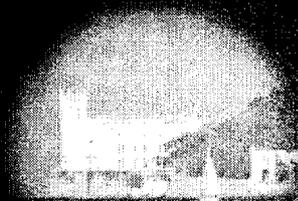


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**A NOTE ON HERMITIAN-EINSTEIN METRICS
ON PARABOLIC STABLE BUNDLES**

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**A NOTE ON HERMITIAN-EINSTEIN METRICS
ON PARABOLIC STABLE BUNDLES**

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Abstract

Let \bar{M} be a compact complex manifold of complex dimension two with a smooth Kähler metric and D a smooth divisor on \bar{M} . If E is a rank 2 holomorphic vector bundle on \bar{M} with a stable parabolic structure along D , we prove that there exists a Hermitian-Einstein metric on $E' = E|_{\bar{M} \setminus D}$ compatible with the parabolic structure, and whose curvature is square integrable.

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1 Introduction

Let \overline{M} be a compact Kähler manifold of complex dimension 2, let ω be a Kähler metric on \overline{M} . Let D be a smooth irreducible divisor in \overline{M} , and let $M = \overline{M} \setminus D$. The restriction of ω to M gives a Kähler metric on M . For simplicity, we assume in this paper that E is a rank 2 holomorphic vector bundle over \overline{M} and let $E' = E|_M$ be the restriction of the bundle E to M .

We define [LN] the notion of a stable parabolic structure on E (along D) and the notion of a Hermitian-Einstein metric on E' with respect to the restriction of the Kähler metric ω to M . We proved in [LN] that there exists a Hermitian-Einstein metric on E' compatible with the parabolic structure. We prove in this paper that there exists in fact a Hermitian-Einstein metric on E' (compatible with the parabolic structure) with the property that the curvature of the metric is square integrable (Theorem 2.2). In the case of a projective surface, the square integrability was proved by Biquard [B (4.2)] using a result of Simpson, while our proof is valid with the Kähler case also.

Once we know the curvature of the H-E metric is in L^2 , it is in fact in L^p for $p > 2$ (Remark 2.4), and hence the metric defines a parabolic bundle on \overline{M} as in [B, Theorem 1.1]. Since the metric is also compatible with the given parabolic structure, both parabolic structures are the same. Therefore proving the result that the curvature form of the H-E metric is in L^2 completes our earlier paper and this is the motivation for this note.

2 The existence of a H-E metric

In this section we shall prove our main theorem. See [LN] for the definitions, such as Hermitian-Einstein metrics, parabolic bundles, etc.

We need the following result proved in [LN], regarding the initial metric K_0 on E' .

Lemma 2.1 ([LN] Lemma 5.2 and Proposition 6.6) *Let $(E, D, \alpha_1, \alpha_2)$ be a parabolic bundle. Then there exists a Hermitian metric K_0 on $E' = E|_M$ such that*

a) *the curvature form of K_0 , F_{K_0} satisfies that $|F_{K_0}|_{K_0} \in L^p(M)$ for any $1 \leq p < p_0$ where $p_0 = \min\{\frac{2}{1-(\alpha_2-\alpha_1)}, \frac{2}{\alpha_2-\alpha_1}\}$ and $|\text{tr} F_{K_0}| \in L^\infty(M)$.*

b) *$\text{par deg } E_* =$ the analytic degree $d(E, K_0)$ and $(E, D, \alpha_1, \alpha_2)$ is parabolic stable if and only if (E, K_0) is analytic stable.*

Theorem 2.2 *Let \overline{M} be a compact Kähler manifold of complex dimension 2 and D a smooth irreducible divisor of \overline{M} . Let E be a rank 2 holomorphic vector bundle on \overline{M} with a parabolic structure $E_* = (E, D, \alpha_1, \alpha_2)$. If E_* is parabolic stable there exists a Hermitian-Einstein metric H on E' compatible with the parabolic structure and whose curvature form is square integrable over M .*

We shall modify Proposition 7.2 in [LN] and its proof to prove the theorem. The main

additional point is the derivation of an L^2 estimate for the curvature of the metrics arising in the heat flow.

Proposition 2.3 *Let $(E, D, \alpha_1, \alpha_2)$ be a parabolic bundle. Then there exists a Hermitian metric $K \in \mathcal{A}_{K_0}$ on $E' = E|_M$ satisfying the heat equation*

$$\begin{cases} K^{-1} \frac{dK}{dt} = -\sqrt{-1} \Lambda F_K^\perp \\ K|_{t=0} = K_0 \quad \text{and} \quad \det K = \det K_0 \end{cases}$$

on M , with $\|F_K|_K\|_{L^2(M)} \leq C$, $\|\Lambda F_K^\perp|_K\|_{L^p(M)} \leq \|\Lambda F_{K_0}^\perp|_{K_0}\|_{L^p(M)}$, and $\Lambda F_K|_K \in L^\infty(M)$ for any $t > 0$, $2 < p < p_0$, where K_0 is the metric constructed in Lemma 2.1, p_0 is the constant in Lemma 2.1, $C > 0$ is a constant depending only on K_0 .

Proof: Let $M_\beta = \{x \in M \mid |\sigma(x)| > \beta\}$, where $0 < \beta < 1$, and consider the above heat equation on M_β with Neumann boundary condition. More precisely, we consider, writing $h = K_0^{-1}K$, the equation

$$(\Delta_{K_0} - \frac{\partial}{\partial t})h = \sqrt{-1}h\Lambda F_{K_0}^\perp - \sqrt{-1}\Lambda\bar{\partial}hh^{-1}\partial_{K_0}h$$

on M_β with $h|_{t=0} = I$, $\det h = 1$, and $\frac{\partial}{\partial n}h|_{\partial M_\beta} = 0$, where $\Delta_{K_0} = -\sqrt{-1}\Lambda\bar{\partial}\partial_{K_0}$, $\frac{\partial}{\partial n}$ denotes the differentiation in the direction perpendicular to the boundary using the operator ∂_{K_0} .

In [LN] we used the Dirichlet boundary condition. We use Neumann boundary condition here so that we have the fact that $\frac{\partial}{\partial n}\Lambda F_K^\perp|_{\partial M_\beta} = 0$, obtained by applying $\frac{\partial}{\partial n}$ to both sides of the heat equation; this fact will enable us to apply the Stokes theorem for deriving the relation (2) below.

It was proved in [S] that this heat equation with Neumann boundary condition has a solution for all time. We denote the solution by K_β for each β . Let $h_\beta = K_0^{-1}K_\beta$.

By an argument similar to the one used in the proof of Proposition 7.2 in [LN], we can show that, there exist a sequence $\beta_i \rightarrow 0$ and a Hermitian metric $K \in \mathcal{A}_{K_0}$ such that for any relatively compact open subset Z , any $\delta > 0$, and any $R > 0$, $K_{\beta_i} \rightarrow K$ in $L^p_{2/1}(Z \times [\delta, R])$ for any $1 < p < \infty$. By the Sobolev embedding, we have $K_{\beta_i} \rightarrow K$ in $C^{1/0}(Z \times [\delta, R])$. Therefore the limit K satisfies the heat equation on $M \times (0, \infty)$ and thus belongs to $C^\infty(M \times (0, \infty))$. We can also show that $\|\Lambda F_K^\perp|_K\|_{L^p(M)} \leq \|\Lambda F_{K_0}^\perp|_{K_0}\|_{L^p(M)}$ and $\Lambda F_K|_K \in L^\infty(M)$ for any $t > 0$, $2 < p < p_0$, as we did in [LN].

Now we derive the L^2 bound of the curvature.

By the formula $\partial_{K_\beta} = \partial_{K_0} + h_\beta^{-1}\partial_{K_0}h_\beta$ and the fact that $\frac{\partial}{\partial n}\Lambda F_K^\perp|_{\partial M_\beta} = 0$ we can see that

$$\frac{\partial}{\partial n_\beta}\Lambda F_K^\perp|_{\partial M_\beta} = 0 \tag{1}$$

where $\frac{\partial}{\partial n_\beta}$ denotes the differentiation in the direction perpendicular to the boundary using the operator ∂_{K_β} .

Because $\det h_\beta = 1$, we have $\text{tr} F_{K_\beta} = \text{tr} F_{K_0}$ for all t , and

$$\frac{d}{dt} F_{K_\beta}^\perp = \frac{d}{dt} F_{K_\beta} = \sqrt{-1} \bar{\partial} \partial_{K_\beta} K_\beta^{-1} \frac{d}{dt} K_\beta = \bar{\partial} \partial_{K_\beta} \Lambda F_{K_\beta}^\perp.$$

Using the above identity we get

$$\begin{aligned} & \frac{d}{dt} \int_{M_\beta} |F_{K_\beta}^\perp|_{K_\beta}^2 dV \\ &= 2\text{Re} \int_{M_\beta} \left(\frac{d}{dt} F_{K_\beta}^\perp, F_{K_\beta}^\perp \right)_{K_\beta} dV \\ &= 2\text{Re} \int_{M_\beta} (\bar{\partial} \partial_{K_\beta} \Lambda F_{K_\beta}^\perp, F_{K_\beta}^\perp)_{K_\beta} dV \\ &= 2\text{Re} \int_{M_\beta} \nabla_{\bar{k}} \nabla_l (F_{K_\beta}^\perp)_{\gamma \bar{i}\bar{i}}^\delta \cdot (F_{K_\beta}^\perp)_{\delta \gamma \bar{k}\bar{l}} dV \\ &= -2\text{Re} \int_{M_\beta} \nabla_l (F_{K_\beta}^\perp)_{\gamma \bar{i}\bar{i}}^\delta \cdot \nabla_{\bar{k}} (F_{K_\beta}^\perp)_{\delta \gamma \bar{k}\bar{l}} dV \quad (\text{by (1) and Stokes theorem}) \\ &= -2\text{Re} \int_{M_\beta} \nabla_i (F_{K_\beta}^\perp)_{\gamma \bar{i}\bar{i}}^\delta \cdot \nabla_{\bar{k}} (F_{K_\beta}^\perp)_{\delta \gamma \bar{k}\bar{l}} dV \quad (\text{by Bianchi identity}) \\ &\leq 0, \end{aligned} \tag{2}$$

Letting $\beta \rightarrow \infty$, using Fatou's lemma, we get

$$\int_M |F_K^\perp|_K^2 dV \leq \int_M |F_{K_0}^\perp|_{K_0}^2 dV.$$

Since $|\text{tr} F_{K_0}| \in L^\infty(M)$ and $\text{tr} F_K = \text{tr} F_{K_0}$, we have

$$\int_M |F_K|_K^2 dV \leq C.$$

This completes the proof of the proposition.

Proof of the main theorem: As in the proof of Theorem 7.3 in [LN], the metric $K(t)$ converges to a Hermitian-Einstein metric H (as $t \rightarrow \infty$) compatible with the parabolic structure. On the other hand, by Proposition 2.3 we have $\| |F_K|_K \|_{L^2(M)} \leq C$. It follows from Fatou's lemma that $|F_H|_H \in L^2(M)$.

Remark 2.4 *Once we know that the curvature form of the H-E metric is in L^2 , then it belongs in fact to L^p , for $p \geq 2$, as implied by the result of Sibner-Sibner [SS Theorem 5.1 and Theorem 5.2] (see [B (4.2)]).*

Remark 2.5 *Conversely, if E' is a holomorphic vector bundle over M and admits a Hermitian-Einstein metric H with $\| |F_H|_H \|_{L^p(M)} < \infty$, for some $p > 2$, one can show (cf. [B], Theorem 1.1) that E' can be extended to a holomorphic vector bundle E over \bar{M} with a parabolic structure along D and such that H is compatible with the parabolic structure. Moreover E is parabolic polystable (cf. [B] and [LN]).*

Remark 2.6 *We can use our existence theorem to derive a Bogomolov Chern number inequality for parabolic bundles (cf. [L]). For the case of projective varieties see Biswas [Bs].*

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