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Nonlinear Physics of Twisted Magnetic Field Lines

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ABSTRACT: Twisted magnetic field lines appear commonly in many different plasma systems, such as magnetic ropes created through interactions between the magnetosphere and the solar wind, magnetic clouds in the solar wind, solar corona, galactic jets, accretion discs, as well as fusion plasma devices. In this paper, we study the topological characterization of twisted magnetic fields, nonlinear effect induced by the Lorentz back reaction, length-scale bounds, and statistical distributions.

1. INTRODUCTION

The nonlinear magnetohydrodynamics (MHD) involves a variety of complex phenomena. It is impossible to construct nontrivial theory by direct analyses of the basic equations. To elucidate a specific phenomenon, we must apply a reduction of the model with appealing to scale separations, singular perturbations, coarse-graining (averaging), etc.

In this paper, we discuss a slow motion (or a steady state) of a low-pressure magnetized plasma. In more specific terms, we consider the following singular limit. The general MHD equations read, in the standard normalized units,

$$\partial_t \boldsymbol{v} = -(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} + \epsilon_A^{-2}(\nabla\times\boldsymbol{B})\times\boldsymbol{B} - \beta\nabla p + \epsilon_R\Delta\boldsymbol{v}, \quad \nabla\cdot\boldsymbol{v} = 0, \tag{1}$$

$$\partial_t \boldsymbol{B} = \nabla\times(\boldsymbol{v}\times\boldsymbol{B}) - \epsilon_L\nabla\times(\nabla\times\boldsymbol{B}). \tag{2}$$

Unknown variables are the magnetic field $\boldsymbol{B}$, the flow velocity $\boldsymbol{v}$ and the pressure p. The Alfvén number ϵ_A, Lundquist number ϵ_L^{-1}, Reynolds number ϵ_R^{-1}, and the beta ratio β are nondimensional positive parameters. The incompressibility condition ($\nabla\cdot\boldsymbol{v} = 0$) may be replaced by an evolution equation for the pressure p in a more sophisticated model.

This system of nonlinear parabolic equations (1)–(2) is a close cousin of the Navier-Stokes system describing neutral fluids (see [1, 2] and papers cited therein). The MHD system includes coupling between the magnetic field and the flow velocity through the nonlinear induction effect and its reciprocal Lorentz force, which adds a considerable complexity to the usual Navier-Stokes system. Surprisingly, however, we observe a more regular and ordered behavior in some MHD systems. Such phenomena are highlighted by a singular perturbation of $\epsilon_A^2 \to 0$, with fixing the time-scale, in the momentum-balance equation (1). This limit is amenable to slow motion of a strongly magnetized low β plasma. The determining equation becomes the force-free condition $(\nabla\times\boldsymbol{B})\times\boldsymbol{B} = 0$, which is equivalent to the Beltrami condition

$$\nabla\times\boldsymbol{B} = \lambda\boldsymbol{B}. \tag{3}$$

Here λ is a scalar function. By the solenoidal condition ($\nabla\cdot\boldsymbol{B} = 0$) and identity $\nabla\cdot(\nabla\times\boldsymbol{B}) = 0$, taking the divergence of the both sides of (3) yields

$$\boldsymbol{B}\cdot\nabla\lambda = 0. \tag{4}$$

Since (4) means that the function λ should be constant along the streamline (field line) of $\boldsymbol{B}$, analysis of the system of equations (3)–(4) requires integration of the streamline equation

$$\frac{d}{ds}\boldsymbol{x} = \boldsymbol{B}(\boldsymbol{x}). \tag{5}$$

The solenoidal condition ($\nabla\cdot\boldsymbol{B} = 0$) parallels Liouville's theorem for the Hamiltonian flow, and hence one can formulate (5) in a canonical form [3]. For a general three-dimensional $\boldsymbol{B}$, the solution of (5) exhibits chaos. Hence, the general analysis of the system (3)–(4) includes an essential mathematical difficulty. Two special cases, however, can be studied rigorously. One is the case where $\boldsymbol{B}$ has an ignorable coordinate (two-dimensional). Then, (5) becomes integrable, and the system (3)–(4) reduces into a nonlinear elliptic equation [4, 5]. The three-dimensional problem involves the non-integrable streamline problem (5), however, it is decoupled from the Beltrami problem (3)–(4), if we assume a constant λ that make (4) trivial.

2. CONSTANT-LAMBDA BELTRAMI FIELD

The constant-λ condition for the Beltrami field is a strong ansatz based on the following physical reasons. The streamline equation (5) in a three-dimensional magnetic field is generally non-integrable, and hence, we may assume that streamlines (magnetic field-lines) are embedded densely in a volume. Since (4) demands that λ is constant along each field line, it is natural to assume a constant λ over such a volume. The theory of energy relaxation also derives the constant-λ condition. Woltjer [6] pointed out the importance of the magnetic helicity

$$K = \frac{1}{2}\int_{\Omega} \boldsymbol{A}\cdot\boldsymbol{B}d\boldsymbol{x}.$$

Here $\nabla\times\boldsymbol{A} = \boldsymbol{B}$, Ω is the entire volume of the plasma and $d\boldsymbol{x}$ is the volume element. The viscous dissipation does not change the helicity K, while the magnetic energy diminishes toward a "ground state". The magnetic field self-organized through this energy relaxation is characterized by a minimizer of the magnetic energy $W = \int_{\Omega} B^2 dx/2$ subject to a given helicity. This variational principle reads as $\delta(W - \lambda K) = 0$, where λ is the Lagrange multiplier. The formal Euler-Lagrange equation, under appropriate boundary conditions, is identical to (3). Taylor [7] formulated an equivalent variational principle, however, his model is based on a different hypothesis to justify the preferential conservation of the helicity. The energy dissipation proceeds faster than the change of the helicity, if the resistive dissipation is dominated by spatially concentrated fluctuation currents (see also Hasegawa [8]). Both effects, the viscous dissipation, resulting in ion heating, and the resistive dissipation, resulting in electron heating, were compared for a specific relaxation process [9].

There are many different observations suggesting the creation of constant-λ force-free fields in astrophysical, space and laboratory plasmas. Magnetic flux tubes (flux ropes), in which field lines are twisted, are produced through interactions between the magnetosphere and interplanetary magnetic fields [10]. In a laboratory plasma, detailed measurements of magnetic fields showed that the field produced after self-organization through turbulence is closely approximated by a solution of (3) [7]. Galactic jets are also considered to have similar configurations of magnetic fields [11].

The Beltrami field plays an essential role in the so-called "dynamo theory". To understand the rapid generation of magnetic fields in astrophysical systems, we have to invoke a "fast dynamo action" that has a growth rate of the magnetic energy independent of the resistivity (see [12] and papers cited therein). In a highly conductive plasma the evolution of the magnetic field $\boldsymbol{B}$ obeys Faraday's law (2) with $\epsilon_L \to 0$. A plasma flow $\boldsymbol{v}$ with chaotic streamlines (maps with positive Lyapunov exponents), which may have a large length-scale, bring about complex mixing of magnetic flux, and the length-scale of the inhomogeneity cascades toward a small scale, resulting in amplification of the magnetic field. If the length-scale reduces down to the dissipative range, and the resistive damping becomes comparable to the induction effect, then the magnetic field energy turns to diminish. In this classical picture of the kinematic dynamo, the magnetic field energy accumulates into small scale fluctuations, and the life-time of the amplified magnetic field is limited by the time-scale of the cascade process. To obtain a larger length-scale and a longer life-time of amplified magnetic fields, an appropriate limitation for the scale reduction should occur. The nonlinear effect of the amplified magnetic fields, that is the Lorentz back-reaction, plays an essential role in this "post-kinematic phase". Here we assume that the plasma achieves a quasi-steady state through the energy relaxation process. Then, the momentum balance equation reduces into (3), and the flow $\boldsymbol{v}$ must be chosen in such a way that $\boldsymbol{B}$ satisfies (3) implicitly. The parameter λ characterizes the length-scales of $\boldsymbol{B}$. Hence, the condition (3) imposes a bound for the length-scale of the field, if the magnitude of λ is restricted by some reason. This bound avoids scale reduction down to the resistive regime, and extends the life-time of the amplified magnetic field.

Through the kinematic dynamo process, the current ($\propto \nabla \times \boldsymbol{B}$) tends to concentrate in small volumes, which may be disconnected. When the sectional length-scale of such a volume becomes small enough, the Lorentz force dominates ($\epsilon_A^2 \ll 1$). Let Ω to be such a "clump" of the magnetic field. Its length-scale is denoted by ℓ_c. This Ω may have a complex topology. We want to find a constant-λ Beltrami field in Ω. If the parameter λ can be chosen such that $|\lambda| \leq \lambda_c = O(\ell_c^{-1})$, then equilibration of the clump into such a Beltrami field results in a lower bound for the length-scale [13]. Here we solve the Beltrami condition (3) for a given helicity and an "external magnetic field". The external component of $\boldsymbol{B}$ is defined by decomposing $\boldsymbol{B} = \boldsymbol{B}_\Sigma + \boldsymbol{h}$, where $\nabla \times \boldsymbol{h} = 0$ and $\nabla \cdot \boldsymbol{h} = 0$. This $\boldsymbol{h}$, which represents the magnetic field rooted outside Ω, is assumed to be a given function. Its complement $\boldsymbol{B}_\Sigma$ is the unknown variable. We define the gauge-invariant helicity by

$$\mathcal{K} = \int_\Omega \boldsymbol{A} \cdot \boldsymbol{B}_\Sigma dx \tag{6}$$

We prove the existence of a solution with $|\lambda| \leq \lambda_c = O(\ell_c^{-1})$ for every $\boldsymbol{h} \neq 0$ and $\mathcal{K}$ in the next section (Theorem3). The nonvanishing $\boldsymbol{h}$ plays the role of symmetry breaking.

3. EXISTENCE THEOREM AND COMPLETENESS THEOREM

The constant-λ Beltrami condition (3) is regarded as an eigenvalue problem with respect to the curl operator. Interestingly, the topology of the domain plays an essential role in this eigenvalue problem.

To study the spectrum the curl derivatives, we need the fundamental theory of vector function spaces. Let Ω ($\subset \boldsymbol{R}^3$) be a bounded domain with a smooth boundary $\partial\Omega = \cup_{i=1}^n \Gamma_i$ (Γ_i is a connected surface). We consider cuts of the domain Ω. Let $\Sigma_1, \cdots, \Sigma_m$ ($m \geq 0$) be cuts such that $\Sigma_i \cap \Sigma_j = \emptyset$ ($i \neq j$), and such that $\Omega \setminus (\cup_{i=1}^m \Sigma_i)$ becomes a simply connected domain. The number m of such cuts is the first Betti number of Ω. When $m > 0$, we define the flux through each cut by

$$\Phi_{\Sigma_i}(\boldsymbol{u}) = \int_{\Sigma_i} \boldsymbol{n} \cdot \boldsymbol{u} ds \qquad (i = 1, 2, \cdots, m),$$

where $\boldsymbol{n}$ is the unit normal vector on Σ_i with an appropriate orientation. By Gauss's formula, $\Phi_{\Sigma_i}(\boldsymbol{u})$ is independent of the place of the cut Σ_i, if $\nabla \cdot \boldsymbol{u} = 0$ in Ω and $\boldsymbol{n} \cdot \boldsymbol{u} = 0$ on $\partial\Omega$.

We denote $L^2(\Omega)$ the Lebesgue space of square-integrable (complex) vector fields in Ω, which is endowed with the standard innerproduct $(\boldsymbol{a}, \boldsymbol{b})$. We define the following subspaces of $L^2(\Omega)$;

$$\begin{aligned} L^2_\Sigma(\Omega) &= \{\boldsymbol{w};\ \nabla \cdot \boldsymbol{w} = 0 \text{ in } \Omega,\ \boldsymbol{n} \cdot \boldsymbol{w} = 0 \text{ on } \partial\Omega,\ \Phi_{\Sigma_i}(\boldsymbol{w}) = 0\ (i = 1, \cdots, m)\}, \\ L^2_H(\Omega) &= \{\boldsymbol{h};\ \nabla \cdot \boldsymbol{h} = 0, \nabla \times \boldsymbol{h} = 0 \text{ in } \Omega,\ \boldsymbol{n} \cdot \boldsymbol{h} = 0 \text{ on } \partial\Omega\}, \\ L^2_g(\Omega) &= \{\nabla\phi\}. \end{aligned}$$

We have an orthogonal decomposition [14]

$$L^2(\Omega) = L^2_\Sigma(\Omega) \oplus L^2_H(\Omega) \oplus L^2_g(\Omega).$$

The space of solenoidal vector fields with vanishing normal component on $\partial\Omega$ is

$$L^2_\sigma(\Omega) = L^2_\Sigma(\Omega) \oplus L^2_H(\Omega).$$

The subspace $L^2_H(\Omega)$ corresponds to the cohomology class, whose member is a harmonic vector field and $\dim L^2_H(\Omega) = m$ (the first Betti number of Ω). When Ω is simply connected, then $m = 0$ and $L^2_H(\Omega) = \emptyset$. We have the following theorems [15].

Theorem 1 *Let $\Omega \subset \boldsymbol{R}^3$ be a smoothly bounded domain. We define a curl operator $\mathcal{S}$ in the Hilbert space $L^2_\Sigma(\Omega)$ by*

$$\mathcal{S}\boldsymbol{u} = \nabla \times \boldsymbol{u}, \quad D(\mathcal{S}) = \{\boldsymbol{u} \in L^2_\Sigma(\Omega)\ ;\ \nabla \times \boldsymbol{u} \in L^2_\Sigma(\Omega)\}.$$

Then $\mathcal{S}$ is a self-adjoint operator. The spectrum of $\mathcal{S}$ consists of only point spectra $\sigma_p(\mathcal{S})$, which is a discrete set of real numbers.

Theorem 2 *In $L^2_\sigma(\Omega)$ we define a curl operator $\tilde{\mathcal{S}}$ by*

$$\tilde{\mathcal{S}} = \nabla \times \boldsymbol{u}, \quad D(\tilde{\mathcal{S}}) = \{\boldsymbol{u} \in L^2_\sigma(\Omega)\ ;\ \nabla \times \boldsymbol{u} \in L^2_\sigma(\Omega)\}.$$

(i) When $\dim L^2_H(\Omega) = 0$, i.e. if Ω is simply connected, then $\tilde{\mathcal{S}} \equiv \mathcal{S}$, and hence, the spectrum $\sigma(\tilde{\mathcal{S}}) = \sigma_p(\tilde{\mathcal{S}})$.

(ii) When $\dim L^2_H(\Omega) > 0$, i.e. if Ω is multiply connected, then $\tilde{\mathcal{S}}$ is an extension of $\mathcal{S}$. The spectrum $\sigma(\tilde{\mathcal{S}})$ consists of only spectra $\sigma_p(\tilde{\mathcal{S}})$, and $\sigma_p(\tilde{\mathcal{S}}) = \mathbf{C}$. Hence, for every $\lambda \in \mathbf{C}$,

$$(\tilde{\mathcal{S}} - \lambda)\boldsymbol{u} = 0 \tag{7}$$

has a nontrivial solution.

Theorem 2 proves the general existence of the constant-λ Beltrami function for every $\lambda \in \mathbf{C}$, if Ω is multiply connected. In the next theorem, we solve the constant-λ Beltrami equation (3) for a given helicity $\mathcal{K}$ and harmonic field $\boldsymbol{h} \in L^2_H(\Omega)$. Now λ is an unknown variable. This problem is related with the magnetic clump discussed in Sec. 2.

We assume that Ω is multiply connected. Let $\{\varphi_j\}$ be the complete set of the eigenfunctions of the self-adjoint curl operator $\mathcal{S}$ (Theorem 1). The corresponding eigenvalues are numbered as

$$\cdots \le \mu_{-2} \le \mu_{-1} < 0 < \mu_1 \le \mu_2 \le \cdots. \tag{8}$$

For every $\boldsymbol{B} \in L^2_\sigma(\Omega)$, we have an orthogonal-sum expansion

$$\boldsymbol{B}(\boldsymbol{x},t) = \sum_j c_j(t)\varphi_j(\boldsymbol{x}) + \boldsymbol{h}(\boldsymbol{x},t), \tag{9}$$

where $\boldsymbol{h} \in L^2_H(\Omega)$. The harmonic field $\boldsymbol{h}$ is a given function, which plays an important role of "symmetry breaking" in the following discussion. The first summation in the right-hand side of (9) is denoted by $\boldsymbol{B}_\Sigma$. The energy of $\boldsymbol{B}$ is given by

$$W = \frac{1}{2}\sum_j c_j^2 + \frac{1}{2}\|\boldsymbol{h}\|^2. \tag{10}$$

There exists $\boldsymbol{g}$ such that $\boldsymbol{h} = \nabla \times \boldsymbol{g}$. The vector potential of $\boldsymbol{B}$ is given by

$$\boldsymbol{A} = \sum_j \frac{c_j}{\mu_j}\varphi_j + \boldsymbol{g}. \tag{11}$$

Denoting $D_j = (\boldsymbol{\varphi}_j, \boldsymbol{g})$, the gauge invariant helicity (6) becomes

$$\mathcal{K} = \frac{1}{2}(\boldsymbol{A}, \boldsymbol{B}_\Sigma) = \frac{1}{2}\sum_j(\frac{c_j^2}{\mu_j} + D_j c_j). \tag{12}$$

For given $\mathcal{K}$ and $\boldsymbol{h}$, we can solve (3) by the variational principle $\delta(W - \lambda\mathcal{K}) = 0$, and obtain

$$c_j = \frac{\lambda\mu_j}{2(\mu_j - \lambda)} D_j \quad (\forall j). \tag{13}$$

The energy and the helicity become

$$W = \sum_j \frac{\lambda^2\mu_j^2}{8(\mu_j - \lambda)^2} D_j^2 + \frac{1}{2}\|\boldsymbol{h}\|^2, \qquad \mathcal{K} = \sum_j \frac{\lambda\mu_j(2\mu_j - \lambda)}{8(\mu_j - \lambda)^2} D_j^2. \tag{14}$$

We can show that $\mathcal{K}$ is a monotone function of λ in the range of $\mu_{-1} < \lambda < \mu_1$ (see definition (8)), if $D_j \neq 0$ $(\exists j)$, viz., if we have a "symmetry breaking" $\boldsymbol{h} \neq 0$. For every $\kappa \in \mathbf{R}$, the equation $\mathcal{K}(\lambda) = \kappa$ has a unique solution in this range of λ. Now we have the following theorem.

Theorem 3 *Let Ω $(\subset \mathbf{R}^3)$ be a multiply connected bounded domain. Assume that $\boldsymbol{h}$ $(\in L^2_H(\Omega))$ is finite. For every κ $(\in \mathbf{R})$, the Beltrami condition (3) has a unique solution $\boldsymbol{B}$ such that its helicity $\mathcal{K} = \kappa$, and λ such that $\mu_{-1} < \lambda < \mu_1$.*

4. STATISTICAL EQUILIBRIUM

Using the phenomenological variational principle $\delta(W - \lambda\mathcal{K}) = 0$ (Sec. 2), we develop a statistical mechanical model that reproduces the constant-λ Beltrami field at the "zero *temperature* limit". A finite *temperature* (in the sense of MHD fluctuation) equilibrium includes fluctuations. The statistical theory predicts the spectra of macroscopic physical quantities such as the energy, helicity, etc.

A key step is to find an invariant measure of the temporal evolution equation. It corresponds to Liouville's theorem in the Hamiltonian dynamics. Montgomery *et al.* [16] used the "Chandrasekhar-Kendall functions", which are the eigenfunctions of the curl in a cylindrical geometry [17], to expand the solenoidal vector fields $\boldsymbol{B}$ and $\boldsymbol{v}$, and defined an infinite-dimensional phase space spanned by the expansion coefficients. The formal Lebesgue measure is shown to be invariant against the nonlinear ideal $(\epsilon_R, \epsilon_L \to \infty)$ dynamics. The completeness theorem of the eigenfunctions (Theorem 1) gave a mathematical justification of the expansion, and generalized the Hilbert-space approach for an arbitrary geometry. An important development in recent work [18] is the treatment of the harmonic magnetic field, which brings about a symmetry breaking associated with a topological constraint. When we consider a multiply connected domain, the harmonic magnetic fields, which are rooted outside the domain, are represented by the cohomology class. If we impose the ideal conducting boundary conditions, these harmonic fields are invariant. The rest orthogonal complement spans the dynamical phase space. The invariant harmonic component plays the role of an externally applied symmetry breaking. Interestingly, this term yields "power-law spectra" of the energy, helicity and helicity fluctuation. It is easy to verify the following proposition.

Proposition 1 (Invariant Measure) *Let $\boldsymbol{v}(x,t)$ be a smooth vector field in Ω. Suppose that $\boldsymbol{B}(x,t)$ obeys*

$$\partial_t \boldsymbol{B} = \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) \quad \text{in } \Omega, \tag{15}$$

$$\boldsymbol{n} \times (\boldsymbol{v} \times \boldsymbol{B}) = 0 \quad \text{on } \partial\Omega. \tag{16}$$

Using the eigenfunctions of the curl operator $\boldsymbol{\varphi}_j$ and the harmonic field $\boldsymbol{h}_\ell$, we write $\boldsymbol{B}(x,t)$ in the form of (9). Then, $dC = \prod_j dc_j$ is an invariant measure.

The ansatz of the variational principle $\delta(W - \lambda\mathcal{K}) = 0$ suggests that two additive quantities W and $\mathcal{K}$ are the relevant state variables that characterize the statistical equilibrium. The possible ensemble consistent with this variational principle is the Boltzmann distribution

$$P(W, \mathcal{K}) \propto \exp[-\beta(W - \lambda\mathcal{K})] \tag{17}$$

where β is interpreted as an inverse temperature of the magnetic field. The helicity and the energy of each mode is $(c_j^2/\mu_j + D_j c_j)/2$ and $c_j^2/2$, respectively. The Boltzmann distribution for the amplitude c_j is

$$P_j \propto \exp\left[-\frac{\beta}{2}\left(c_j^2 - \frac{\lambda}{\mu_j}c_j^2 - \lambda D_j c_j\right)\right]. \tag{18}$$

The ensemble averages of W and $\mathcal{K}$ over the phase space become

$$\langle W \rangle = \sum_j \left[\frac{\mu_j}{2\beta(\mu_j - \lambda)} + \frac{\lambda^2 \mu_j^2}{8(\mu_j - \lambda)^2} D_j^2\right], \tag{19}$$

$$\langle \mathcal{K} \rangle = \sum_j \left[\frac{1}{2\beta(\mu_j - \lambda)} + \frac{\lambda \mu_j (2\mu_j - \lambda)}{8(\mu_j - \lambda)^2} D_j^2\right]. \tag{20}$$

These results are compared with (14). The first term of the right-hand side of (19) and that of (20) are the contributions of the fluctuations. In (19), the energy of the harmonic field, which is constant here, is omitted. This classical statistical model suffers from the Rayleigh-Jeans catastrophe, viz., when we pass the limit of the infinite summation over the all modes, the fluctuation terms diverge. To avoid this divergence, we can appeal to the Bose-Einstein statistics with second-quantizing the mode amplitude c_j and defining bosons MHD fluctuations [18].

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