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Averaged RMHD Equations

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Abstract

A new reduced set of resistive MHD equations is derived by averaging the full MHD equations on specified flux coordinates, which is consistent with 3D equilibria. It is confirmed that the total energy is conserved and the linearized equations for ideal modes are self-adjoint.

Key Words: reduced MHD equation, averaging method, three-dimensional equilibrium

1 Introduction

Recently, several codes have been developed for the magnetohydrodynamic (MHD) linear stability analysis of the three-dimensional (3D) equilibria calculated by the VMEC code[1]. A comparison study of the codes is summarized in [2]. The codes are classified into two categories. One is the 3D approach, which has an advantage that the stability of the exact 3D equilibrium can be calculated without any approximation. However, these codes can only examine ideal modes because they are based on the energy principle, and it is difficult to provide sufficient perturbation space to search for the largest growth rate.

The other is the 2D approach. Some of them employ reduced MHD equations, which can treat not only ideal modes but also resistive ones. The nonlinear growth of the unstable modes can also be studied as an initial value problem. However, understanding the approximation which is used in reducing the 3D equilibrium properties to 2D expressions is essential in these approach. Recently, Todoroki[3] derived a Grad-Shafranov type equation by averaging the 3D equilibrium equation on specified coordinates without any ordering. This means that any 3D equilibrium solution satisfies the equation. In this paper, the averaging method is applied to the derivation of a reduced set of MHD equations, in order to study the stability of the 3D equilibria against the resistive modes as well as the ideal ones. Since the coordinates introduced by Todoroki are not easy to treat in the stability analysis, the use of flux coordinates is considered. In this case, every flux coordinates system cannot give the Grad-Shafranov type equation in the averaging method, because the equation has to be expressed with only metric and surface quantities.

In Section 2, the averaged equilibrium equation in flux coordinates is discussed. The reduced MHD equations based on the averaging method are derived and the basic properties of them are examined in Section 3. In Section 4, numerical results are presented. Conclusions are given in Section 5.

2 Averaged equilibrium equation

In arbitrary flux coordinates (ρ, θ, ζ), where ρ is the label of the magnetic flux and θ and ζ are the poloidal and the toroidal angles, respectively, the magnetic field, $\boldsymbol{B}$, and the current density, $\boldsymbol{J}$, are expressed by

$$\boldsymbol{B} = \nabla\rho \times \nabla \left(\frac{d\chi}{d\rho}\theta - \frac{d\Psi}{d\rho}\zeta + \tilde{\mu}(\rho, \theta, \zeta) \right) \tag{1}$$

and

$$\boldsymbol{J} = \nabla\rho \times \nabla \left(\frac{dI}{d\rho}\theta + \frac{dF}{d\rho}\zeta - \tilde{\nu}(\rho, \theta, \zeta) \right), \tag{2}$$

respectively. Here $\chi(\rho)$ and $\Psi(\rho)$ denote the toroidal and the poloidal magnetic fluxes inside the flux surfaces, respectively, and $I(\rho)$ and $F(\rho)$ are the total toroidal current inside the flux surface and the total poloidal current outside the flux surface, respectively. $\tilde{\mu}$ and $\tilde{\nu}$ are periodic functions with respect to both θ and ζ. By substituting the contravariant components of $\boldsymbol{B}$ and $\boldsymbol{J}$ into the force balance equation, $\nabla P = \boldsymbol{J} \times \boldsymbol{B}$, and averaging the equation in the ζ direction, we can obtain

$$\langle\sqrt{g}\rangle_\zeta \frac{dP}{d\rho} = -\frac{dF}{d\rho}\langle\sqrt{g}B^\zeta\rangle_\zeta - \frac{d\Psi}{d\rho}\langle\sqrt{g}J^\zeta\rangle_\zeta + \langle\frac{\partial\tilde{\nu}}{\partial\zeta}\frac{\partial\tilde{\mu}}{\partial\theta}\rangle_\zeta - \langle\frac{\partial\tilde{\nu}}{\partial\theta}\frac{\partial\tilde{\mu}}{\partial\zeta}\rangle_\zeta, \tag{3}$$

where $\langle f\rangle_\zeta = \oint f d\zeta / \oint d\zeta$ is used. The periodic functions of $\tilde{\mu}$ and $\tilde{\nu}$ must not enter this equation for the averaged equilibrium equation to be expressed with only metric and surface quantities. Therefore, a constraint must be imposed on the flux coordinates to elimination the last two terms,

$$\tilde{\mu} = \tilde{\mu}(\rho, \theta) \quad or \quad \tilde{\nu} = \tilde{\nu}(\rho, \theta). \tag{4}$$

It can easily be shown that the former condition corresponds to that used in Todoroki's approach[3]. Therefore, this condition is used here. In this case, $\langle\sqrt{g}B^\zeta\rangle_\zeta$ and $\langle\sqrt{g}J^\zeta\rangle_\zeta$ can be obtained with the relations of $B_i = g_{ij}B^j$. Then, we obtain a Grad-Shafranov type equation in flux coordinates with $\tilde{\mu} = \tilde{\mu}(\rho, \theta)$, which can be written as

$$\frac{d\Psi}{d\rho}\left[\frac{\partial}{\partial\rho}\left(G_\perp^{\theta\theta}\frac{d\Psi}{d\rho}\right) - \frac{\partial}{\partial\theta}\left(G_\perp^{\rho\theta}\frac{d\Psi}{d\rho}\right) + \frac{\partial(Fh_\theta)}{\partial\rho} - \frac{\partial(Fh_\rho)}{\partial\theta}\right]$$
$$= -\langle\sqrt{g}\rangle_\zeta\frac{dP}{d\rho} - T\frac{dF}{d\rho}, \tag{5}$$

where $G^{\perp}_{ij} = G_{ij} - G_{i\zeta}G_{j\zeta}/G_{\zeta\zeta}$, $h_i = G_{i\zeta}/G_{\zeta\zeta}$, $T = F/G_{\zeta\zeta} - h_\theta d\Psi/d\rho$ and $G_{ij} \equiv \langle g_{ij}/\sqrt{g}\rangle_\zeta$. It is noted that no ordering or approximation is used in the derivation; therefore, any 3D equilibrium solution satisfies this equation. Thus, if a reduced set of MHD equation is derived in flux coordinates with (4), it must be consistent with a 3D equilibrium. An equation similar to (5) is derived by Pustovitov[4], however, he used an ordering with respect to the magnetic field. More recently, he has obtained a similar equation without making any approximation[5] If we take $\tilde{\mu} = 0$ as the special case of the constraints of (4), which corresponds to the condition that the magnetic field line is expressed as a straight line in the flux coordinates, the averaged equilibrium equation is simplified to

$$\frac{d\Psi}{d\rho}\left[\frac{\partial}{\partial\rho}\left(G_{\theta\theta}\frac{d\Psi}{d\rho} + G_{\theta\zeta}\frac{d\chi}{d\rho}\right) - \frac{\partial}{\partial\theta}\left(G_{\rho\theta}\frac{d\Psi}{d\rho} + G_{\rho\zeta}\frac{d\chi}{d\rho}\right)\right]$$
$$= -\langle\sqrt{g}\rangle_\zeta\frac{dP}{d\rho} - \frac{dF}{d\rho}\frac{d\chi}{d\rho}, \qquad (6)$$

because $\sqrt{g}B^\zeta = d\chi/d\rho$.

3 Reduced MHD equations consistent with 3D equilibrium

A reduced set of MHD equations including resistivity is derived by reducing and averaging the full MHD equations in the flux coordinates which give the Grad-Shafranov type equation. In order to express the magnetic differential operator in an algebraic form, the condition of $\tilde{\mu} = 0$ is employed here. The procedure is similar to the one developed by Strauss[6], however, only the following assumptions are used here: First, the magnetic field is expressed as

$$\boldsymbol{B} = \nabla\chi \times \nabla\theta + \nabla\zeta \times \nabla\Psi \qquad (7)$$

with the condition that $\chi_0, \Psi \gg \chi_1$, where the subscripts, 0 and 1 denote the equilibrium and the perturbed parts, respectively, and the variable (except for the metrics) without either of the subscripts includes both of them. Plasma incompressibility and the condition of $v^\zeta \ll v^\rho, v^\theta$ are also assumed to eliminate the compressional modes. Furthermore, the assumption that the pitch number of the equilibrium quantities is much larger than the toroidal mode number of the perturbation is used.

Then, the reduced equations for the poloidal flux, Ψ, the stream function Φ and the plasma pressure P are obtained. These consist of the Ohm's law including resistivity η,

$$\frac{\partial\Psi}{\partial t} = -\langle\frac{\sqrt{g}}{\chi_0'}\boldsymbol{B}\cdot\nabla\rangle_\zeta\Phi + \eta\langle J_\zeta\rangle_\zeta, \qquad (8)$$

the vorticity equation

$$-\frac{\langle\rho_m\sqrt{g}\rangle_\zeta}{\chi_0'}\frac{d}{dt}\frac{\langle\sqrt{g}U^\zeta\rangle_\zeta}{\chi_0'} = -\langle\frac{\sqrt{g}}{\chi_0'}\boldsymbol{B}\cdot\nabla\rangle_\zeta\frac{\langle\sqrt{g}J^\zeta\rangle_\zeta}{\chi_0'} + [\langle\Omega\rangle_\zeta, P], \qquad (9)$$

and the equation of state

$$\frac{dP}{dt} = 0. \tag{10}$$

Here the vorticity is given by

$$\langle\sqrt{g}U^\zeta\rangle_\zeta = \frac{\partial\langle v_\theta\rangle_\zeta}{\partial\rho} - \frac{\partial\langle v_\rho\rangle_\zeta}{\partial\theta} \tag{11}$$

with the relations $\langle v^\rho\rangle_\zeta = (1/\chi_0')\partial\Phi/\partial\theta$, $\langle v^\theta\rangle_\zeta = -(1/\chi_0')\partial\Phi/\partial\rho$ and $\langle v_i\rangle_\zeta = \langle g_{ij}\rangle_\zeta\langle v^j\rangle_\zeta$, and the current density components are given by

$$\langle\sqrt{g}J^\zeta\rangle_\zeta = \frac{\partial\langle B_\theta\rangle_\zeta}{\partial\rho} - \frac{\partial\langle B_\rho\rangle_\zeta}{\partial\theta} \quad \text{and} \quad \langle J_\zeta\rangle_\zeta = G_{\zeta j}\langle\sqrt{g}J^j\rangle_\zeta \tag{12}$$

with the relations $\langle\sqrt{g}B^\rho\rangle_\zeta = -(1/\chi_0')\partial\Psi/\partial\theta$, $\langle\sqrt{g}B^\theta\rangle_\zeta = (1/\chi_0')\partial\Psi/\partial\rho$ and $\langle B_i\rangle_\zeta = G_{ij}\langle\sqrt{g}B^j\rangle_\zeta$. The convective and the magnetic derivatives are expressed as

$$\frac{df}{dt} = \frac{\partial f}{\partial t} - [\Phi, f] \quad \text{and} \quad \langle\frac{\sqrt{g}}{\chi_0'}\boldsymbol{B}\cdot\nabla\rangle_\zeta f = \frac{\partial f}{\partial\zeta} + [\Psi, f], \tag{13}$$

respectively, with the Poisson bracket defined by

$$[f, g] = \frac{1}{\chi_0'}\left(\frac{\partial f}{\partial\rho}\frac{\partial g}{\partial\theta} - \frac{\partial f}{\partial\theta}\frac{\partial g}{\partial\rho}\right). \tag{14}$$

The averaged curvature of a field line is given by

$$\langle\Omega\rangle_\zeta = \frac{\langle\sqrt{g}\rangle_\zeta}{\chi_0'}. \tag{15}$$

In this derivation, the large aspect ratio approximation is not used, therefore, these reduced MHD equations can be applied to a configuration with a small aspect ratio.

As the above assumptions are used in the derivation, the basic properties have to be examined. First, by taking $\partial/\partial t = \Phi = 0$, the equation that

$$\frac{\partial}{\partial\theta}\left(\frac{d\Psi}{d\rho}\frac{\langle\sqrt{g}J^\zeta\rangle_\zeta}{\chi_0'} + \frac{dP}{d\rho}\frac{\langle\sqrt{g}\rangle_\zeta}{\chi_0'}\right) = 0 \tag{16}$$

is obtained. (6) is obtained by setting the constant of integration equal to $-dF/d\rho$. This means that the reduced equations are consistent with 3D equilibrium.

Next, when we defined the volume element, the internal energy, the kinetic energy, and the magnetic energy as

$$d\tau \equiv \langle\sqrt{g}\rangle_\zeta d\rho d\theta d\zeta, \tag{17}$$

$$\mathsf{U} \equiv -\int P d\tau, \tag{18}$$

$$\mathsf{K} \equiv \frac{1}{2}\int \langle\rho_m\rangle_\zeta\langle v^i\rangle_\zeta\langle v_i\rangle_\zeta d\tau, \tag{19}$$

$$\mathsf{M} \equiv \frac{1}{2}\int \langle B^i\rangle_\zeta\langle B_i\rangle_\zeta d\tau, \tag{20}$$

respectively, the energy conservation law is given by

$$\frac{\partial}{\partial t}(\mathsf{K} + \mathsf{M} + \mathsf{U}) = -\eta \int \langle\sqrt{g}J^\zeta\rangle_\zeta \frac{\langle J_\zeta\rangle_\zeta}{\langle\sqrt{g}\rangle_\zeta} d\tau. \tag{21}$$

Finally, the normalized and linearized equations are given by

$$\frac{\partial\Psi}{\partial t} = -\left(\frac{\partial}{\partial\zeta} + \iota\frac{\partial}{\partial\theta}\right)\Phi + \frac{1}{S}\langle J_\zeta\rangle_\zeta, \tag{22}$$

$$-\frac{\partial}{\partial t}\langle\sqrt{g}U^\zeta\rangle_\zeta = -\left(\frac{\partial}{\partial\zeta} + \iota\frac{\partial}{\partial\theta}\right)\langle\sqrt{g}J^\zeta\rangle_\zeta$$

$$-\chi_0[\Psi, \frac{\langle\sqrt{g}J_0^\zeta\rangle_\zeta}{\chi_0'}] + \frac{\beta_0}{2}\chi_0\left[\langle\Omega\rangle_\zeta, P\right], \tag{23}$$

and

$$\frac{\partial P}{\partial t} = -\frac{1}{\chi_0'}\frac{dP_0}{d\rho}\frac{\partial\Phi}{\partial\theta}. \tag{24}$$

Here S and β_0 denote the magnetic Reynolds number and the beta value at the magnetic axis. The linearized equation for ideal modes with a time dependence of $\exp(i\omega t)$ can be written in the following form;

$$\begin{aligned}
&-\omega^2\int\frac{1}{\chi_0'}\left\{\langle g_{\theta\theta}\rangle_\zeta\left|\frac{\partial\Phi}{\partial\rho}\right|^2 - \langle g_{\theta\theta}\rangle_\zeta\left(\frac{\partial\Phi^*}{\partial\rho}\frac{\partial\Phi}{\partial\theta} + \frac{\partial\Phi}{\partial\rho}\frac{\partial\Phi^*}{\partial\theta}\right) + \langle g_{\rho\rho}\rangle_\zeta\left|\frac{\partial\Phi}{\partial\theta}\right|^2\right\}d\tau \\
&= \int\left\{G_{\theta\theta}\left|\frac{\partial\Psi}{\partial\rho}\right|^2 - G_{\rho\theta}\left(\frac{\partial\Psi^*}{\partial\rho}\frac{\partial\Psi}{\partial\theta} + \frac{\partial\Psi}{\partial\rho}\frac{\partial\Psi^*}{\partial\theta}\right) + G_{\rho\rho}\left|\frac{\partial\Psi}{\partial\theta}\right|^2\right. \\
&+ \frac{\langle\sqrt{g}J_0^\zeta\rangle_\zeta}{\chi_0'}\left(\frac{\partial\Phi}{\partial\rho}\frac{\partial^2\Phi^*}{\partial\theta\partial\zeta} + \frac{\partial\Phi^*}{\partial\rho}\frac{\partial^2\Phi}{\partial\theta\partial\zeta}\right) + \iota\left|\frac{\partial\Phi}{\partial\theta}\right|^2\frac{\partial}{\partial\rho}\left(\frac{\langle\sqrt{g}J_0^\zeta\rangle_\zeta}{\chi_0'}\right) \\
&\left. + \frac{\beta_0}{2}\frac{P_0'}{\chi_0'}\frac{\partial\langle\Omega\rangle_\zeta}{\partial\rho}\left|\frac{\partial\Phi}{\partial\theta}\right|^2\right\}d\tau
\end{aligned} \tag{25}$$

The self-adjointness of this equation can easily be confirmed.

4 Numerical Calculation

The ARMS code has been developed to solve (22) $\sim$ (24) by applying the RESORM code[7] to these equations employing the geometrical toroidal angle as ζ. Figure 1 shows the linear growth rates and one of the flow patterns of the resistive and the ideal interchange modes in the equilibria used in [2]. The growth rates of the ideal mode with $m = 3/n = 2$ given by the ARMS code are compared with those by the CHAFAR code[8]. Good agreement between them is obtained. The resistive modes are also calculated and it is shown that they are unstable below the ideal beta limit.

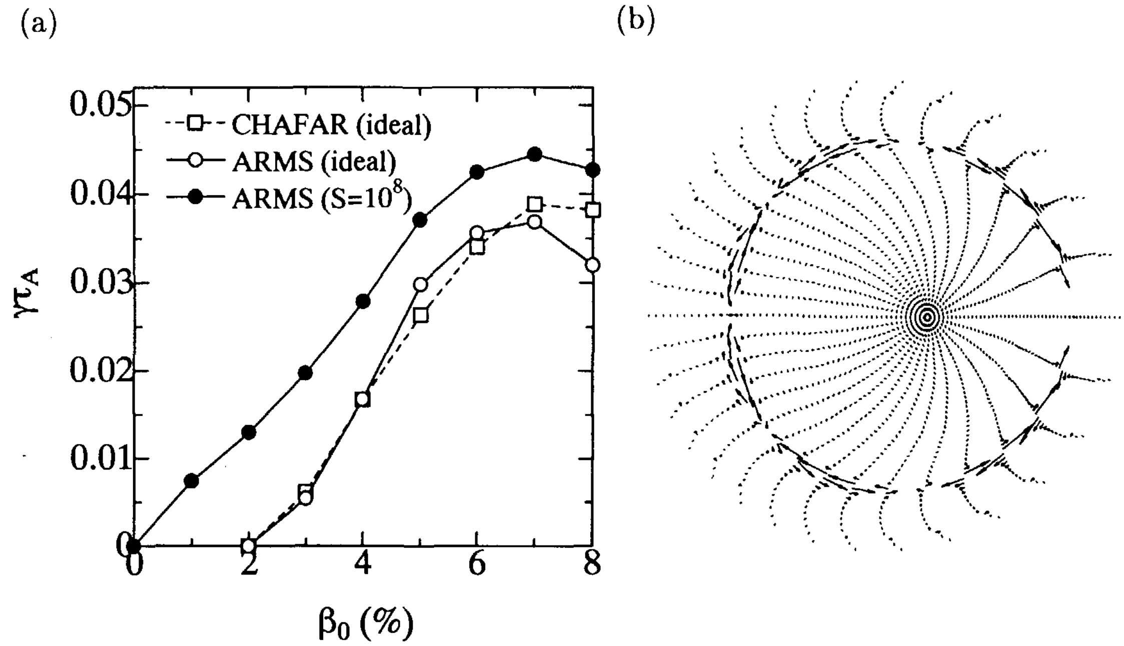

Fig.1 (a) Linear growth rates of ideal and resistive interchange modes normalized by the poloidal Alfven time and (b) the flow pattern at $\beta_0 = 4\%$ and $S = 10^8$.

5 Conclusion

A set of nonlinear reduced MHD equations for resistive modes is derived based on the averaging method without an assumption of large aspect ratio. This set of the equations is consistent with 3D equilibria if the flux coordinates satisfying $\partial\tilde{\mu}/\partial\zeta = 0$ or $\partial\tilde{\nu}/\partial\zeta = 0$ are employed.

It is confirmed that the energy conservation law can be derived and that are self-adjoint the linearized equations for ideal modes. The growth rate of the ideal interchange mode agrees well with the results of the CHAFAR code.

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