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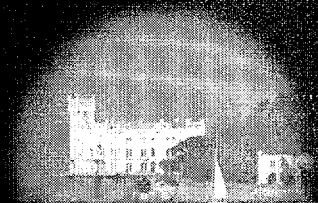
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**QUASI-EQUILIBRIUM INTERPRETATION  
OF AGING DYNAMICS**

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**QUASI-EQUILIBRIUM INTERPRETATION  
OF AGING DYNAMICS**

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**Abstract**

We develop an interpretation of the off-equilibrium dynamical solution of mean-field glassy models in terms of quasi-equilibrium concepts. We show that the relaxation of the “thermoremanent magnetization” follows a generalized version of the Onsager regression postulate of induced fluctuations. We then find the rationale for the equality between the *fluctuation-dissipation ratio* and the rate of growth of the configurational entropy close to the asymptotic state, found empirically in mean-field solutions.

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## I. INTRODUCTION

Aging is a scaling dynamical regime characteristic of glassy systems [1]. In this regime, typical features of equilibrium systems, such as the asymptotic absence of macroscopic heat currents, coexist with non-stationary aspects such as the dependence of the correlation and response functions on the system's "age", i.e. the time spent in the low temperature phase. The solution of mean-field spin glass models [2–5] has given a general framework to understand aging phenomena, and has produced detailed predictions, which have been verified in numerical simulations of long-range [6] and short-range [7] glassy systems. A characteristic prediction of this solution is the existence, at low temperature, of a dynamical regime where extensive quantities depending on the configuration of the system at a single time (one time observables in the following) are well thermalized, (or evolve very slowly), while two time correlation function and susceptibilities exhibit non-stationary scale invariant behavior.

Despite this coherent theoretical scheme, and recent progress in linking aging dynamics to the nature of the equilibrium regime [8], a physical understanding of some fundamental aspects of aging dynamics is still lacking. While the investigation of both equilibrium and asymptotic off-equilibrium regimes give solutions that show unexpected coincidences, all efforts to interpret aging as a quasi-equilibrium condition have been thwarted by facts such as:

1. No matter how large we take the aging time if we then wait long enough the system eventually wanders away from any finite region of phase space [2].
2. Two identical systems starting from the same condition at any given aging time will always come apart as far as possible [9].

In this paper we try to make ends meet. In particular, we try to give a physical and intuitive explanation of the link between the so called "fluctuation dissipation ratio" (FDR) (the factor  $x(q)$ ) and the Parisi function (in Sherrington-Kirkpatrick like models) or the derivative of the configurational entropy close to the threshold state (in p-spin-like models). Our work should be understood as a physical interpretation of mean field aging dynamics in terms of a quasi-equilibrium scenario and not as an alternative derivation of the results of the theory.

We show that a modified version of the Onsager postulate on regression of fluctuations applies to the relaxation of the magnetization in thermoremanent magnetization experiments. The observed anomalies in the response are then analyzed. In this paper we discuss aging mainly in the case of a "one time sector" approximation, which is the dynamic counterpart of the "one step replica symmetry breaking" (1RSB) approximation in the equilibrium analysis. This is exact in models like the p-spin model, while it is only an approximation, and a rather crude one, when continuous replica symmetry breaking is present, like in the Sherrington-Kirkpatrick (SK) model or the random manifold model. We will refer to the first class of models as "p-spin-like" and to the second as "SK-like". We discuss only this "one time sector" approximation to unify the argument, and simplify the notation. But we expect the reader to be able to generalize it without effort.

The picture that emerges from our analysis is simple and intuitive: the age of an aging systems determines the rate of entropy decrease, i.e. the flow rate of heat towards the thermal bath. A small force in the linear response regime cannot change this rate. Acquiring a non zero magnetization means entropy reduction which has then to be compensated by an increase (or reduced decrease) of the free-energy associated with the spin-couplings. As a consequence, the response becomes proportional to the growth of the logarithm of the number of *quasi-states* (to be defined later) with free-energy. It is as if the slow degrees of freedom respond to external forces by sampling states that lie above in free-energy while they are blocked from exploring those that are at the same or at a lower level. This paper presents what we believe are convincing arguments in favor of this assertion. As a byproduct the time scale dependent effective temperatures will appear [10] and their connection with the derivative of the logarithm of the number of quasi-states (or configurational entropy) with respect to free-energy is explained. [11]

We organize the paper as follows: in section II we review some properties of aging in mean-field. In section III we discuss our definition of quasi-equilibrium and how it relates to dynamics. In section IV we recall Onsager regression postulate and generalize it to aging systems. In section V we discuss the origin of the FDR in SK-like models, while we treat the case of p-spin-like models in section VI. Finally we present a summary and the conclusions.

## II. A SHORT REVIEW

We consider a mean-field spin-glass quenched at a given time  $t = 0$  into the glassy phase. For simplicity we will imagine that the degrees of freedom consist of Ising spins  $S_i = \pm 1$ ,  $i = 1, \dots, N$  ( $N \rightarrow \infty$ ). We are interested on the long time dynamics of such a system, i.e. on a regime where it has thermalized for a long time  $t_w$  before any measurements. The long time limit  $t_w \rightarrow \infty$  is always taken after the thermodynamic limit.

The free-energy, and its derivatives with respect to the control parameters (e.g. temperature) tend to some asymptotic time-independent values. More interesting is the behavior of observables depending on two time variables, such as correlation and response functions. Aging behavior manifests itself as an asymptotic non-stationary dependence on time of these quantities.

Let us consider the spin-spin time dependent autocorrelation function

$$C(t, u) = \frac{1}{N} \sum_{i=1}^N S_i(t) S_i(u) \quad t > u \gg t_w \quad (1)$$

A first, short time, dynamical regime is obtained considering the difference  $\tau = t - u$  finite to derive stationary correlations  $C_{st}^*(\tau)$ . Let us denote  $q_{EA}$  the long  $\tau$  limit of  $C_{st}^*$ . In the aging regime  $C(t, u)$  relaxes below  $q_{EA}$ . In the “one time sector” approximation scheme, the correlation function in this regime can be written as [2],

$$C(t, u) = C_{ag}(h(u)/h(t)) \quad (2)$$

where  $h(\cdot)$  is an increasing function not derivable from the present theory, and  $t, u$  are large with  $\lim_{t, u \rightarrow \infty} h(u)/h(t) = \text{finite}$ . The formulae in the two regimes are summarized in:

$$C(t, u) = C_{st}(t - u) + C_{ag}(\mathbf{h}(u)/\mathbf{h}(t)) \quad (3)$$

where  $C_{st}(t - u) = C_{st}^*(t - u) - q_{EA}$  is a monotonically decreasing function equal to  $1 - q_{EA}$  for  $t - u = 0$  and tending to zero for  $t - u \rightarrow \infty$ , while  $C_{ag}(\mathbf{h}(u)/\mathbf{h}(t))$  is equal to  $q_{EA}$  for  $\mathbf{h}(u)/\mathbf{h}(t) = 1$  ( $u = t$ ) and tends to  $q_0$  for  $\mathbf{h}(u)/\mathbf{h}(t) \rightarrow 0$ . In order to simplify the notation we will suppose that the “time reparametrization”  $\mathbf{h}(t)$  is the identity  $\mathbf{h}(t) = t$ , and  $C_{ag}(\mathbf{h}(u)/\mathbf{h}(t)) = C_{ag}(u/t)$ . We will also suppose  $q_0 = 0$ , but this will not affect any of our following arguments.

We stress that the form (3) of the correlation function implies that if we fix the value of  $u$  and let  $t$  run, the time spent at the value of the correlation equal to  $q_{EA}$  is much larger than the time needed to reach it. This is an aspect of the “time scale separation” observed in glassy systems that will play a crucial role in our discussion.

We are also interested in the behavior of the linear response function,

$$R(t, u) = \frac{1}{N} \sum_{i=1}^N \frac{\delta}{\delta h_i(u)} \langle S_i(t) \rangle |_{h=0} \quad t > u \gg t_w \quad (4)$$

and the corresponding integrated function

$$\chi(t, u) = \int_{t_w}^u ds R(t, s) \quad t > u \gg t_w \quad (5)$$

which represent the susceptibility at time  $t$  in a “thermoremanent magnetization” experiment in which a constant small field has acted from the time  $t_w$  up to time  $u$ . By the condition  $u \gg t_w$  we mean  $t_w/u \rightarrow 0$  for  $t_w \rightarrow \infty$ . The linear response theory requires that the limit  $h \rightarrow 0$  be taken *before* sending the time  $u$  to infinity.

While in the stationary regime the fluctuation dissipation relation  $R(t, s) = R_{st}(t - s) = \beta \frac{\partial C(t, s)}{\partial s}$  is verified, in the aging regime the relation is substituted by having a non trivial fluctuation-dissipation ratio (FDR):

$$x(q) = \lim_{\substack{t, s \rightarrow \infty \\ C(t, s) = q}} \frac{TR(t, s)}{\frac{\partial C(t, s)}{\partial s}}, \quad (6)$$

which turns out to coincide with the function  $x(q)$  that appears in the replica approach that in principle applies to equilibrium of different kinds [12,13]. In the one sector scenario  $x(q)$  is constant all through the aging regime.

The FDR  $x(q)$  verifies the mathematical properties of a cumulative probability distribution, a feature that has been explained in recent work where it has been shown that there is a deep connection between the dynamic properties during aging and the property of ergodicity breaking in equilibrium [8]. Using only the hypothesis of equilibration of one time observables (OTO) and the existence of a linear response regime for the correlation functions (stochastic stability), it was proved that the function  $x(q)$  is related to the function  $P(q)$  describing the statistics of equilibrium pure states [14] through the equation:

$$P(q) = \frac{dx(q)}{dq}. \quad (7)$$

The theorem was originally formulated for finite dimensional systems, where the OTOs are guaranteed to thermalize but can be generalized to mean-field long range models of the SK-like class. A different situation is found in p-spin-like models [15] where one of the hypotheses of the theorem is violated because the asymptotic value of the dynamic energy is higher than the one of the states dominating the partition function.

### III. THE QUASI-EQUILIBRIUM HYPOTHESIS

From now on we will work on the “one time sector” approximation described in the previous section.

Let us consider a large time  $u$ , and the corresponding spin configuration  $S_i(u)$ . Our previous observations suggest that the value of  $q_{EA}$  can be used to decompose the spin configuration in a “fast part” and a “slow part” according to

$$S_i(u) = [S_i(u) - m_i(u)] + m_i(u) \quad (8)$$

where the slow variable  $m_i(u)$  can be estimated immediately from the running average

$$m_i(u) \simeq \frac{1}{\Delta t} \int_u^{u+\Delta t} dw S_i(w) \quad \text{with} \quad C(u + \Delta t, u) = q_{EA}. \quad (9)$$

We will see later how to improve on this estimate. By (8) and (9) we obtain correctly the corresponding decomposition of the correlation function in two time domains

$$\langle [S_i(t) - m_i(t)][S_i(u) - m_i(u)] \rangle = C_{st}(t - u) \quad (10)$$

while

$$\langle m_i(t)m_i(u) \rangle = C_{ag}(u/t). \quad (11)$$

any other decomposition, obtained averaging the spins over times such that the value of  $C(u, v)$  is different from  $q_{EA}$ , would mix  $C_{st}$  and  $C_{ag}$ .

Through this decomposition one can define a dynamical notion of “quasi-state” in which the system (almost) equilibrates before relaxing further. The quasi equilibrium hypothesis can be formulated considering the probability distribution of finding the system in a given configuration of the slow and the fast variables at time  $t$  induced by the thermal noise and the flat distribution of initial conditions

$$P_t(\{S_i\}, \{m_i\}) = \langle \prod_i \delta(S_i(t) - S_i) \delta(m_i(t) - m_i) \rangle_{\substack{\text{thermal noise} \\ \text{initial conditions}}} \quad (12)$$

which can be written as:

$$P_t(\{S_i\}, \{m_i\}) = P_t(\{S_i\}|\{m_i\})P_t(\{m_i\}). \quad (13)$$

All the known properties of the dynamical solution, and in particular the short time response to external perturbations, are consistent with the proposition that the conditional probability  $P_t(\{S_i\}|\{m_i\})$  becomes independent of time, and takes asymptotically the form of a restricted Boltzmann measure:\*

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\*To be more precise we should define the measure in such a way that each  $S_i$  has average  $m_i$ . One can check with a detailed calculation that this condition is automatically fulfilled by the measure (14).

$$P(\{S_i\}|\{m_i\}) = \frac{e^{-\beta H(\{S_i\})} \delta(\sum_i S_i m_i - N q_{EA})}{\sum_{\{S_i\}} e^{-\beta H(\{S_i\})} \delta(\sum_i S_i m_i - N q_{EA})}. \quad (14)$$

We stress that the same property manifestly would not hold had we chosen time scales such that  $C(t, u) < q_{EA}$  in the average (9). This proposition is implicit in the definition of the aging regime, where there is no ambiguity in the definition of  $q_{EA}$ . The measure should be restricted to the transverse configuration space projecting out those directions along which the system evolves (*transverse* quasi-states) where the energy landscape is flat or has negative eigenvalues. In these conditions, the free-energy of the quasistate (see below), as well as the value of  $q_{EA}$  entering in (14) are close to their asymptotic values, but still depend on time. We can safely assume that in the asymptotic regime the number of negative directions become vanishingly small. Notice that, if we take two macroscopically different sets of slow variables  $\{m_i\}$  and  $\{m'_i\}$  then, by construction, the corresponding conditional probabilities  $P(\{S_i\}|\{m_i\})$  and  $P(\{S_i\}|\{m'_i\})$  are mutually orthogonal. This can be easily understood from the fact that the mutual overlap among a configuration with non-zero weight in the first distribution and a configuration with non-zero weight in the second one is almost surely smaller than  $q_{EA}$ . It is therefore convenient to think of a discretized  $m_i$ -sphere such that the centers of the neighboring cells correspond to disjoint quasistates. In general, we expect different quasistates to define disjoint regions in configuration space and that the union of all such regions define a partition of all relevant configurations. We will use  $\alpha$  as the index of the quasistate which of course will change with the slow time. In (9) therefore  $m_i(u)$  should rather read  $m_i^\alpha$  with  $\alpha$  a function of  $u$ . The finiteness of  $\Delta t$  limits the accuracy of the running average estimate. To derive a better estimate of  $m_i^\alpha$  we could clone the trajectory from time  $u$  on and take a weighted average along all trajectories.

It is useful to define thermodynamic quantities such as the dynamical free-energy:

$$\mathcal{F}_t = \sum_{\alpha} P_t(\{m_i^\alpha\}) [F(\{m_i^\alpha\}) + T \log(P_t(\{m_i^\alpha\}))] \quad (15)$$

where  $T$  is the temperature of the thermal bath, and

$$F(\{m_i^\alpha\}) = \int \prod_i dS_i P(\{S_i\}|\{m_i^\alpha\}) [H(\{S_i\}) + T \log(P(\{S_i\}|\{m_i^\alpha\}))] \quad (16)$$

is the free-energy of the (transverse)  $\alpha$  quasi-state.

Observe that  $\mathcal{F}_t$  includes the average free-energy of the quasi-states

$$F_t = \sum_{\alpha} P_t(\{m_i^\alpha\}) F(\{m_i^\alpha\}) \quad (17)$$

and a slow entropy term

$$S_t = - \sum_{\alpha} P_t(\{m_i^\alpha\}) \log(P_t(\{m_i^\alpha\})). \quad (18)$$

An explicit computation shows that, due to the disjointness property of the quasistates, the sum  $S_t + F_t$  coincides with the entropy of the distribution  $P_t(\{S_i\})$ .

We can identify  $\mathcal{F}_t$  with the dynamical free-energy  $\int \prod_i dS_i P_t(\{S_i\}) [H(\{S_i\}) + T \log P_t(\{S_i\})]$ .

This last quantity is known to decrease in any process verifying detailed balance. In our case, due to the white average over the initial conditions we expect in addition both  $F_t$  and  $S_t$  to decrease with time.

For typical trajectories extensive quantities are self-averaging and therefore the free-energy  $F_t$  is a well defined function. The asymptotic value,  $F_\infty$ , is the free-energy of the equilibrium state for SK-like systems, and the free-energy of the threshold TAP solutions for the p-spin class.

The role of  $S_t$  in the dynamical relaxation is not immediate. By construction, it does not say anything about the number of quasi-states accessible starting from a generic point of a trajectory at time  $t$ . In fact we expect that the support of  $P_t(\{m_i\})$  decomposes in non-overlapping, mutually inaccessible, regions of phase space that become more and more isolated as time advances.

By inverting the relation  $F_t$  among free-energy and time, we can define  $\mathcal{S}(F) = S_{t(F)}$  and derive that  $\mathcal{S}(F_\infty)$  is the extensive part of the configurational entropy of ground states in SK-like models (zero in this case) and the configurational entropy of the threshold states in the p-spin. The *dynamical* entropy that we defined weights different regions of phase space according to their basins of attraction. We are here using the observation that being the threshold states identical in all their properties they must also have the same basin of attraction. Furthermore with respect to those states which appear in exponentially smaller numbers there is the additional observation that their basin of attraction cannot be so much larger as to compensate for their smaller multiplicity.

For large time, the support of  $P_t(\{m_i\})$  will be in regions of small TAP gradient. We can calculate  $\mathcal{S}(F)$ :

$$S_t = \mathcal{S}(F_t) = \left. \frac{\partial \mathcal{S}(F)}{\partial F} \right|_{F_\infty} (F_t - F_\infty) + \mathcal{S}_\infty \quad (19)$$

In our scenario  $\mathcal{S}(F_t)$  measures the multiplicity of quasistates at  $t$  which as before will have equal basin of attraction. A detailed calculation in the p-spin model is developed in the Appendix. It shows that if we compute the multiplicity of minima of the modulus of the gradient of the TAP free energy then their number and derivative are continuous across the threshold value. Therefore we propose to identify *the quasistates with the minima of the gradient of the TAP free energy*. For SK-like models we conjecture instead  $\mathcal{S}(F)$  to be equal to the number of stable TAP excites states at level  $F$ .

With this identification we now write  $\left. \frac{\partial \mathcal{S}(F)}{\partial F} \right|_{F_\infty}$  equal to  $\beta x$ . It is one of those remarkable coincidences, referred to in the introduction, that the same value of  $x$  appears in the anomalous FDR. We will interpret this coincidence in the following sections.

This identification of the quasistates is crucial. An explicit calculation of the dynamical entropy could check its validity but unfortunately with our present techniques such calculation is not feasible.

A strong hint in favor of it comes however from the study of the equilibrium dynamics at temperatures slightly larger then the dynamical transition temperature  $T_d$ . For  $T - T_d \ll T_d$  there is a similar separation of 2 time scales controlled by the ratio  $\frac{T - T_d}{T_d}$ . This allows the definition of dynamical quasistates along lines similar to the ones followed in the non-equilibrium case. The free-energy obtained by considering the collection of the quasistates of



appropriate energy, should be coherent with the direct computation from the partition function. By an explicit computation that we sketch in the appendix, we verified that, up to second order corrections in  $T - T_d$ , the thermodynamic entropy coincides with the TAP internal entropy plus the configurational entropy of the saddles.

#### IV. REGRESSION OF FLUCTUATIONS AND ONSAGER POSTULATE

In order to discuss the behavior of the response function we consider the set-up of “thermoremanent magnetization” (TRM) experiment [1]. The system is allowed to age in a small field  $h$  acting from time  $t_w$  to time  $u$  such that  $C(u, t_w) \rightarrow 0$ . At later times  $t > u$  one detects the magnetization  $M(t) = \frac{1}{N} \sum_{i=1}^N S_i(t) = h\chi(t, u)$ .<sup>†</sup> Our set-up differs slightly from the one considered usually in the literature, in which the field acts directly from the quenching time. We switch the field on at time  $t_w$  because we find conceptually clearer to discuss the behavior of the magnetization starting from a situation where the system is already in the asymptotic regime. We notice that the response to any arbitrarily varying field  $h(t)$  can be expressed as a linear superposition of TRM magnetizations.

In order to discuss the decay of  $M(t)$  we will show that a generalization of the Onsager postulate of normal regression of fluctuations applies to the dynamical off-equilibrium process [16]. The principle, originally stated for equilibrium systems, concerns the behavior of *macroscopic quantities* and states that in the linear response regime one can not distinguish the regression of a spontaneous fluctuation of a certain quantity from the regression from the same value when imposed through a constraint on the equilibrium measure. Onsager’s postulate means that for a large system, a spontaneous fluctuation must have the characteristic of the *most probable* fluctuations and therefore correspond to constrained minimization of the free-energy. This is equivalent to free-energy minimization in a conjugated field, thus leading to an immediate derivation of the fluctuation-dissipation theorem.

More in detail the argument can be phrased as follows [16]. Consider a thermodynamic system at equilibrium and a given macroscopic (extensive) quantity  $\alpha$  which takes the value zero at equilibrium. Be  $\gamma$  the corresponding conjugate intensive variable. Suppose that at time zero the quantity  $\alpha$  has a small but extensive spontaneous fluctuation  $\alpha_0$ . This will occur with exponentially small probability, but when it occurs the subsequent evolution of  $\alpha(t)$  will be independent of the thermal noise, i.e.  $\alpha(t) = E(\alpha(t)|\alpha_0)$ , where we denoted by  $E(\cdot|\alpha_0)$  the conditional expectation over the trajectories for fixed  $\alpha_0$  at time zero. As  $\alpha_0$  is small, we can write

$$\alpha(t) = E(\alpha(t)|\alpha_0) = A(t)\alpha_0 \tag{20}$$

Denoting by  $E_{\alpha_0}(\cdot)$  the average over the distribution of  $\alpha_0$ , and  $C_\alpha(t) = E(\alpha(t)\alpha(0))$  the correlation function, it follows that  $A(t) = C_\alpha(t)/E_{\alpha_0}(\alpha_0^2)$ . Notice that the typical values of  $\alpha_0$

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<sup>†</sup>It should be kept in mind that  $M(t)$  denotes the magnetization at time  $t$  but can depend on both  $t$  and  $u$ .

entering in the correlation function are of the order  $\sqrt{N}$ , while in the relation (20) we consider values of order  $N$ . The validity of the analysis above relies on the smoothness of the probability distribution of  $\alpha_0$  in the crossover region, assumption which is at the heart of linear response theory.

As one is conditioning (20) by the value of  $\alpha_0$  only, then the overwhelming majority of the configurations  $\mathcal{C}$  giving rise to the fluctuation are the ones “typical” of the restricted canonical distribution

$$\frac{e^{-\beta H(\mathcal{C})} \delta(\alpha(\mathcal{C}) - \alpha_0)}{\int d\mathcal{C} e^{-\beta H(\mathcal{C})} \delta(\alpha(\mathcal{C}) - \alpha_0)}. \quad (21)$$

which is equivalent to

$$\frac{e^{-\beta(H(\mathcal{C}) - \gamma\alpha(\mathcal{C}))}}{\int d\mathcal{C} e^{-\beta(H(\mathcal{C}) - \gamma\alpha(\mathcal{C}))}} \quad (22)$$

in which  $\gamma$  is fixed by:  $\langle \alpha \rangle_\gamma = \alpha_0$ . Since to the linear order in  $\gamma$  we have  $\langle \alpha \rangle_\gamma = \beta\gamma \langle \alpha^2 \rangle_{\gamma=0}$ , it follows that the relaxation of  $\alpha(t)$  induced by the field is given by:

$$\alpha(t) = \beta\gamma C_\alpha(t). \quad (23)$$

which is the fluctuation-dissipation theorem in its integral form.

Here we would like to show how a generalized form of the regression principle holds in aging dynamics where the time scale separation suggests that, besides the fluctuations of the instantaneous magnetization  $M(u)$  one should also consider possible fluctuations of the running global magnetization  $m(u)$ , defined as

$$m(t) = \frac{1}{N} \sum_{i=1}^N m_i(t). \quad (24)$$

We consider the conditional expectation value of the magnetization at time  $t$  given small values of the instantaneous and running magnetizations  $M(u)$  and  $m(u)$ :  $E(M(t)|M(u), m(u))$ . This can again be expanded to the first order:

$$E(M(t)|M(u), m(u)) = A(t, u)[M(u) - m(u)] + B(t, u)m(u). \quad (25)$$

and the functions  $A$  and  $B$  can be fixed by a hypothesis of continuity, leading to

$$E(M(t)|M(u), m(u)) = \left[ \frac{C_{st}(t-u)}{1-qEA} [M(u) - m(u)] + \frac{C_{ag}(u/t)}{qEA} m(u) \right]. \quad (26)$$

where we have used  $\langle (M(u) - m(u))^2 \rangle = \frac{1-qEA}{N}$  and  $\langle m(u)^2 \rangle = \frac{qEA}{N}$ .

Onsager’s argument demonstrates two things:

1. that the decay of a spontaneous fluctuation with time is governed by the correlation function.
2. that a fluctuation induced by a conjugate field will decay as a spontaneous fluctuation if the probability distribution defining the state of the system immediately after the induced

field is turned off is equal to the unperturbed probability distribution projected on the hypersurface defined by the equations:

$$\frac{1}{N} \sum_{I=1}^N m_i(u) = m(u); \quad \frac{1}{N} \sum_{I=1}^N S_i(u) = M(u), \quad (27)$$

where now,  $m(u)$  represents the value of an average as (9) for times immediately *after* the field is turned off.

This second condition is consistent with our scenario of quasi-equilibrium in the dynamical relaxation process. In the next sections we will deal with the problem of computing the slow part of the magnetization induced by a field. We will first discuss the case of SK-like models whose OTOs during the dynamical relaxation tend to the ground state values. Then we will discuss those systems where the asymptotic state is different from the ground state. In this case, the argument is further complicated by the extensive multiplicity of the threshold states.

## V. THE CASE OF SK-LIKE MODELS.

We first recall that the equilibrium analysis of these models [17] in the “one step replica symmetry breaking” approximation determines the multiplicity of states at low free-energy  $F$  [18] as

$$\mathcal{N}(F)dF = e^{\beta x(F-F_{GS})}dF \quad (28)$$

where  $F_{GS}$  is the ground state free-energy and  $x$  is the Parisi parameter in this approximation.

We then quote from the dynamical solution the expression of the magnetization in the TRM experiment described in the previous section:

$$M(t) = C_{st}(t-u)\beta h + C_{ag}(u/t)\beta h x. \quad (29)$$

The comparison of this with equation (26) tells us the following remarkable fact: the action of an external field  $h$  from time  $t_w$  to  $u$  produces at  $u$  a state of the system (in the sense of a measure in the microscopic variables) which is identical to the one we can obtain through infinite realizations of the thermal noise and selection of those trajectories with

$$\begin{aligned} M(u) - m(u) &= \beta h(1 - q_{EA}) \\ m(u) &= \beta h x q_{EA}. \end{aligned} \quad (30)$$

Therefore, thanks to the use of Onsager’s postulate it is enough to calculate the response at time  $u$  immediately after the magnetic field has been turned off. We have assumed  $t_w$  and  $u$  sufficiently large so that the system is in a quasi-state with free-energy  $F$  slightly larger than the one of the ground state. Eqs. (30) separate the response to the magnetic field in two components: a) inside the same quasi-state the more probable configurations will change and b) the quasi-state will change. The response a) is the equilibrium intrastate response and is trivial. To isolate b) we imagine turning off the magnetic field at time  $u$  and then waiting a

finite time  $\Delta t$  such that  $C_{st}(\Delta t)$  is  $q_{EA}$  while still  $\Delta t/u$  is zero. Then we know that the system has gone from one quasi-state at time  $t_w$  to another at time  $u + \Delta t$  both defined with zero magnetic field. The distribution of (zero magnetic field) quasi-states with this free-energy is given by (28). Each of them may have a magnetization, uncorrelated from the free-energy and with variance  $\langle m^2 \rangle = q_{EA}/N$ . The typical number of quasi-states with free-energy density  $F$  and magnetization  $m$  is therefore given by

$$\mathcal{N}(F, m) = e^{\beta x(F - F_{GS})} e^{-N \frac{m^2}{2q_{EA}}} \quad (31)$$

implying that

$$\mathcal{S}(F, m) = \beta x(F - F_{GS}) - \frac{m^2 N}{2q_{EA}} \geq 0 \quad (32)$$

We first note that if we send  $u$  to infinity before sending  $h$  to zero, i.e. we consider fields such that the induced magnetization  $m$  verifies  $\beta x(F_u - F_{GS}) \ll m^2 N / 2q_{EA}$  we can derive the result (30) in a quite straightforward way. In fact, we obtain that a non zero magnetization has to be compensated by an increase of free-energy so as to keep the configurational entropy  $\mathcal{S}(F, m)$  non negative

$$F - F_{GS} = \frac{m^2 N}{2q_{EA}\beta x}. \quad (33)$$

Along this line in the  $F, m$  plane the state with lowest total free-energy  $F - hmN$  has

$$F = F_{GS} + \frac{\beta N h^2 q_{EA} x}{2}. \quad (34)$$

implying  $m = \beta x h q_{EA}$ .

The interpretation of this result is particularly illuminating. Turning on the magnetic field is a way of making energy available to the system. The thermal bath would normally absorb part of this energy. However this is possible only if the entropy of the system decreases in the process. This cannot happen here as by hypothesis the available entropy is much smaller than the one required to increase  $m$ . We conclude that the equilibration must occur only between the magnetic free-energy  $hmN$  and the unperturbed, zero magnetic field  $F$ .

With this argument in mind we can now understand the limit more relevant to the dynamical approach. In this case  $F(u) - F_{GS}$  is extensive and large with respect to the potential energy introduced by the external magnetic field. In this situation there is, formally, enough entropy to allow the magnetization to reach the value of equilibrium with the thermal bath  $m = \beta h q_{EA}$ . However, with the same token one would argue that the thermal bath could have absorbed that entropy to decrease the spin-spin interaction energy. We know that this is not the case, or rather that entropy/heat is absorbed at a certain rate basically determined by the barriers. The external force is uncorrelated with the direction of relaxation of the system, and therefore it is reasonable to assume that the turning on of the magnetic field *will not modify the rate of entropy decrease (heat transfer to the thermal bath)*. We conclude as before that the equilibration must occur between the magnetic free-energy  $hmN$  and  $F$ . In formulas if we call  $F^h(u)$ ,  $F(u)$  the free-energy (associated with the inter spins couplings) that the system would reach in the presence of the magnetic field or in its absence at time  $u$ , then:

$$S(F^h(u), m(u)) = S(F(u)) \quad (35)$$

so that:

$$\beta x(F^h(u) - F_{GS}) - \frac{m(u)^2 N}{2q_{EA}} = \beta x(F(u) - F_{GS}) \quad (36)$$

The previous argument now follows minimizing  $F^h(u) - Nhm(u)$ .

We remark that both entropy reductions refer to the same degrees of freedom, and therefore respond on the same time scale. The result is that the thermal bath acts as if it was uncoupled while the two forms of (free-) energy mutually equilibrate.<sup>‡</sup> In other words the transition time to higher free-energy states is much smaller than the one required to go to equal or lower free-energy states.

This argument is so crucial to our picture that we feel it necessary to try to confirm it with a detailed model of the dynamical process.

Let us imagine the dynamical trajectory from a (large) time  $u$  to a time  $t$  such that  $C(t, u) \approx 0$ . We discretize the dynamics in  $k$  steps such that  $u = t_0 < t_1 < \dots < t_k = t$  such that  $C(t_{i+1}, t_i) = q_{EA} - \epsilon$  with  $\epsilon$  small. At time  $t_i$  the system will have free-energy  $F_i$  and  $F_{i+1} - F_i$  will be small but extensive. The model we make of the dynamical process consists in assuming that when going from time  $t_i$  to time  $t_{i+1}$  the system can access different quasi-states with the lower one at free-energy  $F_{i+1}$  and the higher ones distributed exponentially

$$\mathcal{N}(F) = e^{\lambda(F - F_{i+1})} \quad (37)$$

while the probability of transition to a state with free-energy  $F$  is proportional to

$$e^{-\beta F} \quad (38)$$

The model is consistent for  $\lambda < \beta$ , (otherwise the free-energy would grow with time) and incorporates the following two features:

1. The decrease in extensive free-energy is deterministic

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<sup>‡</sup>This represents an instance of the recent proposal that a system and a thermometer responding in the same time scale will equalize their effective temperatures [10]. In fact the inverse temperature of the magnetic field interaction energy is  $dS/dE_h = d(-m^2/2q_{EA})/d(-mh) = m/(q_{AE}h) = \beta x$ . However our entropic interpretation suggests that time scales will strongly depend on  $\beta x$ , the lower the effective temperature, the slower the evolution of the system; if two different aging systems starting with different effective temperatures and equal time scales are put in contact, they will quickly develop different time scales before equilibrating.

2. If we fix the initial condition, the increase in entropy in a single step is finite.<sup>§</sup> In fact this can be calculated following the lines in [19] for the Random Energy Model, with the result

$$\Delta S = \Gamma'(1) - \frac{\Gamma'(1 - \lambda/\beta)}{\Gamma(1 - \lambda/\beta)} \quad (39)$$

In  $k$  steps the entropy generated will be  $k\Delta S$  and therefore negligible with respect to  $Nm^2/(2q_{EA})$ . Although non-extensive,  $k\Delta S$  can be arbitrarily large, thus explaining the divergence of two cloned trajectories. In this model we have heavily used the self-averaging character of the macroscopic quantities along the trajectories. It is again clear that the only way to develop a magnetization is by compensating it with an increase in the (zero magnetic field) free-energy.

## VI. P-SPIN LIKE MODELS

For p-spin like models even the limit  $t_w \rightarrow \infty$  before  $h \rightarrow 0$  is non-trivial. In fact it is well known that for this kind of systems the properties of the quasi-states encountered in the dynamics are closer and closer to these of the threshold TAP states, which have extensive configurational entropy  $S_{th}$ . If this entropy would be accessible in the dynamical process the equality (19) would be valid with  $S_\infty = S_{th}$ . The condition that the total configurational entropy at time  $u$  be positive would then read

$$S_{th} + \beta x(F_u - F_{th}) - \frac{m^2 N}{2q_{EA}} \geq 0 \quad (40)$$

which could be satisfied even if  $F_u - F_{th}$  is small and negligible in front of  $-\frac{m^2 N}{2q_{EA}}$ . In more physical terms we can say that among the  $e^{S_{th}}$  states there are  $e^{S_{th} - \frac{m^2}{2q_{EA}}}$  states with magnetization  $m$ . If all these states were available to a single trajectory the response would be normal,  $m = \beta h q_{EA}$ . If we want the response to be anomalous we must show that the system, while wandering in phase space has no access to the configurational entropy.\*\*

The logarithm of the number of states in the vicinity of any given TAP state has been computed by Cavagna, Giardinà and Parisi in ref. [20] in the case of the  $p$ -spin model. Below threshold all states are isolated; there are no states closer than a given distance  $q_{EA} - q_{max}$ .  $q_{max}$  is a level-dependent overlap which tends to  $q_{EA}$  at threshold. Right at the threshold, the logarithm of the number of states as a function of the distance ( $q_{EA} - q$ ) is

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<sup>§</sup>The entropy about which we are talking here correspond to a the dynamical probability in which the initial condition is fixed, and is therefore increasing with time.

\*\*A moment of reflection reveals that otherwise the system, wandering in such a large space, would pass to lower lying states and relax below  $F_{th}$ .

$$N\Sigma(q) \propto N(q_{EA} - q)^5. \quad (41)$$

Let us again imagine a discretization of the dynamics in which at each step the system can jump a distance  $\delta = (q - q_{EA})$ . Then after  $n$  steps the log of the number of accessible states would be at most of the order of  $n \delta^5$ .<sup>††</sup> On the other hand the distance traveled will be:  $\Delta = n \delta$  if all the steps are in the same direction and  $\Delta = \sqrt{n} \delta$  if the steps are uncorrelated. In both cases it is easy to see that if we take the limit  $\delta \rightarrow 0$  and  $n \rightarrow \infty$  fixing  $\Delta$  we find that  $\log(\mathcal{N})/N$  goes to zero ( $n \delta^5 \rightarrow 0$ ).

Notice that the argument is based on the scarcity of states in the vicinity of a given state. This should be a generic feature for p-spin like systems other than the  $p$ -spin model.

Having eliminated the configurational entropy from the balance, the argument proceeds as in the case of SK-like models.

Let us conclude by pointing out that threshold states with large magnetization (of order  $\beta h q_{EA}$ ) do exist, but are non-critical in the presence of the field. Therefore with probability one such states would be isolated and unreachable.

## VII. SUMMARY AND CONCLUSIONS

The main point of our analysis has been to give an explanation of the anomalous response function. We have found the physical origin of the equality between the FDR and the growth rate of the configurational entropy close to the asymptotic state. The value of the anomalous response can be traced to the lack of available entropy when the system is close to the low lying states. Our interpretation clarifies the relation among equilibrium properties and off-equilibrium dynamics. For p-spin-like systems we have argued that the extensive configurational entropy of the threshold states does not play any thermodynamical role. We have seen that the classical Onsager's argument on the equivalence between the regression of a spontaneous, noise-caused, fluctuation of the magnetization and the one induced by an external field can be generalized to aging systems.

Our analysis can be summarized by saying that in aging systems the rate of entropy decrease is a function of age and does not change due to small forces. Thus the balance is always between the value of the unperturbed free-energy and the one of the perturbation, without taking into account the thermal bath. We expect this conclusion to hold also in short range systems with aging.

In spin glass materials, one time observables (OTO) equilibrate, and the picture we have developed relates to the structure of configuration space close to the ground state. In aging

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<sup>††</sup>This estimate could correspond to a severe double counting, as one can realize applying the estimate to finite dimensional Brownian motion. In infinite dimensional problems we expect it to give essentially the correct result. But, in any case, we only need it as an upper bound in our argument.

experiment of structural glasses on the other hand, OTOs are far from their asymptotic values. Still, one can observe quasi-scaling aging dynamics on two time observables. The structure of the phase space visited on this time scale can not be related to “true” asymptotic properties of the system. We would like to speculate that even here the fluctuation-dissipation ratio, which could be a slowly varying function of time, is related to the derivative of available phase space with free energy also varying along the dynamical path. This could be true even if the system would eventually reach equilibrium on a different time scale where FDT is asymptotically obeyed.

Finally the case of multiple time sectors or multiple replica symmetry breakings will need trivial modifications.

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## VIII. APPENDIX

The aim of this appendix is two fold. We first show that in the p-spin model the derivative of the configurational entropy of the saddles is continuous at the threshold. Then, we prove that above  $T_d$  the paramagnetic state can be seen, to the first order in  $T - T_d$  as a collection of quasi-states identifiable with the points of least gradient of the TAP free-energy.

Let us start from the expression of the TAP free-energy for the p-spin model [21]

$$F_{TAP}[m_i = \sqrt{q}S_i] = E_0 q^{\frac{p}{2}} - \frac{\beta (1 - p (1 - q) q^{-1+p} - q^p)}{4} - \frac{\log(1 - q)}{2\beta} \quad (42)$$

where  $E_0$  is the angular part of the energy as a function of the angular variables  $S_i$ . It is well known that while one can find stationary points of the angular part for all the values of  $E_0$  in the range  $|E_0| > -E_{GS}$ . Conversely, at finite temperature one finds solutions for the radial parts only in the range  $-E_{th} > |E_0| > -E_{GS}$ . The overwhelming majority of these solution are free-energy minima.

The stationary points of the angular part for  $-E_{th} < |E_0|$  turn out to be saddles, with a number of unstable directions which depends on  $E_0$ . The number of stationary points as a function of  $E_0$  is given by [22]

$$\Sigma(E_0) = \frac{1}{2} \left[ \frac{2 - p}{p} - \frac{2}{p^2 z^2} + \frac{(-1 + p) z^2}{2} - \log\left(\frac{p z^2}{2}\right) \right] \quad (43)$$

where  $z$  is an auxiliary variable given by

$$-\frac{E_0}{-1 + p} - \frac{\sqrt{E_0^2 - E_{th}^2}}{-1 + p} \quad (44)$$

For the saddles  $E_0 > E_{th} = -\left(\sqrt{2} \sqrt{\frac{-1+p}{p}}\right)$  the formula become complex. This is due to the fact that the Hessian which appears in the computation [22] has negative eigenvalues and one



has to compute the absolute value of its determinant. As suggested in [20] this can be done just taking the real part of expression (43), which gives the parabolic shape

$$\Sigma(|E_0| < -E_{th}) = -E_0^2 \frac{(p-2)}{2(p-1)} + \frac{1}{2} \log(p-1) \quad (45)$$

An explicit computation using this formula shows that the  $E_0$ -derivative of (43) and (45) is continuous at the threshold energy.

Let us now pass to our second task. We would like to identify the quasistates close to threshold as points of minima of TAP gradient. Unfortunately we were not able to prove this directly in the aging regime at low temperature, for we do not know how to compute the dynamical entropy. We start then from the observation that for temperatures higher, but close to  $T_d$  one observes slowing down of the dynamics with time scale separation which is less and less ambiguous as  $T \rightarrow T_d$ . So we can define dynamic quasistates even above  $T_c$ , where the role of a small but finite  $T - T_d$  is similar to the role of a finite  $t_w$  in the low temperature dynamics. We put these quasistates in relation with the TAP free-energy, supposing that they coincide with the points of least TAP gradient for fixed internal energy equal to the paramagnetic value  $-\beta/2$ . These are saddle points of the angular part, while the radial part is an inflection point, i.e. we fix  $q$  in the value of the minimum of  $dF_{TAP}/dq$ . By explicit computation from (42) and (45) we find that the total free-energy  $F_{TAP} - T\Sigma(E_0)$  is equal to the paramagnetic free-energy  $-\beta/4$  up to terms which are of the second order in  $T - T_d$ . For instance for  $p = 3$  one finds that  $F_{TAP} - T\Sigma(E_0) = -\beta/4 - 8\sqrt{2/3}(T - T_d)^2$ .

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