CONFORMAL DIRAC STRUCTURES

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Abstract

The Courant bracket defined originally on the sections of a vector bundle $TM \oplus T^*M \to M$ is extended to the direct sum of the 1-jet vector bundle and its dual. The extended bracket allows one to interpret many structures encountered in differential geometry, in terms of Dirac structures. We give a new approach to conformal Jacobi structures.
1 Introduction

In the present paper, we describe globally Jacobi manifolds through the theory of Dirac structures. Dirac structures on manifolds, introduced by T. Courant and A. Weinstein in [C-W] provide a framework for the study of Poisson structures, pre-symplectic manifolds, as well as foliations. Roughly speaking, a Dirac structure on a smooth manifold $M$ is a sub-bundle $L \rightarrow M$ of the vector bundle $TM \oplus T^*M \rightarrow M$, whose total space is maximally isotropic under the canonical symmetric 2-form on $TM \oplus T^*M$ and whose smooth sections are closed under the Courant bracket (see Section 2 below). On the other hand, a Jacobi structure on $M$ is defined by a bivector field $\pi$ and a vector field $E$ satisfying $[\pi,\pi]_s = 2E \wedge \pi, [E,\pi]_s = 0$, where $[.,.]_s$ is the Schouten-Nijenhuis bracket ([Kz]). A Jacobi manifold is a smooth manifold equipped with a Jacobi structure. It is known that Jacobi structures include contact forms, symplectic and Poisson structures (see e.g. [G-L]). Thus a natural question arises: is there a simple characterization of Jacobi manifolds by means of Dirac structures?

The main problem which emerges when one tries to describe a general Jacobi structure using the concept of a Dirac structure on a manifold, comes from the fact that a Jacobi structure on a smooth manifold $M$ involves the vector bundle of the 1-jets of real functions on $M$ instead of the tangent bundle. Hence the first step in resolving the above question is to find a suitable bracket on the sections of the vector bundle $\mathcal{E}^1(M) = (TM \times \mathbb{R}) \oplus (T^*M \times \mathbb{R}) \rightarrow M$ extending the Courant bracket. In Section 3, we introduce such a bracket on sections of $\mathcal{E}^1(M)$, which turns out to be an operation leading to the 1-jet Lie algebroid structure associated to any Jacobi manifold (see [Ke-SB]). Our approach is based on an idea that can be found in [D], where the author generalizes the Courant bracket to the case of complexes over Lie algebras, which is the algebraic counterpart of the de Rham complex of differential forms.

Our principal aim is to develop the theory of conformal Jacobi structures (see [D-L-M]) within the context of Dirac structures. So we expect to find an equivalence relation for Dirac structures generalizing the concept of a conformal Jacobi manifold. Our motivation for this investigation originates from the fact that an important class of manifolds, namely contact manifolds without a globally defined contact 1-form, are conformal Jacobi manifolds in the sense of [D-L-M] but not Jacobi manifolds.

Our presentation goes as follows: first, we introduce basic definitions and give some examples of Dirac structures on manifolds in Section 2.

In Section 3, we consider a skew symmetric bracket $[.,.]$ on the sections of $\mathcal{E}^1(M)$. This bracket is just a straightforward extension of the Courant bracket. Next, we consider $\mathcal{E}^1(M)$ endowed with its canonical symmetric 2-form and we define a Dirac structure of $\mathcal{E}^1(M)$ to be a maximal isotropic sub-bundle $L$ of $\mathcal{E}^1(M)$ whose sections are closed under this bracket $[.,.]$. Some properties of Dirac structures of $\mathcal{E}^1(M)$ are given in Theorem 3.4.

In Section 4, some examples of Dirac structures of $\mathcal{E}^1(M)$, such as homogeneous Poisson
manifolds, Jacobi structures and Nambu manifolds are presented.

Section 5 is devoted to the discussion about admissible functions of a Dirac structure of $\mathcal{E}^1(M)$. We show that there is an one-to-one correspondence between Jacobi structures on a manifold $M$ and Dirac structures of $\mathcal{E}^1(M)$ for which any smooth function on $M$ is admissible.

In Section 6, we introduce an equivalence relation among Dirac structures of $\mathcal{E}^1(M)$ and we use it to define a conformal Dirac structure. This section contains our main results (Theorem 6.1 and Theorem 6.4).

2 Courant bracket - Dirac structures on manifolds

Let $M$ be a smooth manifold. Consider the vector bundle $TM \oplus T^*M \to M$. The Courant bracket on the space $\Gamma(TM \oplus T^*M)$ of smooth sections of $TM \oplus T^*M$ is the skew symmetric operation $[\cdot, \cdot]_c$ such that for any $X_1 + \alpha_1, X_2 + \alpha_2 \in \Gamma(TM \oplus T^*M)$, we have

$$[X_1 + \alpha_1, X_2 + \alpha_2]_c = [X_1, X_2] + \mathcal{L}_{X_1} \alpha_2 - \mathcal{L}_{X_2} \alpha_1 + \frac{1}{2} d(i_{X_2} \alpha_1 - i_{X_1} \alpha_2),$$

where $i_X$ is the interior product by the vector field $X$, $d$ is the exterior derivative and $\mathcal{L}_X = d \circ i_X + i_X \circ d$ is the Lie derivation by $X$. The Courant bracket does not satisfy the Jacobi identity in spite of the fact that there is a striking similarity between this bracket and the Lie bracket on the double $\mathcal{G} \oplus \mathcal{G}^*$ of a Lie bialgebra $(\mathcal{G}, \mathcal{G}^*)$ (see [Dr]). In fact, if we denote by $(\cdot, \cdot)_+$ the canonical symmetric 2-form on $TM \oplus T^*M$, this induces an operation on $\Gamma(TM \oplus T^*M)$ defined by

$$(X_1 + \alpha_1, X_2 + \alpha_2)_+ = \frac{1}{2} (i_{X_2} \alpha_1 + i_{X_1} \alpha_2).$$

Hence we get

$$[[e_1, e_2], e_3]_c + c.p. = \frac{1}{3} d([e_1, e_2]_c, e_3)_+ + c.p.$$

for any $e_1, e_2, e_3 \in \Gamma(TM \oplus T^*M)$. Here the notation $c.p.$ denotes the other terms obtained by cyclic permutations of indices 1,2 and 3. The Courant bracket was systematized by Liu, Weinstein and Xu in [L-W-X].

In what follows, for simplicity, we will omit the base space of considered vector bundles when this is implied (i.e. a vector bundle $E \to M$ will be often replaced by its total space $E$).

**Definition 2.1** Let $L$ be a sub-bundle of $TM \oplus T^*M$. Then $L$ is said to be a Dirac structure on the manifold $M$ if $L$ is maximally isotropic for the bilinear form $(\cdot, \cdot)_+$ and the set $\Gamma(L)$ of smooth sections of $L$ is closed under the Courant bracket.

Let us give the basic examples of Dirac structures:
Example 2.2 Pre-symplectic 2-forms.

Let $\omega$ be a differential 2-form on $M$. It can be considered as a map $\omega : TM \to T^*M$ defined by $\omega(Y) = i_Y \omega$. Let us denote by $L_\omega$ its graph. Then, $L_\omega$ is a Dirac structure on $M$ if and only if $\omega$ is a closed 2-form.

Example 2.3 Poisson structures.

A Poisson structure on a manifold $M$ is given by a bivector field $\pi \in \Gamma(L^2(TM))$ such that $[\pi, \pi] = 0$, where $[,]$ is the Schouten-Nijenhuis bracket on the space of polyvector fields (see [Kz]).

Any bivector field $\pi$ defines a skew symmetric map, that we denote also by $\pi : T^*M \to TM$, whose extension to the sections is given by $\pi(\alpha, \beta) = \pi(\alpha, \beta)$ for any two 1-forms $\alpha, \beta$. The graph $L_\pi$ of that map is a Dirac structure on $M$ if and only if $\pi$ is a Poisson structure.

Example 2.4 Involutive distributions.

Let $F$ be sub-bundle of $TM$ and denote by $F^\perp$ its annihilator in $T^*M$. We consider $L = \{Y + \xi \in F, \xi \in F^\perp\}$. So $L$ is a Dirac structure on $M$ if and only if $\Gamma(F)$ is closed under the Lie bracket of vector fields.

3 Extension of the Courant bracket

Let $M$ be a smooth manifold. We shall extend the Courant bracket to the smooth sections of the vector bundle $E^1(M) \to M$, where

$$E^1(M) = (TM \times \mathbb{R}) \oplus (T^*M \times \mathbb{R}).$$

The smooth sections of $E^1(M)$ are of the form $e = (X, f) + (\alpha, g)$, where $X$ is a smooth vector field on $M$, $\alpha$ is a 1-form, $f$ and $g$ are smooth functions on $M$. In what follows, we use the notation $X \cdot f$ instead of $i_X df$ for any vector field $X$ and for any smooth function $f$. Consider the following bracket on $\Gamma(E^1(M))$:

$$[(X_1, f_1) + (\alpha_1, g_1), (X_2, f_2) + (\alpha_2, g_2)] = \left([X_1, X_2], X_1 \cdot f_2 - X_2 \cdot f_1\right) + \left(\mathcal{L}_{X_1} \alpha_2 - \mathcal{L}_{X_2} \alpha_1 + \frac{1}{2}d(i_{X_2} \alpha_1 - i_{X_1} \alpha_2)\right) + f_1 \alpha_2 - f_2 \alpha_1 + \frac{1}{2}(g_2 df_1 - g_1 df_2 - f_1 dg_2 + f_2 dg_1),$$

$$X_1 \cdot g_2 - X_2 \cdot g_1 + \frac{1}{2}(i_{X_2} \alpha_1 - i_{X_1} \alpha_2 + f_1 g_2 - f_2 g_1),$$

where the $(X_i, f_i) + (\alpha_i, g_i)$ are in $\Gamma(E^1(M))$. In order to simplify our notation, we use the same symbol $[,]$ for both the Lie bracket on vector fields and the bracket on sections of $E^1(M)$. By identifying $(X_i, 0) + (\alpha_i, 0)$ with $X_i + \alpha_i$, we get:

$$[X_1 + \alpha_1, X_2 + \alpha_2] = \left([X_1, X_2], 0\right) + \left(\mathcal{L}_{X_1} \alpha_2 - \mathcal{L}_{X_2} \alpha_1 + \frac{1}{2}d(i_{X_2} \alpha_1 - i_{X_1} \alpha_2)\right),$$
So the projection of the second member of this equality onto $TM \oplus T^*M$ gives the Courant bracket of $X_1 + \alpha_1$ and $X_2 + \alpha_2$.

**Remark 3.1** To any sub-bundle $L \subset TM \oplus T^*M$, we may associate the sub-bundle $\tilde{L}$ of $\mathcal{E}^1(M)$ whose fibre at $x$ is given by

$$\tilde{L}_x = \{(X(x), 0) + (\alpha(x), f(x)) \mid X + \alpha \in \Gamma(L), \ f \in C^\infty(M)\}.$$ 

In fact $L$ is maximally isotropic for the 2-form $\langle ., . \rangle_+$ if and only if $\tilde{L}$ is a maximal isotropic sub-bundle with respect to the extension of $\langle ., . \rangle_+$ to $\mathcal{E}^1(M)$ defined by:

$$\left\langle (X_1, \mu_1) + (\alpha_1, \lambda_1), (X_2, \mu_2) + (\alpha_2, \lambda_2) \right\rangle_+ = \frac{1}{2} \left( i_{X_2} \alpha_1 + i_{X_1} \alpha_2 + \lambda_1 \mu_2 + \lambda_2 \mu_1 \right)$$

for any $(X_i, \mu_i) + (\alpha_i, \lambda_i)$ elements of $\mathcal{E}^1(M)$, with $i = 1, 2$. Furthermore, $\Gamma(L)$ is closed under the Courant bracket if and only if $\Gamma(\tilde{L})$ is closed under the bracket $[.,.]$.

**Definition 3.2** Let $L$ be a sub-bundle of $\mathcal{E}^1(M)$. Then $L$ is be said to be a Dirac structure of $\mathcal{E}^1(M)$ if $L$ is maximally isotropic for the bilinear form $\langle ., . \rangle_+$ and $\Gamma(L)$ is closed under $[.,.]$.

It follows from Remark 3.1 that Dirac structures on $M$ are in one-to-one correspondence with Dirac structures of $\mathcal{E}^1(M)$ which are subsets of $TM \oplus (T^*M \times \mathbb{R})$. Hence, any closed 2-form (resp. foliation, Poisson structure) can be identified with a Dirac structure of $\mathcal{E}^1(M)$.

Let us fix some notations.

**Notations.** For any $e_1, e_2, e_3 \in \Gamma(\mathcal{E}^1(M))$, we set

$$T(e_1, e_2, e_3) = \frac{1}{3} \langle [e_1, e_2], e_3 \rangle_+ + c.p \quad \text{and} \quad \rho(e)h = X \cdot h$$

for any $e = (X, f) + (\alpha, g) \in \Gamma(\mathcal{E}^1(M))$ and $h \in \mathcal{C}^\infty(M)$.

**Proposition 3.3** With the above notations, we have:

(i) $[[e_1, e_2], e_3] + c.p. = (0, 0) + \left( dT(e_1, e_2, e_3), T(e_1, e_2, e_3) \right)$ for any $e_1, e_2, e_3 \in \Gamma(\mathcal{E}^1(M))$,

(ii) $[e_1, fe_2] = f[e_1, e_2] + (\rho(e_1)f)e_2 - \langle e_1, e_2 \rangle_+ \left( (0, 0) + (df, 0) \right)$ for any smooth function $f$ and for any $e_1, e_2 \in \Gamma(\mathcal{E}^1(M))$.

The proof of Proposition 3.3 is straightforward. We shall use this proposition to show that any Dirac structure of $\mathcal{E}^1(M)$ is a Lie algebroid. Recall that a vector bundle $\mathcal{E}$ over a smooth manifold $M$ is said to be a Lie algebroid if there is a Lie bracket $[.,.]_\mathcal{E}$ on $\Gamma(\mathcal{E})$ and a bundle map $\varrho : \mathcal{E} \to TM$, extended to a map between sections of these bundles, such that

1) $\varrho([X,Y]_\mathcal{E}) = [\varrho(X), \varrho(Y)]$, 

2) $[\varrho(X), \varrho(Y)] = \varrho([X,Y]_\mathcal{E})$ for any $X, Y \in \Gamma(\mathcal{E})$. 

3) The linear map $\varrho : \mathcal{E} \to TM$ is an $\mathcal{E}$-linear map, i.e., $\varrho(X_1 + X_2) = \varrho(X_1) + \varrho(X_2)$ and $\varrho(\lambda X) = \lambda \varrho(X)$ for any $X \in \Gamma(\mathcal{E})$ and $\lambda \in \mathbb{R}$.

4) The linear map $\varrho : \mathcal{E} \to TM$ is a Lie algebra homomorphism, i.e., $\varrho([X,Y]_\mathcal{E}) = \varrho(X) + \varrho(Y)$ for any $X, Y \in \Gamma(\mathcal{E})$.

5) $\varrho$ is a smooth bundle map.

6) $\varrho$ is integrable, i.e., for any $X, Y \in \Gamma(\mathcal{E})$, there exists a smooth function $f : M \to \mathbb{R}$ such that $\varrho(X)_x = \varrho(Y)_x$ for any $x \in M$. 

7) $\varrho$ is a Lie algebra homomorphism, i.e., $\varrho([X,Y]_\mathcal{E}) = [\varrho(X), \varrho(Y)]$ for any $X, Y \in \Gamma(\mathcal{E})$.

8) $\varrho$ is a Lie subalgebra homomorphism, i.e., $\varrho([X,Y]_\mathcal{E}) = [\varrho(X), \varrho(Y)]$ for any $X, Y \in \Gamma(\mathcal{E})$.

9) $\varrho$ is a Lie algebra homomorphism, i.e., $\varrho([X,Y]_\mathcal{E}) = [\varrho(X), \varrho(Y)]$ for any $X, Y \in \Gamma(\mathcal{E})$.

10) $\varrho$ is a Lie algebra homomorphism, i.e., $\varrho([X,Y]_\mathcal{E}) = [\varrho(X), \varrho(Y)]$ for any $X, Y \in \Gamma(\mathcal{E})$.
2) \([X, fY]_\xi = f[X, Y]_\xi + (\rho(X)f)Y\)

for any \(X, Y\) smooth sections of \(\xi\) and for any smooth function \(f\) defined on \(M\). Then \(\rho\) is called the anchor of the Lie algebroid.

**Theorem 3.4** Let \(L\) be an isotropic sub-bundle of \((\xi^1(M), \{\cdot, \cdot\}+)\) such that \(\Gamma(L)\) is closed under the bracket \([\cdot, \cdot]\) of \(\Gamma(\xi^1(M))\). Then \((L, [\cdot, \cdot], \rho_L)\) is a Lie algebroid. In particular, any Dirac structure of \(\xi^1(M)\) is a Lie algebroid.

**Proof:** By applying the first property of Proposition 3.3, we obtain the Jacobi identity on \(\Gamma(L)\).

Moreover, the second property of Proposition 3.3 implies that

\[ [e_1, f e_2] = f[e_1, e_2] + (\rho(e_1)f)e_2 \quad \forall e_1, e_2 \in \Gamma(L), \quad \forall f \in C^\infty(M). \]

\[ \square \]

**Remark 3.5** Let \((A, A^*)\) be a Lie bialgebroid (i.e. \(A\) and \(A^*\) are endowed with Lie algebroid structures such that the differential \(d_*\) on \(\Gamma(A^\wedge A^*)\), induced from the Lie algebroid structure on \(A^*\) is a derivation of the Lie bracket on \(\Gamma(A)\)). In [L-W-X], the authors define a bracket on the direct sum \(E = A \oplus A^*\) such that \(E\) is a Courant algebroid (see LWX for details). This bracket is different from the extended Courant bracket that we define above. It is clear that \((TM, T^*M)\) is a bialgebroid, where \(\Gamma(TM)\) is equipped with the Lie bracket and \(T^*M\) is equipped with the null Lie algebroid bracket. But Property (i) of Proposition 3.3 shows that the extended Courant bracket defined on \(\Gamma(\xi^1(M))\) does not induce a Courant algebroid structure on \(\xi^1(M)\).

### 4 Further examples of Dirac structures

We begin the section by giving a correspondence between Jacobi manifolds and some special Dirac structures. Next, we show that apart from Jacobi manifolds, there are other interesting examples of Dirac structures.

#### 4.1 Jacobi manifolds.

A *Jacobi structure* on a manifold \(M\) is given by a pair \((\pi, E)\) formed by a bivector field \(\pi\) and a vector field \(E\) such that ([L])

\[ [E, \pi]_s = 0, \quad [\pi, \pi]_s = 2E \wedge \pi, \]

where \([\cdot, \cdot]_s\) is the Schouten-Nijenhuis bracket on the space of polyvector fields. A manifold endowed with a Jacobi structure is said to be a Jacobi manifold. When \(E\) is zero, we get a Poisson structure.
Any smooth section \( P \) of \( \Lambda^2(TM \times R) \) is defined by a pair \((\pi, E)\), where \( \pi \) is a bivector field and \( E \) is a vector field on \( M \). Indeed, \( P \) can be regarded as a vector bundle map \( P: T^*M \times R \to TM \times R \) whose extension to the sections of \( T^*M \times R \) has the following form:

\[
P(\alpha, g) = (\pi\alpha + gE, -i_E\alpha)
\]

for any 1-form \( \alpha \) and for any smooth function \( g \), where \( \pi\alpha \) is the vector field defined by \( i_{\pi\alpha}\beta = \pi(\alpha, \beta) \) for any 1-form \( \beta \). The graph \( L_P \) of \( P \) is a Dirac structure of \( \mathcal{E}^1(M) \) if and only if \((\pi, E)\) is a Jacobi structure. In such a case, we may identify \( L_P \) with \( T^*M \times R \) equipped with the Lie algebroid structure that is associated to the Jacobi manifold. Namely, we consider the map \( \Phi: \Gamma(T^*M \times R) \to L_P \) defined by \( \Phi(\alpha, f) = (\pi\alpha + gE, -i_E\alpha) + (\alpha, f) \). Then \( \Phi \) a Lie algebroid isomorphism. Here \( L_P \) is endowed with the extended bracket \([.,.]\), while \( \Gamma(T^*M \times R) \) is equipped with the following bracket introduced in [Ke-SB]

\[
\{(\alpha, f), (\beta, g)\}_{(\pi, E)} = \left( (L_{\pi}\beta - L_{\pi}\alpha - d(\pi(\alpha, \beta))) + fL_E\beta - gL_E\alpha - i_E(\alpha \wedge \beta), \\
-\pi(\alpha, \beta) + \pi(\alpha, df) - \pi(\beta, df) + fE \cdot g - gE \cdot f \right).
\]

We refer the reader to [H-M] for discussions on morphisms of Lie algebroids. One may observe that \((T^*M \times R, TM \times R)\) is not a Lie bialgebroid when \((M, \pi, E)\) is a generic Jacobi manifold (otherwise we obtain the Poisson case).

### 4.2 Nambu manifolds.

Let \( M \) be an \( n \)-dimensional smooth manifold. A Nambu structure on \( M \) of order \( p \) (\( 2 < p \leq n \)) is defined by a \( p \)-vector field \( \Pi \) which satisfies the Fundamental Identity (see [T])

\[
\{f_1, \ldots, f_{p-1}, \{g_1, \ldots, g_p\}_n\} = \sum_{k=1}^{p}\{g_1, \ldots, g_{k-1}, \{f_1, \ldots, f_{p-1}, g_k\}_n, g_{k+1}, \ldots, g_p\}_n
\]

for any \( f_1, \ldots, f_{p-1}, g_1, \ldots, g_p \in C^\infty(M) \), where \( \{ \}_n \) is defined by

\[
\{f_1, \ldots, f_p\}_n = \Pi(df_1, \ldots, df_p) \quad \forall f_1, \ldots, f_p \in C^\infty(M).
\]

A manifold equipped with such a structure is called Nambu manifold. In fact, Nambu structures of order 2 are nothing but Poisson structures. We shall show that the local structure of Nambu manifolds gives rise to Dirac structures:

**Proposition 4.1** Locally, any Nambu structure of order \( n - 2 \) on an \( n \)-dimensional manifold corresponds to a family of Dirac structures.
To prove this proposition, we shall use the characterization of Nambu manifolds by means of differential forms which is given in [D-Z]. Precisely, let Ω be a volume form on $M$, any $p$-vector field $\Pi$ corresponds to a $(n - p)$-form $\omega = i_\Pi \Omega$. In [D-Z], the authors proved that $\Pi$ is a Nambu structure if and only if

$$(i_\Omega \omega) \wedge \omega = 0 \quad \text{and} \quad (i_A \omega) \wedge d\omega = 0$$

(1)

for any $(n-p-1)$-vector field $A$. Obviously, $\omega$ depends on the volume form $\Omega$. A differential form which satisfies the relations (1) is called a co-Nambu form. For the proof of Proposition 4.1, we need also the following two lemmas:

**Lemma 4.2** (see [GJ]) A $p$-vector field $\Pi$ is a Nambu structure of order $p$ if and only if for any point $x$ where $\Pi(x) \neq 0$, there exists a local system of coordinates $(x_1, \ldots, x_n)$ defined in a neighborhood of $x$ such that

$$\Pi = \frac{\partial}{\partial x_1} \wedge \cdots \wedge \frac{\partial}{\partial x_p}.$$  

**Lemma 4.3** Any pair $(\omega, \eta)$ formed by a differential 2-form and a closed 1-form such that $d\omega = \eta \wedge \omega$, corresponds to a Dirac structure $L_{(\omega, \eta)}$ of $\mathcal{E}^1(M)$.

**Proof:** For any 2-form $\omega$ and any closed 1-form $\eta$, we consider the sub-bundle $L_{(\omega, \eta)} \subset \mathcal{E}^1(M)$ whose fibre at a point $x$ is:

$$L_{(\omega, \eta)}(x) = \{ (X, -i_X \eta)_x + (i_X \omega + f \eta, f)_x / X \in \Gamma(TM), f \in C^\infty(M) \}.$$  

It is not hard to check that $L_{(\omega, \eta)}$ is a maximal isotropic sub-bundle of $\mathcal{E}^1(M)$. Furthermore, by a simple (but long) computation, we may prove that $L_{(\omega, \eta)}$ is a Dirac structure of $\mathcal{E}^1(M)$ if and only if we have the relation

$$d\omega = \eta \wedge \omega.$$  

(2)

This gives the lemma. 

**Remark 4.4** Dirac structures of the type $L_{(\omega, \eta)}$ embrace *locally conformal symplectic structures*. Recall that a locally conformal symplectic structure on an $2m$-dimensional manifold $M$ is given by a pair $(\omega, \eta)$, where $\omega$ is a non-degenerate 2-form and $\eta$ is a closed 1-form such that $d\omega = \eta \wedge \omega$. We have the definition:

**Definition 4.5** A *locally conformal pre-symplectic structure* is given by a 2-form $\omega$ and a closed 1-form such that $d\omega = \eta \wedge \omega$. 

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Proof of Proposition 4.1: Let \( \Pi \) be a Nambu structure of order \( n - 2 \) on an \( n \)-dimensional manifold. Denote \( \mathcal{U} = \{ x \in M / \Pi(x) \neq 0 \} \). Taking into account the fact that \( \Pi \) is a Nambu structure on \( M \) if and only if it is such on \( \mathcal{U} \), we may restrict ourselves to \( \mathcal{U} \). According to Lemma 4.2, around any point \( x_0 \) of \( \mathcal{U} \) we can write: \( \Pi = \partial / \partial x_1 \wedge \ldots \wedge \partial / \partial x_{n-2} \) for a suitable system of coordinates \((x_1, \ldots, x_n)\) defined on an open set \( \mathcal{V}_{x_0} \) containing \( x_0 \). To any function \( f \) such that \( f(x) \neq 0 \) on \( \mathcal{V}_{x_0} \), we associate the local volume form \( \Omega_f = f dx_1 \wedge \ldots \wedge dx_n \). We obtain \( i_{\Pi} \Omega_f = \pm dx_{n-1} \wedge dx_n \). Denote \( \omega_f = i_{\Pi} \Omega_f \). Then we have \( d \omega_f = \pm d|ln|f| \wedge \omega_f \). Applying Lemma 4.3, we deduce the existence of a Dirac structure of \( \mathcal{E}^1(\mathcal{V}_{x_0}) \) associated to any pair \((\omega_f, d|ln|f|)\). Thus, any function \( f : \mathcal{V}_{x_0} \to \mathbb{R} \) which vanishes nowhere, defines a Dirac structure \( L(\omega_f, d|ln|f|) \).

Conversely, assume that \( L(\omega, d|ln|f|) \) is Dirac structure, where \( f \) is a function which does not vanish on \( M \) and \( \omega \) is a decomposable 2-form (i.e. there are two independent differential 1-forms \( \omega_1, \omega_2 \) such that \( \omega = \omega_1 \wedge \omega_2 \)). Then \( \omega \) is a co-Nambu 2-form, moreover \( g \omega \) is also a co-Nambu 2-form for any non-vanishing function \( g \). From Lemma 4.3, we obtain that any pair \((g \omega, d|ln|g)\) corresponds to a Dirac structure. Using a local volume form \( \Omega \) defined on an open set \( \mathcal{V} \), we get a Nambu structure \( \Pi \) on \( \mathcal{V} \) characterized by \( i_{\Pi} \Omega = \omega \). Therefore, Proposition 4.1 is obtained.

4.3 Homogeneous Poisson manifolds.

A homogeneous Poisson manifold \((M, \pi, Z)\) is a Poisson manifold \((M, \pi)\) with a vector field \( Z \) on \( M \) such that \([Z, \pi]_s = -\pi\), where \([.,.]_s\) is the Schouten-Nijenhuis bracket defined on polyvector fields. This kind of structures were studied by Dazord, Lichnerowicz and Marle in [D-L-M]. We shall give a new characterization of these structures. Namely, we have the following result:

Proposition 4.6 Let \( \pi \) be a bivector field and let \( Z \) be a vector field on \( M \). Consider the sub-bundle \( L(\pi, Z) \) whose fibre at a point \( x \) is given

\[
L(\pi, Z)(x) = \{ (\pi\alpha - f Z, f) + (\alpha, i_Z\alpha)_x / \alpha \in \Gamma(T^*M), \, f \in C^\infty(M) \}.
\]

Then \((M, \pi, Z)\) is homogeneous Poisson manifold if and only if \( L(\pi, Z) \) is a Dirac structure of \( \mathcal{E}^1(M) \).

Proof: By an easy computation, we see that \( L(\pi, Z) \) is maximally isotropic. Notice that the set of smooth sections of \( L(\pi, Z) \) is spanned by \((Z, -1) + (0, 0)\) and elements of the type \((\pi\alpha, 0) + (\alpha, i_Z\alpha)\), where \( \alpha \) is an arbitrary differential 1-form. By using Property (ii) of Proposition 3.3, we see that it is sufficient to study the bracket of such sections. On one hand for any differential 1-form, we have
\[(\pi \alpha, 0) + (\alpha, i_Z \alpha), (Z, -1) + (0, 0)\] = 
\[- (\nabla, \pi \alpha + \pi L \alpha, 0) \]
\[- (L \alpha - \alpha, i_Z (L \alpha - \alpha))\].

Using the definition of the Schouten-Nijenhuis bracket, we get

\[ [Z, \pi] \alpha = [Z, \pi \alpha] - \pi L \alpha \quad \forall \alpha \in \Gamma(T^*M). \tag{3} \]

It follows that \([(\pi \alpha, 0) + (\alpha, i_Z \alpha), (Z, -1) + (0, 0)]\) belongs to \(\Gamma(L(\pi, Z))\) if and only if

\[ [Z, \pi] \alpha = -\pi \alpha. \]

On the other hand, from a direct calculation of the bracket

\[ [(\pi \alpha_1, 0) + (\alpha_1, i_Z \alpha_1), (\pi \alpha_2, 0) + (\alpha_2, i_Z \alpha_2)], \]

we deduce that this bracket is a smooth section of \(L(\pi, Z)\) for any two differential 1-forms \(\alpha_1, \alpha_2\) if and only if \([\pi, \pi] = 0\) and \([Z, \pi] = -\pi\).

This completes the proof of Proposition 4.6.

Before ending this section, let us remark that any exact differential 2-form on a manifold \(M\) is a Dirac structure of \(\mathcal{E}^1(M)\) which is a graph of a certain map. More precisely, let \(B\) be a smooth section of \(\Lambda^2(T^*M \times \mathbb{R})\). So, \(B\) is given by a pair \((\omega, \alpha)\) formed by a 2-form \(\omega\) and a 1-form \(\alpha\). We may define a vector bundle map \(\phi_B : TM \times R \to T^*M \times \mathbb{R}\) extended to sections, such that

\[ \phi_B(X, f) = (i_X \omega + f \alpha, -i_X \alpha) \]

for any smooth vector field \(X\) and for any \(f \in C^\infty(M)\). The graph of \(\phi_B\) is a Dirac structure of \(\mathcal{E}^1(M)\) if and only if \(\omega = d\alpha\).

5 The Jacobi algebra of admissible functions

Definition 5.1 Let \(L\) be a maximal isotropic sub-bundle of \((\mathcal{E}^1(M), \langle ., . \rangle)\). A function \(f\) is said to be \(L\)-admissible if there exist a vector field \(X_f\) and a smooth function \(\varphi_f\) on \(M\) such that \(e_f = (X_f, \varphi_f) + (df, f)\) is a smooth section of \(L\).

Remark that \(e_f\) is unique up to a smooth section of \(L \cap (TM \times \mathbb{R})\). Let \(\langle ., . \rangle_-\) denote the skew symmetric 2-form on \(\mathcal{E}^1(M)\) defined by:

\[ \langle (X_1, \mu_1) + (\alpha_1, \lambda_1), (X_2, \mu_2) + (\alpha_2, \lambda_2) \rangle_- = \frac{1}{2} \left( i_{X_2} \alpha_1 - i_{X_1} \alpha_2 + \lambda_1 \mu_2 - \lambda_2 \mu_1 \right) \]
for any \((X_i, \mu_i) + (\alpha_i, \lambda_i)\) in \(\mathcal{E}^1(M)\). Define the following bracket on the space of \(L\)-admissible functions:

\[
\{f, g\} = -\langle e_f, e_g \rangle_-
\]

for any two \(L\)-admissible functions \(f\) and \(g\). This bracket is well defined. Indeed, if \(e'_f = (X'_f, \varphi'_f) + (df, f)\) is another element of \(\Gamma(L)\) then \(e_f - e'_f\) is in \(\Gamma(L \cap (TM \times \mathbb{R}))\). Therefore for any \(L\)-admissible function \(g\), we have

\[
\langle e_f - e'_f, e_g \rangle_+ = \langle e_f - e'_f, e_g \rangle_- = 0.
\]

Hence \(\langle e_f, e_g \rangle_- = \langle e'_f, e_g \rangle_-\). Since \(\langle e_f, e_g \rangle_+ = 0\), we can rewrite

\[
\{f, g\} = ix_f dg + g \varphi_f = -(ix_g df + f \varphi_g).
\]

It should be mentioned that the bracket \(\{., .\}\) is local, this means that the support of \(\{f, g\}\) is contained in the intersection of the supports of \(f\) and \(g\). Precisely, \(\{f, g\}(x)\) depends only on the values of \(df(x), dg(x), f(x)\) and \(g(x)\). One may notice that our definition of an \(L\)-admissible function is a little bit more enlarged than the one given by Courant in [C]. Here, we do not need the fact that \(\Gamma(L)\) is closed under \(\{., .\}\) in the definition of \(\{., .\}\). We have the following result:

**Proposition 5.2** Let \(L\) be a Dirac structure of \(\mathcal{E}^1(M)\). Then the space of \(L\)-admissible functions is a Lie algebra.

**Lemma 5.3** Let \(L\) be an isotropic sub-bundle of \(\mathcal{E}^1(M)\) for the symmetric form \(\langle ., . \rangle_+\). If \(e_f = (X_f, \varphi_f) + (df, f)\) and \(e_g = (X_g, \varphi_g) + (dg, g)\) are two smooth sections of \(L\), then

\[
[e_f, e_g] = ([X_f, X_g], X_f \cdot \varphi_g - X_g \cdot \varphi_f) + (d\{f, g\}, \{f, g\}),
\]

where \(\{f, g\} = -\langle e_f, e_g \rangle_-\).

**Proof:** Using the definition of the bracket \(\{., .\}\) on \(\mathcal{E}^1(M)\), we get

\[
[e_f, e_g] = ([X_f, X_g], X_f \cdot \varphi_g - X_g \cdot \varphi_f) + \frac{1}{2} \left( d(X_f \cdot g - X_g \cdot f + g \varphi_f - f \varphi_g), X_f \cdot g - X_g \cdot f + g \varphi_f - f \varphi_g \right).
\]

This is equivalent to

\[
[e_f, e_g] = ([X_f, X_g], X_f \cdot \varphi_g - X_g \cdot \varphi_f) - (d\langle e_f, e_g \rangle_-, \langle e_f, e_g \rangle_-)
\]

\[
= ([X_f, X_g], X_f \cdot \varphi_g - X_g \cdot \varphi_f) + (d\{f, g\}, \{f, g\})
\]

This proves the lemma.
Proof of Proposition 5.2: Let \( L \subset \mathcal{E}^1(M) \) be a Dirac structure. It is clear that the corresponding bracket \( \{.,.\} \) is \( \mathbb{R} \)-bilinear and skew symmetric. Moreover, using Lemma 5.3, we get:

\[
\langle [e_f, e_g], e_h \rangle_+ = \frac{1}{2} \left( i_{[X_f, X_g]} dh + h( X_f \cdot \varphi_g - X_g \cdot \varphi_f ) + \{h, \{f, g\} \} \right) \\
= \frac{1}{2} \left( \mathcal{L}_{X_f} \mathcal{L}_{X_g} h - \mathcal{L}_{X_g} \mathcal{L}_{X_f} h + h( X_f \cdot \varphi_g - X_g \cdot \varphi_f ) + \{h, \{f, g\} \} \right)
\]

for any \( L \)-admissible functions \( f, g \) and \( h \). If we add and withdraw the term \( \varphi_f X_g \cdot h - \varphi_f X_g \cdot h \) in the second member of this last equality, we obtain:

\[
\langle [e_f, e_g], e_h \rangle_+ = \frac{1}{2} \left( X_f \cdot \{g, h\} - X_g \cdot \{f, h\} + \varphi_f X_g \cdot h - \varphi_g X_f \cdot h + \{h, \{f, g\} \} \right) \\
= \{f, \{g, h\}\} - \{g, \{f, h\}\} + \{h, \{f, g\}\}.
\]

By using the definition of a Dirac structure, we have:

\[
\langle [e_f, e_g], e_h \rangle_+ = \{f, \{g, h\}\} - \{g, \{f, h\}\} + \{h, \{f, g\}\} = 0.
\]

Therefore, the space of \( L \)-admissible functions forms a Lie algebra.

Proposition 5.2 extends a result proven in [C-W] for Dirac structures on \( M \) (i.e. for Dirac structures of \( \mathcal{E}^1(M) \) which are subsets of \( TM \oplus (TM \times \mathbb{R}) \)). Now let us give a remark that will be useful later.

Remark 5.4 Let \( L \) be a Dirac structure of \( \mathcal{E}^1(M) \). The product \( fg \) of two \( L \)-admissible functions \( f \) and \( g \) may not be \( L \)-admissible. But, if \( (Y_g, \theta_g) + (dg, 0) \) is in \( \Gamma(L) \), then for any \( L \)-admissible function \( f \), the product \( gf \) is also \( L \)-admissible. Indeed, the section

\[
g \left( (X_f, \varphi_f) + (df, f) \right) + f \left( (Y_g, \theta_g) + (dg, 0) \right) = (gX_f + fY_g, \quad g\varphi_f + f\theta_g) + (d(gf), gf)
\]

belongs to \( \Gamma(L) \). By induction, we obtain that \( g^n f \) is \( L \)-admissible for any integer \( n \geq 1 \). On the other hand, If \( f, g, h \) and \( gh \) are \( L \)-admissible functions, then we have

\[
\{f, gh\} = g\{f, h\} + \{f, g\} h - gh \varphi_f,
\]

where \( (X_f, \varphi_f) + (df, f) \in \Gamma(L) \). This holds since we have:

\[
\{f, gh\} = gX_f \cdot h + hX_f \cdot g + gh \varphi_f \\
= gX_f \cdot h + h\varphi_f + (X_f \cdot g + g\varphi_f) h - gh \varphi_f \\
= g\{f, h\} + \{f, g\} h - gh \varphi_f.
\]
From now on we denote by $A^\infty(L)$ the set of $L$-admissible functions, whenever $L$ is maximal isotropic sub-bundle of $(\mathcal{E}^1(M), \langle \cdot, \cdot \rangle_+)$.

The rest of this section is devoted to the case where the constant functions are $L$-admissible, for a given Dirac structure $L \subset \mathcal{E}^1(M)$. In such a case, there exists a smooth vector field $E$ such that $(E, 0) + (0, 1)$ belongs to $\Gamma(L)$. Therefore, if $e_f = (X_f, \varphi_f) + (df, f)$ is in $\Gamma(L)$, then

$$\langle (E, 0) + (0, 1), e_f \rangle_+ = 0.$$ 

This gives $\varphi_f = -E \cdot f$. Hence Equation (4) implies

$$\{f, gh\} = g\{f, h\} + \{f, g\} h + ghE \cdot f.$$ 

It follows from this above equality, the existence a bi-differential operator on $A^\infty(L)$ of order at most one in each of its argument, denoted by $P$, such that $P(f, g) = \{f, g\}$. This shows that $A^\infty(L)$ is a Jacobi algebra.

Recall that a Jacobi algebra is an associative commutative algebra $A$ with unit 1 (over $\mathbb{R}$ or $\mathbb{C}$) endowed with a skew symmetric bi-differential operator $P$, which defines a Lie algebra structure on $A$. When $P$ vanishes on constants, we get a Poisson algebra. A Jacobi structure $(\pi, E)$ on a smooth manifold $M$ is equivalent to having a Jacobi algebra structure on $C^\infty(M)$ ([Gr]). We summarize our discussion in the following form:

**Theorem 5.5** There exists an one-to-one correspondence between Jacobi structures on $M$ and Dirac structures $L$ of $\mathcal{E}^1(M)$ such that $A^\infty(L) = C^\infty(M)$.

## 6 Conformal Dirac structures

In this section, we introduce the concept of a conformal Dirac structure which includes conformal Jacobi structures. Conformal Jacobi structures are known to be natural generalizations of local Lie algebras considered by Kirillov in [K]. First, we establish the following result:

**Theorem 6.1** Let $L$ be a Dirac structure of $\mathcal{E}^1(M)$. Assume that $\Gamma(L)$ contains an element of the form $(Y_a, \theta_a) + (da, 0)$, with $a(x) \neq 0$ for any $x \in M$. Then the space $A^\infty(L)$ of $L$-admissible functions is endowed with two Lie algebra brackets $\{\cdot, \cdot\}$ and $\{\cdot, \cdot\}^a$ linked by:

$$\{f, g\}^a = \frac{1}{a}(af, ag)$$

for any two $L$-admissible functions $f$ and $g$.

**Proof:** The bracket $\{\cdot, \cdot\}$ is defined on $A^\infty(L)$ by:

$$\{f, g\} = -(e_f, e_g)_-,$$

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where \( e_f = (X_f, \varphi_f) + (df, f) \) and \( e_g = (X_g, \varphi_g) + (dg, g) \) are sections of \( L \). By Remark 5.4, the functions \( af \) and \( ag \) are \( L \)-admissible. Thus, the expression \( \{af, ag\} \) makes sense. Moreover, Proposition 5.2 ensures that \( \{.,.\} \) is a Lie bracket. Hence it remains to prove that \( \{.,.\}^a \) satisfies the Jacobi identity. For any \( L \)-admissible functions \( f_1, f_2 \) and \( f_3 \), we have

\[
\{ \{f_1, f_2\}^a, f_3 \}^a = \frac{1}{a} \{a \{f_1, f_2\}^a, af_3 \} = \frac{1}{a} \{\{af_1, af_2\}, af_3 \}.
\]

The Jacobi identity for the bracket \( \{.,.\} \) implies

\[
\{ \{f_1, f_2\}^a, f_3 \}^a + c.p. = \frac{1}{a} \{\{af_1, af_2\}, af_3 \} + c.p. = 0.
\]

This concludes the proof of Theorem 6.1.

In fact, \( \{.,.\}^a \) is related to a specific isotropic sub-bundle of \( (\mathcal{E}^1(M), \{.,.\}_+) \) which depends on \( a \). To define properly this sub-bundle, we need the following lemma:

**Lemma 6.2** Let \( L \) be a maximal isotropic sub-bundle of \( (\mathcal{E}^1(M), \{.,.\}_+) \). Let \( a \) be a smooth function on \( M \) that vanishes nowhere. Fix a vector field \( Y \) and consider the sub-bundle \( \hat{L} \subset \mathcal{E}^1(M) \) whose fibre at a point \( x \in M \) is given by

\[
\hat{L}_x = \text{span} \left\{ ((aX + fY)_x, (a\varphi - i_Y \alpha)_x) + (\alpha_x, f)_x / (X, \varphi) + (\alpha, f) \in \Gamma(L) \right\}.
\]

Then \( \hat{L} \) is maximally isotropic.

**Proof:** Consider two elements of \( \Gamma(\hat{L}) \) denoted by \( \delta_1 \) and \( \delta_2 \) such that

\[
\delta_n = (aX_n + f_n Y, a\varphi_n - i_Y \alpha_n) + (\alpha_n, f_n), \quad \forall n = 1, 2
\]

with \( e_n = (X_n, \varphi_n) + (\alpha_n, f_n) \in \Gamma(L) \). We have

\[
\langle \delta_1, \delta_2 \rangle_+ = a \langle e_1, e_2 \rangle_+.
\]

Since \( L \) is isotropic, we deduce that \( \langle \delta_1, \delta_2 \rangle_+ = 0 \). This proves that \( \hat{L} \) is isotropic under \( \{.,.\}_+ \). This is a maximal isotropic sub-bundle of \( \mathcal{E}^1(M) \). Indeed, suppose that \( L' \) is isotropic sub-bundle of \( \mathcal{E}^1(M) \) which contains \( \hat{L} \). Then for any \( \delta_1 \in \Gamma(\hat{L}) \) and for any \( e' = (Z, h) + (\beta, k) \in \Gamma(L') \), we have

\[
0 = 2 \langle \delta_1, e' \rangle_+ = i(aX_1 + f_1 Y) \beta + iZ \alpha_1 + hf_1 + k(a\varphi_1 - i_Y \alpha_1) = \langle e_1, (Z - kY, h + i_Y \beta) + (a\beta, ak) \rangle_+.
\]
where \( e_1 = (X_1, \varphi_1) + (\alpha_1, f_1) \) is the section of \( L \) corresponding to \( \delta_1 \). Since \( L \) is maximally isotropic, we deduce that \( (Z - kY, h + i_Y \beta) + a(\beta, k) \) is in \( \Gamma(L) \). Hence we obtain the existence of a section \((X', \varphi') + (\alpha', f') \in \Gamma(L) \) such that
\[
(Z - kY, h + i_Y \beta) + a(\beta, k) = a(X', \varphi') + a(\alpha', f').
\]
This means that
\[
e' = (aX' + f'Y, a\varphi' - i_Y a') + (\alpha', f').
\]
Therefore \( \tilde{L} = L' \). This proves the lemma.

**Proposition 6.3** Let \( L \) be a maximally isotropic sub-bundle of \( \mathcal{E}^1(M), (, ,) \) and let \( a \) be a smooth function on \( M \) that vanishes nowhere. Assume that \((Y_a, \theta_a) + (da, 0)\) is a smooth section of \( L \). Consider the sub-bundle \( L^a \subset \mathcal{E}^1(M) \) whose fibre at a point \( x \in M \) is given by
\[
L^a_x = \text{span}\{(aX + fY)_x, (a\varphi - i_Y a)_x\} + (\alpha_x, f_x) \in \Gamma(L)\}.
\]
Then the corresponding bracket on \( \mathcal{A}^\infty(L^a) \) coincide with \( \{, ,\}^a \).

**Proof:** We have to prove that if \( \delta_f = (Y_f, \psi_f) + (df, f) \) and \( \delta_g = (Y_g, \psi_g) + (dg, g) \) are in \( \Gamma(L^a) \), then
\[
\{f, g\}^a = -\langle \delta_f, \delta_g \rangle_-. \]
Obviously, a function \( f \) is \( L \)-admissible if and only if it is \( L^a \)-admissible. More precisely, any section \((X_f, \varphi_f) + (df, f) \in \Gamma(L) \) corresponds to a section \( \delta_f \) of \( L^a \) given by
\[
\delta_f = (X_{af}, \varphi_{af} - f\theta_a - Y_a \cdot f) + (df, f).
\]
where
\[
X_{af} = aX_f + fY_a, \quad \varphi_{af} = a\varphi_f + f\theta_a
\]
and \( e_{af} = (X_{af}, \varphi_{af}) + (da, a) \in \Gamma(L) \). We have a similar expression for \( \delta_g \). Therefore,
\[
2\langle \delta_f, \delta_g \rangle_- = X_{ag} \cdot f - X_{af} \cdot g + f\varphi_{ag} - g\varphi_{af} + i_Y (g\varphi_f - f\varphi_g).
\]
This is equivalent to the following equation:
\[
2\langle \delta_f, \delta_g \rangle_- = \frac{1}{a} \left( \langle e_{af}, e_{ag} \rangle_+ + i_Y (gX_{af} - fX_{ag})da \right) + i_Y (g\varphi_f - f\varphi_g).
\]
Now, we replace $X_{af}$ and $X_{ag}$ by their value and use the fact that

$$i_{Ya} da = ((Ya, \theta_a) + (da, 0), (Ya, \theta_a) + (da, 0))_+ = 0.$$ 

Then we obtain

$$\langle \delta_f, \delta_g \rangle_+ = -\frac{1}{a} \{af, ag\} + \frac{1}{2} \left(i_{(gX_f-fX_g)} da + i_{Ya} (gdf-fdg)\right).$$ 

Since $fe_g - ge_f$ is in $\Gamma(L)$, we have

$$\langle fe_g - ge_f, (Ya, \theta_a) + (da, 0) \rangle_+ = i_{(gX_f-fX_g)} da + i_{Ya} (gdf-fdg) = 0.$$ 

We deduce that

$$-(\delta_f, \delta_g)_+ = \frac{1}{a} \{af, ag\} = \{f, g\}_a.$$ 

Now we are ready to state our second main result:

**Theorem 6.4** Let $L$ be a Dirac structure of $\mathcal{E}^1(M)$ and let $a$ be a smooth function on $M$ which vanishes nowhere. Assume that $\Gamma(L)$ is spanned by elements of the form $e_f = (X_f, \varphi_f) + (df, f)$ and there exists a smooth section of the type $(Ya, \theta_a) + (da, 0)$ in $\Gamma(L)$. Then the sub-bundle $L^a$ whose fibre at a point $x \in M$ is given by

$$L^a_x = \text{span}\left\{(aX_f + fYa, a\varphi_f - Ya \cdot f)_x + (df, f)_x / (X_f, \varphi_f) + (df, f) \in \Gamma(L)\right\}$$

is a Dirac structure of $\mathcal{E}^1(M)$.

To prove Theorem 6.4, we use the following lemma:

**Lemma 6.5** Let $L$ be a maximally isotropic sub-bundle of $\mathcal{E}^1(M)$ with respect to $\langle ., . \rangle_+$. Then $L$ is a Dirac structure if and only if for any $e_1, e_2$ and $e_3 \in \Gamma(L)$, we have:

$$\langle [e_1, e_2], e_3 \rangle_+ = 0.$$ 

**Proof:** This lemma is an immediate consequence of the fact that $L$ is a maximally isotropic sub-bundle of $\mathcal{E}^1(M)$.

**Proof of Theorem 6.4:** $\Gamma(L^a)$ is spanned the elements of the form $\delta_f = (Y_f, \psi_f) + (df, f)$, where $Y_f = aX_f + fYa$, $\psi_f = a\varphi_f - Ya \cdot f$ and $e_f = (X_f, \varphi_f) + (df, f) \in \Gamma(L)$. Moreover, it follows from Lemma 6.2 that $L^a$ is a maximal isotropic sub-bundle under $\langle ., . \rangle_+$. So, taking into account Lemma 6.5, we have only to prove that

$$\langle [\delta_{f_1}, \delta_{f_2}], \delta_{f_3} \rangle_+ = 0$$

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for any $L$-admissible functions $f_1, f_2$ and $f_3$. Using Proposition 6.3, we obtain

$$\langle [\delta_{f_1}, \delta_{f_2}], \delta_{f_3} \rangle_+ = \{\{f_1, f_2\}^a, f_3\}^a + c.p.,$$

where $\{f_i, f_j\}^a = -\langle \delta_{f_i}, \delta_{f_j} \rangle$. According to Theorem 6.1, $\{\cdot, \cdot\}^a$ is a Lie bracket on $A^\infty(L)$.

Thus, the result is proved.

Remark 6.6 In Theorem 6.4, if $\Gamma(L)$ is not spanned by the elements of the form $e_f = (X_f, \varphi_f) + (df, f)$, one gets a maximal isotropic sub-bundle $L^a$ (see Lemma 6.2). But, it is not clear whether the set of sections of $L^a$ is closed under the bracket $[\cdot, \cdot]$ of $\mathcal{E}^1(M)$. However, all the hypothesis of Theorem 6.4 are satisfied when $L$ corresponds to a locally conformal pre-symplectic structure or a Jacobi structure. We propose the definition:

Definition 6.7 Let $a$ be a smooth function on $M$ which vanishes nowhere. Let $L$ and $L'$ be two Dirac structures of $\mathcal{E}^1(M)$. Assume that there exists a smooth section of the form $(Y_a, \theta_a) + (da, 0)$ in $\Gamma(L)$ and the smooth sections of $L'$ are of the type $(aX + fY_a, a\varphi - i_{Y_a}a) + (\alpha, f)$, where $(X, \varphi) + (\alpha, f)$ belongs to $\Gamma(L)$. Then, $L'$ is said to be $a$-conformal to $L$.

This suggests to define a relation $\mathcal{R}$ among Dirac structures of $\mathcal{E}^1(M)$ by: $L' \mathcal{R} L$ if and only if there exists a smooth function $a$ satisfying $a(x) \neq 0$, $\forall x \in M$ and $L'$ is $a$-conformal to $L$. In such a case, we say that $L'$ is conformally equivalent to $L$ on $M$.

Proposition 6.8 The relation $\mathcal{R}$ is an equivalence relation.

Proof: It is clear that $L$ is 1-conformal to itself, where 1 is the constant function equal to 1 at any point of $M$. Now, assume that $L'$ is $a$-conformal to $L$. By definition, there exists a smooth section $(Y_a, \theta_a) + (da, 0) \in \Gamma(L)$. Since $Y_a \cdot a = 0$ and any smooth section of $L'$ can be obtained from an element of $\Gamma(L)$, we deduce that $(aY_a, a\theta_a) + (da, 0)$ belongs to $\Gamma(L')$. This implies that $(Z_a, \eta_a) + (d(\frac{1}{a}), 0) \in \Gamma(L')$, where

$$Z_a = -\frac{1}{a}Y_a \quad \text{and} \quad \eta_a = -\frac{1}{a}\theta_a.$$  

Moreover we have

$$(X', \varphi') + (\alpha, f) \in \Gamma(L') \iff (\frac{1}{a}X + fZ_a, \frac{1}{a}\varphi - i_{Z_a}a) + (\alpha, f) \in \Gamma(L).$$

This proves that $L$ is $\frac{1}{a}$-conformal to $L'$. Furthermore, it is easy to see that if $L''$ is $b$-conformal to $L'$ and $L'$ is $a$-conformal to $L$, then $L''$ is $ab$-conformal to $L$. So, we obtain the proposition.

This gives rise to the definition:
Definition 6.9 A conformal Dirac structure on $M$ is defined by an open cover $\{U_i, i \in I\}$ of $M$ and a collection of Dirac structures $L_i \subset \mathcal{E}^1(U_i)$ such that if $U_i \cap U_j$ is not empty then $L_i$ is conformally equivalent $L_j$ upon $U_i \cap U_j$, where $\mathcal{E}^1(U_i) = (TU_i \times \mathbb{R}) \oplus (T^*U_i \times \mathbb{R})$.

Consider another open cover $\{V_r, r \in R\}$ of $M$ such that each $V_r$ is endowed with a Dirac structure $\tilde{L}_r \subset \mathcal{E}^1(V_r)$ and whenever $V_r \cap V_s \neq \emptyset$, $\tilde{L}_r$ is conformally equivalent to $\tilde{L}_s$. We say that this datum and $\{(U_i, L_i), i \in I\}$ define the same conformal Dirac structure on $M$ if we have:

$$U_i \cap V_r \neq \emptyset \implies L_i \text{ is conformally equivalent to } \tilde{L}_r, \forall (i, r) \in I \times R.$$

Remark 6.10 Obviously the equivalence class (with respect to $\mathcal{R}$) of any Dirac structure $L \subset \mathcal{E}^1(M)$ defines a conformal Dirac structure on $M$. But it is not hard to find a conformal Dirac structure which is not globally defined on a manifold $M$.

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