JP0055043

Two-Fluid Equilibria with Flow

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Abstract

The formalism is developed for flowing two-fluid equilibria. The equilibrium system is governed by a pair of second order partial differential equations for the magnetic stream function and the ion stream function plus a Bernoulli-like equation for the density. There are six arbitrary surface function. There are separate characteristic surfaces for each species, which are the guiding-center surfaces. This system is a generalization of the familiar Grad-Shafranov system for a single-fluid equilibrium without flow, which has only one equation and two arbitrary surface functions. In the case of minimum energy equilibria, the six surface functions take on particular forms.

I. Introduction

We consider a two-fluid plasma for which the constituents of the system are the electron and ion fluids (hydrogen), and the electromagnetic fields. The objective is to establish the formalism for finding axisymmetric equilibria. This is a significant generalization of the equilibrium problem for a non-flowing single fluid (MHD) which is governed by the Grad-Shafranov (GS) equation. The GS system is a single second order partial differential equation for the magnetic stream function ψ; its characteristic surfaces satisfy $\psi = const$; and it involves two arbitrary surface functions, the toroidal field function $\phi(\psi)$ and the pressure function $p(\psi)$, where. The generalized equilibrium system should also be expressible in terms of stream functions, characteristic surfaces, and arbitrary surface functions. However since we are considering a two fluid with flow, the corresponding system is necessarily much more complicated. The task then is to identify the appropriate stream functions, the characteristic surfaces, and the arbitrary surface functions.

II. Equilibrium state equations

Expression in terms of scalar functions. Axisymmetry with the steady Faraday's law, Gauss's law of magnetism, and the steady continuity equations imply the existence of scalar functions of (r,z) such that

$$\mathbf{E} = -\nabla V_{es}\,; \qquad \mathbf{B} = \frac{\hat{\theta}}{r}\phi + \frac{\hat{\theta}}{r}\times\nabla\psi\,; \qquad n_\alpha \mathbf{u}_\alpha = \frac{\hat{\theta}}{r}\phi_\alpha + \frac{\hat{\theta}}{r}\times\nabla\psi_\alpha \qquad (1,2,3)$$

where the seven scalar functions (of r,z) are the electrostatic potential V_{es}, the toroidal field and flow functions ϕ and ϕ_α [$\alpha = i$(ions), e(electrons)], and the poloidal field and flow functions ψ and ψ_α ($\alpha = i, e$).

Fluid-field coupling. The fluid and field behavior are coupled through Ampere's and Gauss's laws, and the equations of motion of each species. The toroidal and poloidal components of the steady Ampere's law are

$$\Delta^* \psi = \frac{4\pi}{c}\sum_\alpha q_\alpha \phi_\alpha, \qquad \phi = -\frac{4\pi}{c}\sum_\alpha q_\alpha \psi_\alpha, \tag{4,5}$$

respectively. Gauss's law is

$$\nabla^2 V_{es} = -\sum_\alpha q_\alpha n_\alpha \tag{6}$$

The equations of motion take on a particularly simple form in terms of the generalized vorticity, $\mathbf{\Omega}_\alpha = m_\alpha \nabla\times\mathbf{u}_\alpha + (q_\alpha/c)\mathbf{B}$, which proved to be an important vector quantity in the modern (two-fluid) relaxation theory [1,2].

$$\nabla\left(m_\alpha u_\alpha^2/2 + q_\alpha V_{es}\right) + \frac{\nabla p_\alpha}{n_\alpha} = \mathbf{u}_\alpha \times \mathbf{\Omega}_\alpha. \tag{7}$$

Equation of state. An equation of state can greatly simplify the equation of motion. Two reasonable examples are barotropic species, $p_\alpha = p_\alpha(n_\alpha)$ and isothermal characteristic surfaces, $T_\alpha = T_\alpha(F)$ where the surfaces $F = const$ remain to be determined. In both cases we can define a useful function h_α. In the barotropic case $h_\alpha \equiv \int dp_\alpha/n_\alpha$, and in the isothermal surface case $h_\alpha \equiv kT_\alpha(F)\ln(n_\alpha)$. In the former case each equation of motion becomes

$$(barotropic) \qquad \nabla\left(h_\alpha + m_\alpha u_\alpha^2/2 + q_\alpha V_{es} + m_\alpha V_g\right) = \mathbf{u}_\alpha \times \mathbf{\Omega}_\alpha. \tag{8}$$

In the latter case a slightly less general form follows

$$(isothermal) \quad \left(\frac{\hat{\theta}}{r}\times\nabla F\right)\cdot\left[\nabla\left(h_\alpha + m_\alpha u_\alpha^2/2 + q_\alpha V + m_\alpha V_g\right) - \mathbf{u}_\alpha \times \mathbf{\Omega}_\alpha\right] = 0 \tag{9}$$

Surface functions. The three principal directions for each species of a two fluid are the θ direction, the direction of $\mathbf{\Omega}_\alpha$, and the mutually perpendicular direction, $\hat{\theta}\times\mathbf{\Omega}_\alpha$. These directions are different for the electrons and ions. Since $\nabla\cdot\mathbf{\Omega}_\alpha = 0$ one can express the generalized vorticity in terms of *new* toroidal and poloidal functions Φ_α, Ψ_α:

$$\mathbf{\Omega}_\alpha = \frac{q_\alpha}{c}\left(\frac{\hat{\theta}}{r}\Phi_\alpha + \frac{\hat{\theta}}{r}\times\nabla\Psi_\alpha\right) \tag{10}$$

From Eqs. 2,3 these are given by:

$$\Phi_\alpha = \phi + \frac{m_\alpha c}{q_\alpha n_\alpha}\Delta_n^* \psi_\alpha; \qquad \Psi_\alpha = \psi - \frac{m_\alpha c}{q_\alpha n_\alpha}\phi_\alpha \tag{11,12}$$

where the density-weighted GS operator is defined by $\Delta^* F = nr^2\nabla\cdot(\nabla F/nr^2)$. The ordinary GS operator Δ^* has the same form but without the density. The characteristic surface functions for each species are defined by $\Psi_\alpha = const$. Observe that the ion and electron surfaces are not the same. This choice is justified by the simplification in the components of the equations of motion in these directions. The component in the θ direction (angular momentum) leads to $\mathbf{\Omega}_\alpha \cdot \nabla\psi_\alpha = 0$, which implies

$$\psi_\alpha = G_\alpha(\Psi_\alpha) \tag{13}$$

where G_i and G_e are arbitrary surface functions. The parallel component ($\mathbf{\Omega}_\alpha$ direction) leads to Bernoulli equations for each species

$$h_\alpha + m_\alpha u_\alpha^2/2 + q_\alpha V_{es} + m_\alpha V_g = H_\alpha(\Psi_\alpha) \tag{14}$$

where H_i and H_e are arbitrary surface functions. The perpendicular component ($\nabla \Psi_\alpha$ direction) is

$$\frac{q_\alpha}{c} G'_\alpha \Phi_\alpha - \frac{q_\alpha}{c}\phi_\alpha = nr^2\left[H'_\alpha + kT'_\alpha(1-\ln n)\right] \tag{15}$$

where $(..)'$ denotes the derivative of a surface quantity with respect to Ψ_α. This form applies for isothermal surfaces. For barotropic species the kT'_α term is absent.

The surface function Ψ_α is identical to rP_θ, the angular momentum. In an axisymmetric geometry, the free motion of particles preserves rP_θ. Thus, the surfaces $\Psi_\alpha =$ *const* are surfaces along which free motion of particles takes place. This is significant because the *parallel* thermal conduction (coefficient $K_{\parallel\alpha}$) is in this direction. Thus in a hot plasmas with large $K_{\parallel\alpha}$, the temperature of species α should be nearly uniform on the surfaces $\Psi_\alpha =$ *const*; this is precisely the isothermal surface case.

III. Systems of equations for flowing two-fluid equilibria

Reducing assumption. Thus far we have made no reducing assumption beyond the basic two-fluid model. Suppose we assume quasineutrality ($n_i = n_e = n$) and massless electrons ($m_e \to 0$); then the system becomes

$$\Delta^* \psi = \frac{4\pi e}{c}(\phi_i - \phi_e) \qquad \phi = -(4\pi e/c)(\psi_i - \psi_e) \tag{16,17}$$

$$\psi_i = G_i(\Psi_i) \qquad \psi_e = G_e(\Psi_e) \tag{18,19}$$

$$h_i + m_i u_i^2/2 + eV_{es} + m_i V_g = H_i(\Psi_i) \qquad h_e - eV_{es} = H_e(\psi) \tag{20,21}$$

$$\frac{m_i}{n} G'_i \Delta^*_n \psi_i = -\frac{e}{c}\phi G'_i + \frac{e}{c}\phi_i + nr^2\left[H'_i + kT'_i(1-\ln n)\right] \tag{22}$$

$$0 = \frac{e}{c}\phi G'_e - \frac{e}{c}\phi_e + nr^2\left[H'_e + kT'_e(1-\ln n)\right] \tag{23}$$

which are the toroidal and poloidal Ampere's laws, the two angular momenta, the two Bernoulli equations, and the two perpendicular momentum equations, respectively. Recall that the surface functions are defined in Eq. 12. This system includes the arbitrary surface functions are $G_i(\Psi_i)$, $G_e(\psi)$, $H_i(\Psi_i)$, $H_e(\psi)$, and (in the isothermal surface case) $T_i(n)$, $T_e(n)$. The eight unknowns are ψ, ϕ, ψ_i, ϕ_i, ψ_e, ϕ_e, V_{es}, and n.

Minimum energy equilibria. The theory of two-fluid minimum energy states was developed elsewhere [SI3-PoP]. This theory is based on the minimization of the magnetofluid energy given fixed values of certain invariants. The magnetofluid energy is the sum of the total magnetic energy and the flow kinetic energy in the system.

$$W_{mf} = \int d\tau \left(B^2/8\pi + m_i n u_i^2/2\right) \tag{24}$$

The proper invariants for a two-fluid are the self helicities for each species, and (for axisymmetric system boundary) the global angular momentum:

$$K_\alpha = (c^2/8\pi e^2)\int d\tau\, \mathbf{P}_\alpha \cdot \nabla \times \mathbf{P}_\alpha \qquad L_\theta = \int d\tau\, r m_i n u_{i\theta} \tag{25,26}$$

respectively; where $\mathbf{P}_\alpha = m_\alpha \mathbf{u}_\alpha + q_\alpha \mathbf{A}/c$ is the canonical momentum for species $\alpha = i$(ions), e(electrons).

The minimum energy equilibria is a much more restrictive set so that the surface functions G_α, H_α, T_α ($\alpha = i,e$) are no longer arbitrary. In fact they depend on the Lagrange multipliers used in the constrained minimization procedure.

$$G_\alpha(\Psi_\alpha) = \lambda_\alpha \frac{c}{4\pi q_\alpha}\Psi_\alpha; \qquad H_\alpha(\Psi_\alpha) = -(q_\alpha/c)\Omega\Psi_\alpha + const; \qquad (27,28)$$

$$T_\alpha(\Psi_\alpha) = const \qquad (29)$$

This leads to the simplified system for a relaxed equilibrium composed of two second order partial differential equations for ψ *and* ψ_i:

$$\Delta^* \frac{e\psi}{m_i c} = \frac{1}{\ell_c^2}\left[\left(1-\lambda_e^2\ell_c^2\right)\frac{e\psi}{m_i c} - \left(\frac{1}{\lambda_i \ell_c^2}+\lambda_e\right)\frac{\psi_i}{n} - \Omega r^2\right] \qquad (30)$$

$$\frac{1}{n}\Delta_n^* \psi_i = \frac{1}{\ell_c^2}\left[\left(\frac{1}{\lambda_i}+\lambda_e\ell_c^2\right)\frac{e\psi}{m_i c} + \left(1-\frac{1}{\lambda_i^2\ell_c^2}\right)\frac{\psi_i}{n} - \frac{\Omega}{\lambda_i}r^2\right] \qquad (31)$$

and a Bernoulli equation for the density (in the isothermal surface case) is

$$\frac{2k(T_i+T_e)}{m_i}\ln n + \frac{|\nabla\psi_i|^2}{n^2 r^2} + \left(\frac{e\psi}{m_i c} - \frac{1}{\lambda_i \ell_c^2}\frac{\psi_i}{n} - \Omega\right)^2 = const \qquad (32)$$

Here $\ell_c = c/\omega_{pi} \propto n^{-1/2}$ is the collisionless skin depth, and $h(n) \equiv k(T_i+T_e)\ln n$. The auxiliary equations that accompany this system are

toroidal field $$\frac{e\phi}{m_i c} = -\lambda_e \frac{e\psi}{m_i c} - \frac{1}{\ell_c^2}\frac{\psi_i}{n} \qquad (33)$$

electron flow $$\frac{\psi_e}{n} = -\lambda_e \ell_c^2 \frac{e\psi}{m_i c}; \qquad \frac{\phi_e}{n} = \lambda_e^2\ell_c^2\frac{e\psi}{m_i c} + \lambda_e\frac{\psi_i}{n} + \Omega r^2 \qquad (34,35)$$

ion flow $$\frac{\phi_i}{n} = \frac{e\psi}{m_i c} - \frac{1}{\lambda_i \ell_c^2}\frac{\psi_i}{n} \qquad (36)$$

IV. Discussion of 2D flowing equilibria

These equations are the generalization of the GS system for the case of a flowing two-fluid plasma. The following properties, particularly of minimum energy equilibria stand out. (1) The pressure falls with rising velocity according to the Bernoulli relation (Re Eq. 32): thus a minimum energy state of any kind will have velocity rising toward the plasma edge. (2) The relaxed state of a pure FRC (no toroidal field, $\phi = 0$) with only toroidal flow ($\psi_i = 0$) is likely to have a size scale of order ℓ_c (at most a few times ℓ_c), since $\Delta^* \sim 1/\ell_c^2$ (by inspection of Eq.30). This suggests that such FRCs can only be two-fluid stable if the "s" parameter is limited to values of order unity. (3) However, a pure FRC ($\phi = 0$) may have poloidal flow if it is exactly matched by the poloidal flow of the electrons (cp Eqs. 33,34). In this case the size of the FRC may be somewhat larger than ℓ_c is the terms in square brackets on the right sides of Eqs. 30,31 are small, at least over most of the plasma. This would imply a particular relationship between the magnetic flux function ψ and the ion poloidal flux function ψ_i. If so then large-s FRCs that are two-fluid stable may be possible.

Future work on two-fluid, flowing equilibria will address the following topics. (1) A numerical solver for 2D (r,z) equilibria (both relaxed and partially relaxed) will be developed. The most promising approach to this appears to be the successive over-relaxation technique. These equilibria will then be available to test for stability and other properties. (2) The possibility of intermediate cases between pure FRCs (no toroidal magnetic field) and pure spheromaks (zero beta) can be investigated by adjusting the parameters of a relaxed equilibrium, λ_i, λ_e, and Ω. This may help to resolve the question why in experiments there appears to be a clear bifurcation between FRC and spheromaks.

References

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