcrossings of 2-strand braids (see fig. 2) whose closures give these $(2n-1)_1$-torus knots. Or, with the form $(2n-1,2)$ of these same knots, we have the correspondence $n \longleftrightarrow w \equiv 2n-1$, where w means the winding number around the body (tube) of torus (winding number around the hole of torus being equal to 2 for all $n \geq 3$).

In other words, to compare with empirical data, one has to fix appropriately the parameter q: it appears that to each number n, $n \geq 3$, there corresponds a prime root of unity $q = q(n) = e^{i\pi/(2n-1)}$. The latter turns into zero the polynomial $P_n(q) \equiv [n]_q - [n-1]_q$ that coincides with respective Alexander polynomial of the torus $(2n-1)_1$-knot [4,7]. In a sense, the polynomial $P_n(q)$ through its root $q(n)$ determines the strength of deformation at every fixed n and, due to this, serves as *defining polynomial* for the corresponding mass sum rule (and quarkonium).

Let us remark that a (purely heuristic) picture which assigns knot-like structures to some of fundamental particles was proposed in [15].

4. Octet baryon mass formulas: q-deformation lifts 'degeneracy'

The approach similar to that in [4] was applied to baryons $\frac{1}{2}^+$, including charmed ones, by adopting again $U_q(u_4)$ for the 4-flavor symmetry. However, unlike the case of vector mesons treated with 'compact' q-algebras $U_q(u_{n+1})$ of dynamical symmetry, here the irreps of 'non-compact' dynamical symmetry, realized by the q-algebra $\mathcal{A} \equiv U_q(u_{4,1})$, were first exploited [5].

For the standard GMO sum rule [10] for baryon octet masses

$$m_N + m_\Xi = \frac{3}{2} m_\Lambda + \frac{1}{2} m_\Sigma \tag{14}$$

known to hold with 0.58% accuracy, a q-deformed analogue was derived [5,7] which contains (14) as its classical $q = 1$ limit. This was first performed within the concrete irrep[1] $\mathcal{D}_0 \equiv D^+_{12}(p-1, p-3, p-4; p, p-2)$ (p is some fixed integer not entering final results). Note that for state vectors of octet baryons, embedded together with entire 20-plet of $U_q(su_4)$ into this dynamical irrep of $\mathcal{A}$, the Gel'fand-Zetlin basis vectors are applied: mass operator in its $U_q(su_3)$- and $U_q(su_4)$-breaking terms involves bilinearly those representation operators of $\mathcal{D}_0$ which are extra as regards the 'compact' subalgebra $U_q(su_4)$.

Evaluating octet baryon masses within this dynamical irrep $\mathcal{D}_0$ results in the q-analogue

$$m_N + \frac{1}{[2]-1} m_\Xi = \frac{[3]}{[2]} m_\Lambda + \Big(\frac{[2]}{[2]-1} - \frac{[3]}{[2]}\Big) m_\Sigma + \frac{A_q}{B_q} C_{\rm mass} \tag{15}$$

where $C_{\rm mass} = m_\Xi - m_\Lambda - [2](m_\Sigma - m_N)$,

$$A_q = ([2]-2)[2]^3([2]^2-3), \qquad B_q = ([2]^3 - [2]^2[4] + 3[5] - [3])([2]-1). \tag{16}$$

In this q-deformed mass relation, most significant is the remarkable rightmost term with A_q/B_q as prefactor. Due to A_q, this term vanishes for some values of q including the 'classical' one $q = 1$. The polynomial A_q, by its zeros, *determines* both the GMO (at $q = 1$, i.e. $[2]_q = 2$) and, at $q \neq 1$, some other mass sum rules. That is, A_q plays the key role: *it puts on equal footing* the GMO sum rule and some others. Namely,

at $q = 1$, Eq.(15) reduces to the standard GMO sum rule;
at $q = e^{i\pi/2}$ (now $[2]_q = 0$) we get the equality $m_\Sigma = m_\Lambda$ (rough one, empirically);
at $q = e^{i\pi/6}$ (then $[2]_q = \pm\sqrt{3}$) the relation (15) yields new mass sum rule

$$m_N + \frac{1+\sqrt{3}}{2} m_\Xi = \frac{2}{\sqrt{3}} m_\Lambda + \frac{9-\sqrt{3}}{6} m_\Sigma. \tag{17}$$

[1] Necessary details concerning irreps of the algebra $U_q(u_{4,1})$ are given in [5,7] too.

Comparison with data for baryon masses [13] shows the precision of 0.22% (2739.5 MeV versus 2733.4 MeV) - this is significantly better than in the case of GMO.

Such possibility to obtain new mass sum rules which are theoretically on equal footing with the GMO one, inspires to search for dynamical representations, either of the 'noncompact' $U_q(u_{4,1})$ or the compact $U_q(u_5)$ version of (q-deformed) dynamical symmetry, capable to yield relations like (15) but with differing sets of zeroes for relevant A_q. And, what is really surprising, the new sum rule (17) obtained within the specific dynamical irrep $\mathcal{D}_0$ of $U_q(u_{4,1})$ is still not the best one. It can be proved [16], using the compact dynamical $U_q(u_5)$, that among the admissible dynamical irreps there exist an entire series (labelled by an integer k, $0 \leq k < \infty$) of irreps capable to produce even more accurate MSR.

Proposition. An infinite series of q-deformed relations of the form (15), labelled by an integer n, can be obtained which differ in their defining polynomials: in n-th relation, the quantity $A_q = A_q(n)$ has, besides the two obligue roots $q = 1$ and $q = e^{i\pi/2}$ (i.e., $[2] = 0$), the additional root $q = e^{i\pi/n}$ exhibited by $[n] = 0$. The set of roots of the corresponding $B_q = B_q(n)$ has zero intersection with that of $A_q(n)$.

This can be shown by pointing out those concrete dynamical irreps of $U_q(u_5)$ which, after computations, yield just the relations mentioned in the proposition.

To find most optimal choise it is sufficient to analyse agreement of (15) (at vanishing term $\frac{A_q}{B_q} C_{mass}$) with data, using a kind of *'discrete' fitting procedure*. The results are shown in the table.

$\theta = \frac{\pi}{n}$	LHS, MeV	RHS, MeV	(RHS −LHS), MeV	$\frac{\lvert RHS-LHS \rvert}{RHS}$, %
π/∞	4514.0	4540.2	26.2	0.58
$\pi/30$	4518.31	4543.73	25.42	0.56
$\pi/12$	4546.41	4566.61	20.2	0.44
$\pi/9$	4581.54	4595.9	14.36	0.31
$\pi/8$	4607.77	4618.16	10.39	0.23
$\pi/7$	4653.58	4656.85	3.26	0.07
$\pi/6$	4744.88	4734.41	-10.47	0.22
$\pi/5$	4970.0	4928.82	-41.18	0.84

This table clearly reflects the fact of existence of infinite discrete set of mass formulas labelled by an integer n ($6 \leq n < \infty$), namely

$$m_N + \frac{1}{[2]_{q_n} - 1} m_\Xi = \frac{[3]_{q_n}}{[2]_{q_n}} m_\Lambda + \Big(\frac{[2]_{q_n}}{[2]_{q_n} - 1} - \frac{[3]_{q_n}}{[2]_{q_n}}\Big) m_\Sigma, \tag{18}$$

each of which shows better agreement with data than the classical GMO one (here $q_n = e^{i\pi/n}$).

Now it is easy to see: <u>the best choice</u> corresponds to $q = q_7 = e^{i\theta_8}$, $\theta_8 \equiv \pi/7$. <u>In this case</u>

$$[7]_{q_7} = 0, \qquad [3]_{q_7} = \frac{[2]_{q_7}}{[2]_{q_7} - 1},$$

and we get the new MSR in the form

$$m_N + \frac{m_\Xi}{[2]_{q_7} - 1} = \frac{m_\Lambda}{[2]_{q_7} - 1} + m_\Sigma \tag{19}$$

which shows (2582.6 MeV versus 2584.4 MeV) remarkable 0.07% accuracy! Moreover, such excellent precision is combined with an apparent simplicity: equality of the coefficients at m_N

and m_Σ, as well as those at m_Ξ and m_Λ. Due to that, Eq.(19) takes equivalent form (recall that $[2]_{q_7} = q_7 + (q_7)^{-1} = 2\cos\frac{\pi}{7}$)

$$m_\Xi - m_N + m_\Sigma - m_\Lambda = (2\cos\frac{\pi}{7})(m_\Sigma - m_N) \qquad (20)$$

which is of interest being very similar to the decuplet mass formula. see Eq.(21) below.

5. Decuplet baryon masses: essentially nonlinear $SU(3)$-breaking effects

In the case of baryons $\frac{3}{2}^+$ from the $SU(3)$ decuplet it is known that convensional (first order) symmetry breaking yields equal spacing rule (ESR) for masses of isoplet members in 10-plet [10]. Empirical data show that actually there is noticeable deviation from ESR:

$m_{\Sigma^*} - m_\Delta$	$m_{\Xi^*} - m_{\Sigma^*}$	$m_\Omega - m_{\Xi^*}$
152.6 MeV	148.8 MeV	139.0 MeV

It was shown in [6] (in analogy to octet case) that use of the q-algebras $U_q(su_n)$, instead of $SU(n)$, provides natural and simple improvement of situation. From evaluations of decuplet masses in two particular irreps of the dynamical $U_q(u_{4,1})$, the q-deformed mass relation

$$(1/[2]_q)(m_{\Sigma^*} - m_\Delta + m_\Omega - m_{\Xi^*}) = m_{\Xi^*} - m_{\Sigma^*}, \qquad [2]_q \equiv q + q^{-1}, \qquad (21)$$

was obtained. As proven in [6], this mass relation is a *universal one* - it follows within any admissible irrep (such that contains $SU_q(3)$-decuplet embedded in *20*-plet of $SU_q(4)$) of the dynamical $U_q(u_{4,1})$. Taking empirical masses [13] $m_\Delta = 1232$ MeV, $m_{\Sigma^*} = 1384.6$ MeV, $m_{\Xi^*} = 1533.4$ MeV , and $m_\Omega = 1672.4$ MeV we see that the formula (21), to be successfull, requires $[2]_q \simeq 1.95$. But this never holds for real q. On the contrary, pure phase $q = e^{i\theta}$ (in this case $[2]_q = 2\cos\theta$) with fixed $\theta = \theta_{10} \simeq \frac{\pi}{14}$ provides remarkable agreement with data.

Observe the apparent similarity of eq.(21) with mass relation

$$(1/2)(m_{\Sigma^*} - m_\Delta + m_\Omega - m_{\Xi^*}) = m_{\Xi^*} - m_{\Sigma^*} \qquad (22)$$

obtained earlier in different contexts: within tensor method of eightfold way [17]; in additive quark model with most general quark–quark pair interaction [18]; within the diquark-quark model of [19]; in the framework of modern chiral perturbation theory [20]. Such almost model-independence of (22) is based on the fact that each of these approaches takes into account in its specific way not only first, but also the 2nd order of $SU(3)$ symmetry breaking.

Universality of the q-deformed MSR (21) holds even in a wider sense: it extends to all admissible irreps (also containing *20*-plet of $SU_q(4)$) of the 'compact' dynamical $SU_q(5)$. Let us exemplify the decuplet masses by taking simplest although typical instance of irrep. Say, within most symmetric dynamical irrep $\{4000\}$ of $SU_q(5)$ calculation yields $m_\Delta = m_{10} + \beta$, $m_{\Sigma^*} = m_{10} + [2]\beta + \alpha$, $m_{\Xi^*} = m_{10} + [3]\beta + [2]\alpha$, $m_\Omega = m_{10} + [4]\beta + [3]\alpha$, and these obviously satisfy (21).

All these masses can be comprised by the formula with explicit dependence on hypercharge:

$$m_B = m(Y(B)) = m_{10} + [1-Y]\,\alpha + [2-Y]\,\beta. \qquad (23)$$

In the $q = 1$ limit it reduces to familiar formula

$$m_B = \tilde{m}_{10} + a\,Y \qquad (24)$$

with *linear dependence on hypercharge* Y (or strangeness); here $a = -\alpha - \beta$, $\tilde{m}_{10} = m_{10} + \alpha + 2\beta$. On the contrary, one can see that formula (23) *involves highly nonlinear dependence* of mass on hypercharge (it is Y alone which causes $SU(3)$ breaking in the decuplet case). Indeed, for q-number $[N]$ we have $[N] = q^{N-1} + q^{N-3} + \ldots + q^{-N+3} + q^{-N+1}$, ($N$ terms in total) that visualizes essential Y-nonlinearity of (23). Here is the principal difference between (21) and (22): as already noted, the latter accounts only terms which are linear and quadratic in Y.

6. Baryon masses and the Cabibbo angle

As it was found in the case of mesons, the parameter q, being pure phase at any n, is closely related with the issue of (singlet) mixing. Below we argue that q is connectible with the fundamental mixing angle encountered in particle theory - the Cabibbo angle θ_C.

It is known for a long time that mass relations may involve the Cabibbo angle. At the constituent (quark) level this is exhibited by the relation [11]

$$\tan^2\theta_C = m_d/m_s, \tag{25}$$

while at the composite level of pseudoscalar mesons this is seen, e.g., from the formula [21]

$$\tan^2\theta_C = \frac{m_\pi^2}{m_K^2\frac{F_K}{F_\pi} - m_\pi^2},$$

or from Weinberg's formula [22]

$$\frac{m_d}{m_s} = \frac{m_{K^0}^2 + m_{\pi^+}^2 - m_{K^+}^2}{m_{K^0}^2 - m_{\pi^+}^2 + m_{K^+}^2}$$

combined with (25). There exists even wider variety of formulas involving, besides meson masses from the octet 0^-, some *additional dimensionless parameter(s)* such as F_K/F_π, together with (or instead of) θ_C. Among these, most relevant for us is the relation [23]

$$m_\pi^2 + 3\frac{F_\eta^2 m_\eta^2}{F_\pi^2} = 4\frac{F_K^2 m_K^2}{F_\pi^2} \tag{26}$$

which generalizes the GMO mass formula. This is to be compared with pseudoscalar version (involving masses squared) of our q-deformed meson mass relation (4):

$$m_\pi^2 + \frac{[3]}{2[2]-[3]}m_{\eta_8}^2 = \frac{2[2]}{2[2]-[3]}m_K^2. \tag{27}$$

With appropriately fixed $q = q_{ps}$ Eq.(27) is satisfied, *without introducing singlet mixing angle*, if one puts the (mass of) physical η-meson in place of η_8 - just this is meant in what follows.

Besides common feature of (26) and (27) (both give the standard GMO in the corresponding limits $\frac{F_K}{F_\pi} \to 1, \frac{F_\eta}{F_\pi} \to 1$ and $q \to 1$), there is essential difference: the q-deformed one depends solely on q whereas Eq.(26) contains *two independent ratios*. However, with an additional constraint

$$1 + 3F_\eta^2/F_\pi^2 = 4F_K^2/F_\pi^2 \tag{28}$$

we have the juxtapositions

$$\frac{F_K^2}{F_\pi^2} \longleftrightarrow \frac{\frac{1}{2}[2]}{2[2]-[3]}, \qquad \frac{F_\eta^2}{F_\pi^2} \longleftrightarrow \frac{\frac{1}{3}[3]}{2[2]-[3]}. \tag{29}$$

Hence, Eqs.(26) and (27) become correspondent to each other.

On the other hand, the ratio F_K/F_π is related with the Cabibbo angle, see e.g., [24]. Due to (28), the same is true for F_η/F_π. From this fact combined with the above correspondence we conclude: the realistic value q_{ps} is *directly connectible* with the Cabibbo angle.

Now let us return to our q-dependent mass formulas for baryons: Eq.(19) in the octet $\frac{1}{2}^+$ sector and Eq.(21) in the decuplet $\frac{3}{2}^+$ sector. In our opinion, it is natural to extend to these baryonic cases the conclusion just made about possible connection $q \longleftrightarrow \theta_C$. In other words, we consider the values $\theta = \theta_{10} \simeq \pi/14$ (decuplet case) and $\theta = \theta_8 = \pi/7$ (octet case) for deformation parameter $q = e^{i\theta}$ as functions of θ_C: $\theta_{10} = f(\theta_C)$, $\theta_8 = g(\theta_C)$.

It is really surprising that the simplest choice

$$\theta_{10} = \theta_C \qquad \text{and} \qquad \theta_8 = 2\theta_C$$

provides excellent validity of mass sum rules (21) and (19). Now, adopting $\theta_8 = 2\theta_{10}$ as *exact equality* we get two implications:

(i) Since $\pi/7$ is strictly fixed value for θ_8, it is tempting to suggest the exact value $\pi/14$ for the Cabibbo angle;

(ii) Excluding (due to equality $\theta_8 = 2\theta_{10}$) the deformation parameter from the Eqs.(21) and (19) we obtain the following new octet–decuplet mass formula:

$$\frac{m_\Omega - m_{\Xi^*} + m_{\Sigma^*} - m_\Delta}{m_{\Xi^*} - m_{\Sigma^*}} = \left(3 + \frac{m_\Xi - m_\Lambda}{m_\Sigma - m_N}\right)^{1/2}, \tag{30}$$

which is satisfied with remarkable precision.

7. Concluding remarks

Let us recall again most important features of the presented q-deformed hadron mass formulas:

(i) universality of the q-deformed decuplet mass formula, Eq. (21);

(ii) possibility of optimal choice (at strictly fixed $q = e^{i\pi/7}$, cf. Eq.(19)) from infinite set of mass sum rules (18), thanks to degeneracy lifting caused by the *q-deformation* of $SU(n_f)$;

(iii) topological meaning behind meson 1^- q-deformed mass relations (3), embodied in knot invariants ascribed to quarkonia and in possibility to label flavors by a winding number.

(iv) in all three cases, q is pure phase (roots of unity); cases with baryons suggest a relation $q \leftrightarrow \theta_C$ realized in simplest form, and the exact value $\pi/14$ for θ_C.

As shown in [25], it is possible to relate θ_C with ratio of *subquark* (not quark) masses. However, the problem of determination of (the value and genuine origin) of θ_C *independently of values of quark masses* still remains to be solved. Just in this context the idea that space-time symmetries and/or internal symmetries may actually appear through quantum groups/algebras, we hope, will be very useful.

The last remark concerns appearance of knot invariants in connection with vector mesons. The fact that knot invariants are closely connected with quantum algebras is well-known [26]. However, our use of q-algebras to the issue of hadron masses and mass formulas gives a hint of which concrete torus knots are to be assigned to which concrete vector quarkonia.

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