NUMERICAL AND SPECTRAL INVESTIGATIONS OF NOVEL INFINITE ELEMENTS

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Abstract

Exterior problems of time-harmonic acoustics are addressed by a novel infinite element formulation, defined on a bounded computational domain. For two-dimensional configurations with circular interfaces, the infinite element results match well both analytical values and those obtained from other methods like DtN. Along with the numerical performance of this formulation, of considerable interest are its complex-valued eigenvalues. Hence, a spectral analysis of the present scheme is also performed here, using various infinite elements.

Introduction

Numerical methods for solving exterior problems of time-harmonic acoustics are based on either integral representations (surveyed by Shaw [6]) or domain-based computation (see [4]). Here, a novel approach to the latter method, with infinite elements, is evaluated.

Infinite Element Formulation

The exterior boundary-value problem of acoustics involves finding the spatial component of the acoustic pressure, which satisfies the Helmholtz equation in an unbounded domain. It is further subject to specified boundary conditions at inner boundaries and the Sommerfeld radiation condition at infinity.

To obtain a solution to this boundary value problem, the unbounded domain is partitioned by a smooth artificial boundary into a bounded inner domain and its unbounded complement, the outer domain. The original solution is similarly decomposed into inner and outer fields. The inner field is approximated by standard finite element functions, and the outer field by unconventional infinite element functions. A simple functional is defined to weakly enforce continuity between the two fields at the interface. By setting its first variation equal to zero, a symmetric formulation is obtained [2,3].

In this formulation, infinite elements mesh only the finite element interface. Further, weak continuity between the finite elements and the infinite elements at the artificial boundary is automatically ensured within the framework of this scheme. Consequently, *the infinite element mesh need not be the restriction of the finite element mesh to the artificial boundary.* This flexibility may offer a significant advantage in practice.

The infinite element approximation of the outer field is achieved by means of continuous outgoing interpolatory functions which satisfy the homogeneous Helmholtz equation exactly. The basic approximation is piecewise linear in the circumferential direction. For higher order approximations, the basic interpolation can be enriched by a bubble, or based on a trigonometric series. The types of infinite elements are thus termed "linear," "enriched," and "trigonometric." A more elaborate elucidation of these concepts can be found in Harari et. al. [3,5].

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Numerical Experiments

In order to validate the procedure outlined above, we consider problems of infinite circular cylinders of radius a, to which analytical solutions exist. Soft boundary conditions are specified on the wet surface. A circular interface is located at $R = 2.5\,a$. Linear quadrilateral finite elements mesh the inner domain and infinite elements, linear unless stated otherwise, the outer domain (Figure 1). The geometrically non-dimensionalized wave number is $ka = 1$ and the resolution is approximately 13 nodes per wavelength in the finite element mesh.

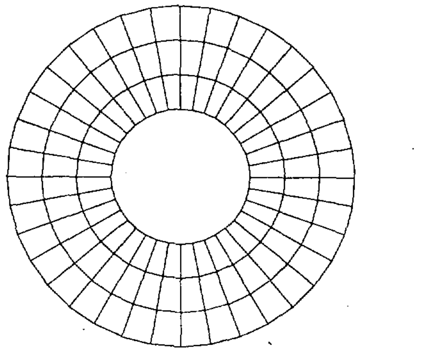

Figure 1: The computational domain

1. Circumferentially harmonic radiation

For a circumferentially harmonic load $\cos n\theta$, the exact solution is $u = H_n^{(1)}(kr)\cos n\theta / H_n^{(1)}(ka)$. For the fifth circumferential mode, $n = 4$, we present contours of the real parts of the analytical and numerical solutions in Figure 2 for linear infinite elements. The inner and outer solutions along the interface are compared to DtN [1] in Figure 3. Both these figures demonstrate the good performance of the formulation.

2. Radiation from a sector of a cylinder

Next we consider the problem with a constant, inhomogeneous loading on a sector of the cylinder. For our comparisons, an arc of 40° is considered. Figure 4 shows the contours of the real parts of the analytical and numerical solutions, while Figure 5 depicts the solution along the interface. The accuracy of the formulation is well manifest in these figures.

The errors relative to interpolated analytical solutions u^I, measured in the energy norm, are tabulated in Table 1, corresponding to various infinite elements.

Spectral Analysis

A spectral analysis of a numerical formulation establishes how closely it represents certain physical char-

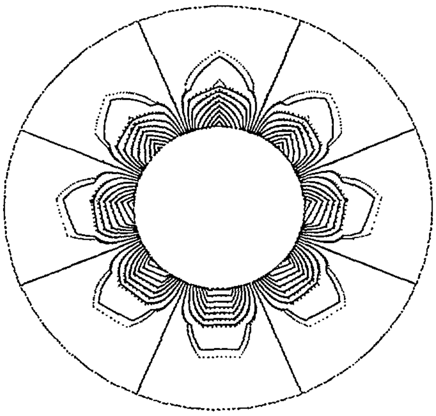

Figure 2: Harmonically loaded cylinder (solid contours represent numerical solution , dashed contours represent analytical solution)

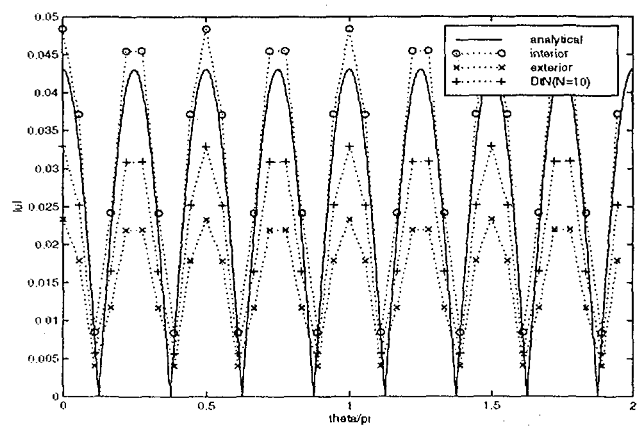

Figure 3: Harmonically loaded cylinder: pressure at the interface

acteristics of the problem it models. It can be shown that the complex eigenvalues of continuous problems in bounded domains, which are equivalent to exterior problems, have *negative-definite imaginary parts*, related to proper enforcement of the radiation condition. The performance of iterative solvers on the matrix equations, depends on the condition number and, more generally, the clustering of eigenvalues.

An eigenvalue analysis of the present scheme is conducted in comparison to the well-established DtN method [1]. Two forms of the coefficient matrix are considered. The eigenvalues and condition number κ of the *full* matrices are relevant to the performance of iterative solvers. Due to the weak coupling, the terms related to the infinite elements are statically *condensed* out of the coefficient matrix, before investigating the sign of the imaginary part of its eigenvalues.

Table 1: Relative errors in energy norm

Problem	Lin.	Enr.	Trg.	DtN
Harmonic	5.56%	5.52%	6.03%	5.62%
Sector	10.98%	10.85%	11.90%	10.86%

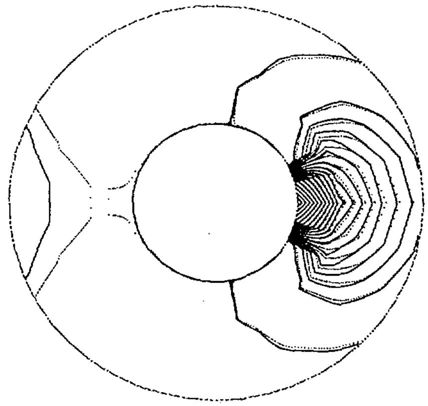

Figure 4: Radiation from a sector of a cylinder (solid contours represent numerical solution, dashed contours represent analytical solution)

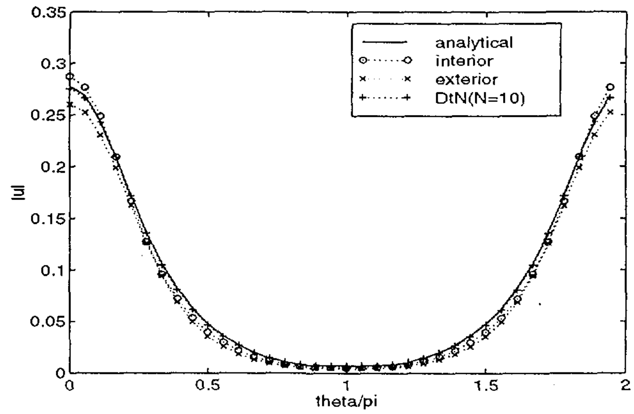

Figure 5: Radiation from a sector of a cylinder: pressure at the interface

The eigenvalues of the previous problem of an infinite circular cylinder are examined. Figure 6(a) shows the eigenvalues of the full and condensed matrices for the linear infinite element. A comparison of eigenvalues between the linear infinite element (condensed) and the DtN scheme in Figure 6(b) reveals the qualitative agreement of the two methods. As required by the physics of the problem, the eigenvalues of both these methods have negative imaginary components.

Higher-order infinite elements are also studied. The eigenvalues from the full and condensed matrices of higher-order infinite elements are compared to those of the linear infinite element in Figure 7. It can be seen that the eigenvalues of higher-order infinite elements possess a few positive imaginary parts, which is uncharacteristic of the continuous problem. Attempts continue to explain this spurious behavior, given the numerical accuracy exhibited by these elements.

Conclusions

A novel infinite element approach to solving exterior problems of time-harmonic acoustics is investigated and validated. Numerical results demonstrate the good performance of this scheme. Linear infinite elements exhibit the proper spectral behavior. Attempts continue to understand the seemingly spurious eigenvalues of higher-

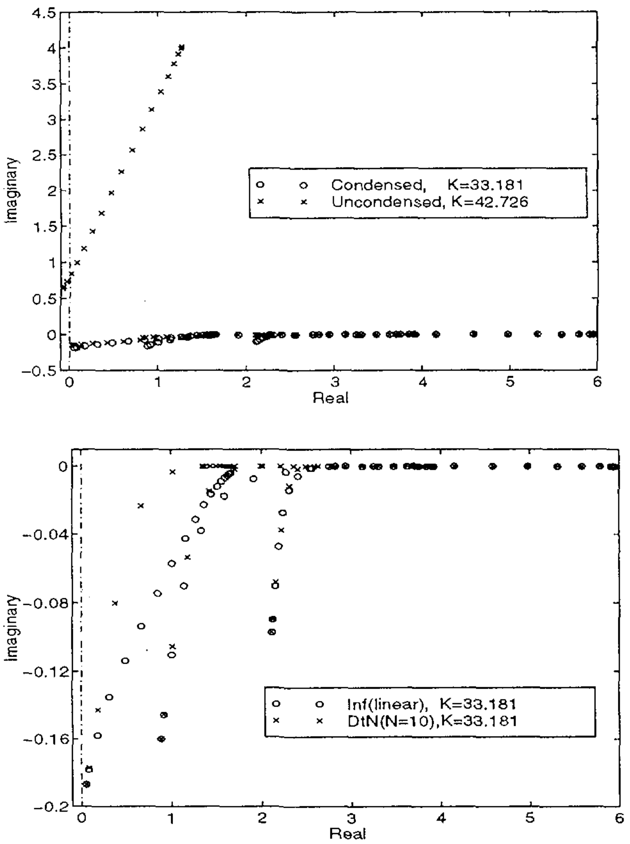

Figure 6: Eigenvalues of linear infinite elements: (a) condensed vs. full; (b) condensed vs. DtN

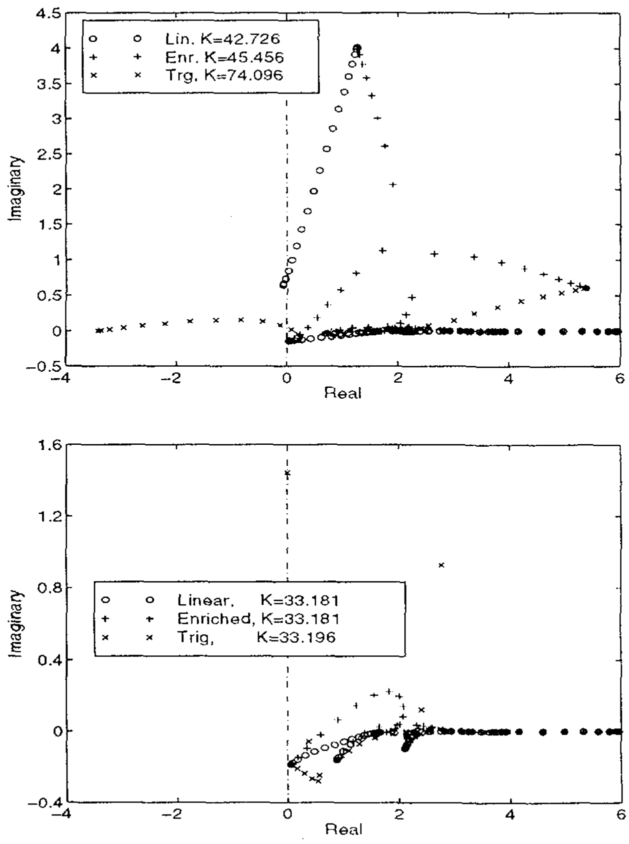

Figure 7: Eigenvalues: (a) full; (b) condensed

order infinite elements.

Acknowledgment

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