THE MOTION OF A DEFORMABLE BODY IN BOUNDED FLUID

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ABSTRACT

The Hamiltonian formalism for the motion of a deformable body in an inviscid irrotational fluid is generalized for the case of the motion in a bounded fluid. We found that the presence of the boundaries in a liquid leads to the chaotization of the body's motion. The "memory" effect connected with a free surface boundary condition is also accounted for.

INTRODUCTION

The classical hydrodynamic problem of the motion of submerged bodies, (Lamb 1945 §3-5) has recently received renewed interest stimulated by the rapidly growing industry of underwater vehicles (UV) operating in deep sea. For this purpose, the problem of a non-linear control of rigid autonomous UV, based on a simple potential hydrodynamic model in an *unbounded* space, has been extensively and rigorously elaborated in recent years by a number of investigators. It is important to stress that the dynamical model of UV operating in an *unbounded* otherwise quiescent fluid, should be extended towards more realistic hydrodynamical circumstances in nature, involving, say, canals, rivers, ports, harbors, offshore structures and also the proximity of free-surfaces or interfaces.

HAMILTONIAN FORMALISM

A corresponding dynamical system arising from the motion of a deformable body with a velocity $\mathbf{U}$ and angular velocity $\mathbf{\Omega}$ in an *unbounded* potential flow field (otherwise at rest) is connected with the group $E(3)$ of motions of the Euclidean space R^3 (see, for example, Leonard & Marsden 1997). This group represents the configuration space of the dynamical system comprised by a deformable body moving in a potential stream and referred to a coordinate system attached to the moving body. On the phase space of the system there exist the following 6 coordinates M_1, M_2, M_3, p_1, p_2, p_3 which represent the corresponding generalized impulses in the Hamiltonian formalism

$$\mathbf{p} \equiv (v\rho_b \hat{1} + \hat{T})\mathbf{U} + \hat{Z}\mathbf{\Omega} + \mathbf{K}_d(t),$$

$$\mathbf{M} \equiv \hat{Z}^T\mathbf{U} + (\hat{I} + \hat{R})\mathbf{\Omega} + \mathbf{P}_d(t).$$

These are in fact just the linear momentum (angular momentum) of the body plus the Kelvin impulse (Kelvin impulse-couple) induced in the fluid due to the motion and deformation of the body. We introduce here the 3 by 3 added-mass tensors $\hat{T}$, $\hat{Z}$ and $\hat{R}$ which correspond to the translational, coupled and rotational motion respectively (Lamb 1945). We also denote by $\mathbf{K}_d$ ($\mathbf{P}_d$) the deformation Kelvin-impulse (Kelvin impulse-couple) (see, for details, Miloh & Galper 1993) and select the density of the fluid to be unity, i. e. $\rho_f = 1$.

It can be shown (see also Galper & Miloh 1995) that also for more general cases (deformable body embedded in a non-uniform ambient flow field moving near some external or moving boundaries) the dynamical equations of the body's motion still remain Hamiltonian with the generalized impulses given again as the generalized impulses of the body plus the generalized impulses *induced in the fluid due to the body's motion.*

The presence of some rigid boundaries breaks the $E(3)$ group of symmetry of the system. The first integrals $\mathbf{p}^2$ and $\mathbf{M} \cdot \mathbf{p}$ which are valid for the motion in an unbounded space are no longer conserved and the corresponding spatial coordinates expressed in the body attached coordinate system, should be considered.

We choose as the generalized coordinates for the body the position of its centroid $\mathbf{X}$ referred to the attached coordinate system. We also introduce an orthogonal operator $\hat{Q}(t)$, which instantaneously connects the body-fixed and the laboratory coordinate systems. The conjugated generalized impulses are represented by the *total* angular and linear momentum of the system ("body" + "fluid"). The kinetic energy of the system expressed in terms of the generalized coordinates and impulses serves as the Hamiltonian given by

$$H(\mathbf{X}, \hat{Q}, \mathbf{p}, \mathbf{M})$$
$$= \frac{1}{2} \left| \begin{array}{c} \mathbf{p} - \mathbf{K} \\ \mathbf{M} - \mathbf{P} \end{array} \right| \cdot \hat{J}^{-1} \left| \begin{array}{c} \mathbf{p} - \mathbf{K} \\ \mathbf{M} - \mathbf{P} \end{array} \right|$$
$$+\varphi(\mathbf{X}, \hat{Q}, t),$$

where all tensorial geometrical parameters and Kelvin impulses (impulse-couples) now depend on the space variables. The reason being that the Green function of the body combined with the rigid boundaries depend on the position and orientation of the body. We denote here by $\varphi(\mathbf{X}, \hat{Q}, t)$ the energy of the fluid induced by the pure deformations of the body. We have also introduced in the above the following 6-by-6 symmetric added-mass tensor of the body

$$\hat{J} \equiv \left| \begin{array}{cc} v\rho_b \hat{1} + \hat{T} & \hat{Z} \\ \hat{Z}^T & \hat{I} + \hat{R} \end{array} \right|,$$

Note that the proposed formalism allows us to consider along the same line boundaries (interfaces or free surfaces) with different physical boundary conditions (i.e., porous boundaries, elastic structures, etc). It is important only to have linear boundary conditions on the boundaries without time-derivatives (i.e., to pose boundary conditions without memory).

EQUATIONS OF MOTION

If the body or the boundary hold an axisymmetric property (we imply that an axisymmetric body preserve the symmetry in the course of deformations) then one can define the unit vector $\mathbf{h}$ aligned along the axis of symmetry (similar to a "Poisson vector" used for the description of a body's orientation in vacuum (Marsden & Ratiu 1995)). We can replace now $\hat{Q}$ by $\mathbf{h}$ expressed in the body attached coordinate system as an orientation variable. Correspondingly, the equation of the motion for $\mathbf{X}$ and $\mathbf{h}$ are given in accordance with the reference to the rotating coordinate system as

$$\frac{d\mathbf{X}}{dt} + \mathbf{\Omega} \wedge \mathbf{X} = \mathbf{U}$$

$$\frac{d\mathbf{h}}{dt} + \mathbf{\Omega} \wedge \mathbf{h} = 0.$$

The fact that the momentum (angular momentum) induced in the fluid by the moving body can be linearly expressed in terms of the instantaneous values of the body's velocities (translational and rotational) and also the corresponding purely geometric parameters (Green function) depending only on the geometry of the body and of the boundaries, strongly simplified the derived equations of motion. They are found to be a system of 12 coupled ODE (which means that there is no memory effect in the system) given by

$$\dot{\mathbf{p}} + \mathbf{\Omega} \wedge \mathbf{p} = -\nabla_{\mathbf{X}} H,$$

and

$$\dot{\mathbf{M}} + \mathbf{\Omega} \wedge \mathbf{M} + \mathbf{U} \wedge \mathbf{p} = (\mathbf{h} \wedge \nabla_{\mathbf{h}}) H$$

These equations resemble mathematically the equations for the geodesical trajectory of a charged point on the manifold $E(3)$ endowed by the Riemann metric (defined by the boundary) moving in an effective magnetic and electric field.

INTEGRABILITY

We recall that for a rigid body moving in an unbounded space there is a classification of integrability of the motion of the body as a function of its shape. It has been shown analytically by Kozlov & Onichenko (1982) that except for a number of "degenerate" shapes, such as, for example, bodies of revolution (Holmes et. al 1997), the Kirchhoff equations are, in principle, nonintegrable and, therefore, the motion of the body may be chaotic (see also Aref & Jones 1993). In particular, it was shown by numerical experiments, that chaos manifests itself both in the orientation dynamics of the body and in the geometry of its trajectory in space. For a rigid body moving in the vicinity of the boundaries the integrability of the motion is defined by the combined shape of the body and the boundaries. Thus, for a rigid body moving in an unbounded space the only translational motion without rotation (rectilinear due to the D'Alambert paradox), or a rotational motion with a fixed centroid, are fully integrable. For the case of a motion in the vicinity of the boundaries, the integrability of the above mentioned two particular cases depends on the geometry of the boundaries. Moreover, generally speaking, nor the motion without rotation neither that with pure rotational motion are integrable. Hence, one can conclude immediately that the presence of boundaries in the fluid leads to a further *chaotization* of the motion.

MEMORY EFFECT

Consider now the motion of a deformable body moving beneath a free surface with the standard linearized boundary conditions applied on the free surface. By invoking the Laplace transform we can express the energy of the system ("body"+"liquid") as a corresponding time-integration of the body's velocities combined with the space- and time-dependent added-masses. The resulting variational problem of determining the body's trajectory, should be resolved by using the causality principle. As an example, it can be shown that for a sphere penetrating a free surface, the derived equation of motion can be integrated. Note finally, that it is just the memory effect which leads to the well-known "ricochet" phenomena (see also Birkhoff & Gaywood 1949).

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POSSIBLE AND IMPOSSIBLE SOLUTIONS OF THE NAVIER-STOKES EQUATIONS

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ABSTRACT

Flow of isochoric constant-viscosity fluids obeys continuity and the Navier-Stokes equations. They are difficult to solve being nonlinear with a nonslip boundary condition at solid walls. Berker presented many solutions, but some of them, e.g. irrotational velocity, contradict the nonslip condition. Radial flow, possible between two nonparallel planes, is shown to be impossible in a cone, though an approximate solution exists. Parallel (equidistant) streamlines are possible only if rectilinear, concentric or coaxial circles, or helices of equal inclination on coaxial cylinders. Two-way flows resemble ideal and Stokes flows. The author presents some spatial jets impacting on a fixed or parallelly moving boundary. A general unsteady spatial solution near a plane boundary is expressed as power series of z, distance from the wall, which shows most boundary layer solutions to be valid only up to z^2 terms. Uniform steady-state flow at a constant piezometric gradient in the x-direction, between nonparallel planes, has a definite solution only up to second-order terms in (y,z), due to undefined boundary condition at ∞. Acceleration averaged over time gives insight into the properties of pseudoturbulent or chaotic (turbulent) flows. Turbulent shear is redefined and Reynolds (turbulent) stresses loose their physical meaning.

INTRODUCTION

Laminar flow of constant density ρ and kinematic viscosity ν fluids in a barotropic force field, obeys the Navier-Stokes equations (NSE) relating velocity $\mathbf{V}(u,v,w)$ and pressure p (or piezometric potential $P = g\,Z + p/\rho$, where Z - elevation, g - gravity acceleration, to the space vector $\mathbf{r}(x,y,z)$ and time t, through the continuity equation:

$$\operatorname{div} \mathbf{V} = 0 \qquad (1)$$

The acceleration is:

$$\mathbf{A} = \mathbf{V}_t + \operatorname{curl} \mathbf{V} \times \mathbf{V} + \operatorname{grad}(V^2/2) = -\operatorname{grad} P - \nu \operatorname{curl}\operatorname{curl} \mathbf{V} \qquad (2)$$

The NSE has to obey a nonslip boundary condition (BC) at solid walls. Eliminating P from (2) we get the compatibility equation:

$$\operatorname{curl} \mathbf{V}_t + \operatorname{curl}(\operatorname{curl} \mathbf{V} \times \mathbf{V}) + \nu \operatorname{curl}^3 \mathbf{V} = \mathbf{0} \qquad (3)$$

Agrawal [2] gives a large number of exact solutions of (1) and (2) or (3), but many of them, e.g. irrotational $\mathbf{V}$, do not satisfy any real BCs. The difficulty is due in part to the nonlinearity of NSE, but mainly to the nonslip BC. Approximate solutions may be obtained by numerical methods or by neglecting part [3], [16] or all [6] of small quadratic terms, leading to Stokes' equation. The results are useful at low Reynold numbers (Re), but sometimes of dubious validity. A number of special solutions are given later.

RADIAL FLOW IN A CONE

Radial flow between two nonparallel planes is possible [5], yet radial flow in a cone is impossible [2]. Proof: In spherical coordinates (R,θ,φ), $\mathbf{V}$ has only a radial component u, with $v = 0$, $w = 0$. By (1), $U = f(\theta,\varphi,t)/R^2$. Introduced into (2):

$$\left.\begin{aligned} A_1 &= u_t + uu_R = -P_R + \nu\,[(R^2 u_R)_R / R^2 - 2u/R^2]; \\ A_2 &= 0 = -P_\theta / R + \nu\,.\,2u_\theta / R; \\ A_3 &= 0 = -P_\varphi / R\sin\theta + \nu\,.\,2u_\varphi / R^2 \sin\theta \end{aligned}\right\} \qquad (4)$$

Eliminating P we get:

$$2f^2 + 4\nu Rf - R^3 f = \text{function of } (R,t)$$

As R and f are mutually independent, the only solution is $f = f(t)$. As $f = 0$ on the cone $\theta = \theta_o$, we have $f(t) \cong 0$, hence $\mathbf{V} = \mathbf{0}$. The difference from two-dimensional flow is due to the different topologies. [6] find nevertheless an approximate solution, $u = k(\cos^2\theta - \cos^2\theta_o)/R^2$, neglecting all quadratic terms. This shows the danger of applying blindly approximate solutions.

PARALLEL FLOW BETWEEN TWO MUTUALLY INCLINED PLANES

The unidirectional velocity u(y,z) between two planes inclined at 45°, for a given grad P is [9]:

$$u_{yz} + u_{zz} = -k; \quad u = kz(y - z)[1 + c(yz + y^2)] \qquad (5)$$

$k = P_x/2\nu$; c – an arbitrary constant.

There exist an infinite number of solutions of the Poisson equation fulfilling nonslip BCs on the planes. A single solution may be said to exist only up to quadratic terms. This apparent contradiction is due to the undefined BC at ∞. One should be wary of boundary layer solutions when BC at ∞ is not exactly known.

EQUIDISTANT OR PARALLEL STREAMLINES

Irmay [12] shows that equidistant; i.e. parallel streamlines in steady-state flow are possible only if they are rectilinear and unidirectional, or concentric or coaxial circles, or equally inclined helices on coaxial cylinders.