Письма в ЭЧАЯ №3[100]-2000 *Particles and Nuclei, Letters No.3[100]-2000*

УДК 539.1.01

EHRENFEST FORCE IN INHOMOGENEOUS MAGNETIC FIELD

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The Ehrenfest force in an inhomogeneous magnetic field is calculated. It is shown that there exist the such (very rare) topologically nontrivial physical situations when the Gauss theorem in its classic formulation fails and, as a consequence, apart from the usual Lorentz force an additional, purely imaginary force acts on the charged particle. This force arises only in inhomogeneous magnetic fields of special configurations, has a *purely quantum origin, and disappears in the classical limit.*

The investigation has been performed at the Bogoliubov Laboratory of Theoretical Physics, JINR.

Сила Эренфеста в неоднородном магнитном поле

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Вычисляется сила Эренфеста в неоднородном магнитном поле. Показывается, что существуют такие (очень редкие) топологически нетривиальные ситуации, когда теорема Гаусса в ее классической формулировке неприменима и, как следствие, помимо обычной силы Лоренца, добавочная, чисто мнимая сила действует на заряженную частицу. Такая сила возникает только в неоднородных магнитных полях специальных конфигураций, имеет чисто квантовое происхождение и исчезает в классическом пределе.

Работа выполнена в Лаборатории теоретической физики им. Н.Н.Боголюбова ОИЯИ.

After discovery of the so-called «topological effects» (Aharonov-Bohm effect in quantum mechanics [1], solitons, instantons, monopoles, Chern-Simons, etc., in the field theory — see [2] for review) we began to fully realize importance of boundary conditions and of space topology in physics. In particular, it became clear that we often cannot neglect the full derivative without risking to lose an important (and even the most significant) part of relevant information. Inspired by these arguments we could attempt to look for some new topologically nontrivial physical situations where the full derivative plays a crucial role and gives rise to observable phenomena. For this purpose we might consider observable quantities explicitly including space integrations (for example, some quantum-mechanical averages) with a nonsimply-connected integration area and pay special attention to the full derivative under the space integral symbol.

So, let us evaluate the quantum-mechanical average force (the Ehrenfest force — see, for example, [3], chapter 10) acting on a particle with an electrical charge q which is put in an electromagnetic (for a moment arbitrary) field $\vec{E} = \vec{\text{grad}}\, A_0 - (1/c)(\partial\vec{A}/\partial t)$, $\vec{H} = \vec{\text{rot}}\vec{A}$, or in terms of the field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$: $E_i = F_{i0}$, $H_i = \frac{1}{2}\epsilon_{ijk} F_{jk}$. The quantum-mechanical Hamiltonian of the particle has the standard form [1]

$$\hat{H} = \frac{1}{2m}\left(\hat{\vec{p}} - \frac{q}{c}\vec{A}\right)^2 + q\varphi + V, \tag{1}$$

where q and m are the electric charge and mass of the particle, $\hat{p}_i = -i\hbar\partial_i$ is the momentum operator, $e\varphi = -A_0$ and V are the electrostatic and nonelectrostatic parts of the interaction potential, respectively.

The average quantum-mechanical force reads

$$\bar{F}_i = m\frac{d^2\bar{x}_i}{dt^2}, \tag{2}$$

where we use the notation

$$\bar{L} \equiv N\int d^3x\psi^*\hat{L}\psi, \quad N^{-1} \equiv \int d^3x\psi^*\psi. \tag{3}$$

In accordance with the theorem on the average value [2]

$$d\bar{L}/dt \equiv (d/dt)\langle\psi|\hat{L}|\psi\rangle = N\int d^3x\,\psi^*\left(d\hat{L}/dt\right)\psi,$$

where

$$\frac{d\hat{L}}{dt} = \frac{\partial\hat{L}}{\partial t} + \frac{1}{i\hbar}[\hat{L}, \hat{H}], \tag{4}$$

we have

$$\bar{F}_i = N\int d^3x\,\psi^*\left(m\frac{d^2\hat{x}_i}{dt^2}\right)\psi, \tag{5}$$

where the operator $d^2\hat{x}_i/dt^2$ has to be calculated by using Eq. (4) (see, for example, [4], problem 7.3).

Calculating the simple commutators of the respective operators with the Hamiltonian (1) one easily obtains $\hat{F}_i = \hat{F}_i^{pot} + \hat{F}_i^{magn}$, where $\hat{F}_i^{pot} = -\partial V/\partial x_i + E_i$, and

$$\hat{F}_i^{magn} = \frac{q}{2c}[F_{ij}\hat{v}_j + \hat{v}_j F_{ij}], \tag{6}$$

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[1] We neglect here by the spin and relativistic effects.

[2] Strictly speaking, taking into account the specific nature of the material presented below, we ought to add the term $dN/dt \int d^3x\psi^*\hat{L}\psi$ in r.h.s of this equation. However, we omit this term having in mind that we will use as a starting point wave functions with conserved normalization – the wave functions of instantaneous approximation, «frozen» at the moment when the magnetic field is switched on.

where

$$\hat{v}_i = \frac{1}{m}\left(\hat{p}_i - \frac{e}{c}A_i\right) \tag{7}$$

is the velocity operator $\hat{v}_i = d\hat{x}_i/dt = -i\hbar[\hat{x}_i, H]$ and $\hat{F}_i^{pot}$ and $\hat{F}_i^{magn}$ are the operators of potential and magnetic forces, respectively.

We will study, here, only the nontrivial magnetic part of the interaction. Thus, the expression for the average magnetic force $\bar{F}_i^{magn} = N\int d^3x\,\psi^*\hat{F}_i^{magn}\psi$ reads

$$\bar{F}_i^{magn} = N\frac{q}{2c}\int d^3x\,\psi^*\left[F_{ij}\hat{v}_j + \hat{v}_jF_{ij}\right]\psi \tag{8}$$

and, evidently, transforms into the expression for the usual Lorentz force $F_i^L = (q/c)F_{ij}v_j$ in the classical limit $\hbar \to 0$.

Till now all our consideration was quite standard. However, further we will show that there exist such physical situations when the magnetic force (8) has an nontrivial purely imaginary part that does not equal zero only in inhomogeneous magnetic fields of special configurations and that disappears in the classical limit $\hbar \to 0$.

Let us calculate the imaginary part of the average magnetic force $\bar{F}_i^{magn}$ (if any) which we denote by ΔF_i^{magn}:

$$\Delta F_i^{magn} = i\,\mathrm{Im}\,\bar{F}_i^{magn}. \tag{9}$$

Conjugating Eq. (8) and using the explicit form of the velocity operator (7) we get

$$\left(\bar{F}_i^{magn}\right)^* = N\frac{q}{2c}\int d^3x\,\psi\left[F_{ij}\hat{v}_j^* + \hat{v}_j^*F_{ij}\right]\psi^* = \bar{F}_i^{magn} + \frac{i\hbar q}{mc}N\int d^3x\,\partial_j\left(\psi^*F_{ij}\psi\right)$$

and, therefore

$$\Delta F_i^{magn} = -i\frac{q\hbar}{2mc}\frac{\int d^3x\,\partial_j\left(\psi^*F_{ij}\psi\right)}{\int d^3x\,\psi^*\psi}. \tag{10}$$

One can see that the integrand in (10) is a full derivative and at first sight it seems evident that in accordance with the Gauss theorem

$$\int\limits_V d^3x\,\partial_i a_i = \int\limits_\Sigma d\vec{\sigma}\,\vec{a}(\sigma) \tag{11}$$

r.h.s. of Eq. (10) equals zero if the wave functions decrease at the space infinity fast enough. However, we will show, further, that r.h.s. of (10) may differ from zero even if the wave functions *decrease at infinity exponentially.* But, first of all, let us consider mathematical example illuminating the essence of the problem.

Paradox

Let us consider the following mathematical object

$$I = \int d^3x\,\partial_i\left[\rho(\vec{x}^2)(x_jF_{ij})\right], \qquad (i,j = 1,2,3) \tag{12}$$

where ρ is an arbitrary function of $\vec{x}^2$. Using the equality

$$\partial_i \rho(\vec{x}^2) = 2x_i \frac{d\rho}{d\vec{x}^2} \equiv 2x_i \rho'(\vec{x}^2)$$

and the identity $F_{ii} \equiv 0$ we get, instead of (12), the equation

$$I = \int d^3x \left[2\rho'(\vec{x}^2)\,(x_i F_{ij} x_j) + \rho(\vec{x}^2) x_j\,(\partial_i F_{ij})\right].$$

However, because of the antisymmetry of the tensor F_{ij} we have $x_i F_{ij} x_j \equiv 0$, and the first term in r.h.s of the last equation equals zero identically. Thus, we have

$$I = \int d^3x\, \rho(\vec{x}^2) x_j\,(\partial_i F_{ij}). \tag{13}$$

Let us remember, now, that the field strength F_{ij} satisfies the Maxwell equations

$$\partial_j F_{ij} \equiv (\,\vec{\text{rot}}\vec{H})_i = J_i, \tag{14}$$

where the generalized current density $\vec{J} = (4\pi/c)\,\vec{j} + (1/c)\,\partial\vec{E}/\partial t$ is the sum of the usual and displacement current densities. Making use of (14) we immediately get, instead of (13), the equation

$$I = \int d^3x\, \rho(\vec{x}^2)\,(x_j J_j), \tag{15}$$

and the statement is that this expression does not always equal zero even if one chooses here as $\rho(\vec{x}^2)$ such perfectly decreasing at infinity function as the Gaussian exponential:

$$\rho(\vec{x}^2) = \mathrm{e}^{-\alpha\vec{x}^2} \qquad (\alpha > 0). \tag{16}$$

Let us present two examples proving this statement.

First example (quasi-stationary case).

It is easy to see that if the displacement current $(1/c)\,\partial\vec{E}/\partial t$ would not present in the Maxwell equation (14), then, choosing as an $\vec{j}$ in (14), (15) the usual expression

$$\vec{j}(t,\vec{x}) = q\,\vec{v}(t)\,\delta^{(3)}\big(\vec{x} - \vec{r}(t)\big) \tag{17}$$

for the current density of a charge q moving with a velocity $\vec{v}(t) \equiv d\vec{r}/dt$ along the trajectory $\vec{r}(t)$, one at once would obtain nonzero result $I = (4\pi q/c)\,exp\big(-\alpha\vec{r}^2(t)\big)\,\big(\vec{v}(t)\vec{r}(t)\big)$. However, the question immediately arises — could the displacement current contribution cancel this expression?

Let us consider now, as an example, the case of a charged particle moving with a nonrelativistic constant velocity and see that in this case the cancellation does not occur. We will follow here to the textbooks by L.D. Landau and E.M. Lifshitz [5] (chapter 5, section 38) and by V.G. Levich [6] (chapter 3, sections 19, 20), where it is shown that electrodynamical description of a single charge q moving with a nonrelativistic constant velocity $\vec{v}(t) = v$ just reduces to the particular case of the well-known (see, for example [6], chapter 3) and widely used *quasi-stationary approximation*, where one, from the very beginning, omits the displacement current $(1/c)\partial\vec{E}/\partial t$ and the term $-(1/c)\partial\vec{H}/\partial t$ in the respective Maxwell equations for $\mathrm{r\vec{o}t}\vec{H}$ and $\mathrm{r\vec{o}t}\vec{E}$ and the sources $\vec{j}$ and ρ depend on t as on a parameter.

Indeed, using the condition $v \ll c$, one can keep in the expansion in powers of v/c of the exact expression for the electric field created by the moving charge

$$\vec{E}(t,\vec{x}) = \frac{q\vec{R}}{R^3}\frac{1 - v^2/c^2}{\left(1 - \sin^2\theta\, v^2/c^2\right)^{3/2}},$$

where θ is the angle between $\vec{R} \equiv \vec{x} - \vec{v}t$ and $\vec{v}$, only the main contribution

$$\vec{E} = q\,\frac{\vec{x} - \vec{v}t}{|\vec{x} - \vec{v}t|^3}\left(1 + O(v^2/c^2)\right).$$

Thus, the expression for the magnetic field

$$\vec{H} = (1/c)\left(\vec{v} \times \vec{E}\right)$$

up to corrections of order $O(v^3/c^3)$ reads (see [5], chapter 5, section 38)

$$\vec{H} = \frac{q}{c}\,\frac{\vec{v} \times (\vec{x} - \vec{v}t)}{|\vec{x} - \vec{v}t|^3}. \tag{18}$$

However, the last expression is nothing else but the solution[3] of the equation

$$\mathrm{r\vec{o}t}\vec{H} = \frac{4\pi}{c}\,q\,\vec{v}\,\delta^{(3)}(\vec{x} - \vec{v}t), \tag{19}$$

and we arrive at nonzero r.h.s. of Eq. (13) (as if we, in the spirit of the quasi-stationary approximation, at once omit the displacement current in (14) and would deal with (17) for the particular choice $\vec{r}(t) = \vec{v}(t)t$) :

$$I = 4\pi q\,(v/c)\,vt\,\exp\left(-\alpha v^2 t^2\right) + O(v^3/c^3) \tag{20}$$

because the corrections in the expansion in powers of a small dimensionless parameter can never cancel the main contribution.

Second example (stationary case).

To avoid even the slightest doubt in correctness of application of the quasi–stationary approximation (actually, one can make any expansion under the symbol of a convergent integral — factor $\exp(-\alpha\vec{x}^2)$ in the integral I) in the just considered case, let us consider another example. We will try to find the configuration of the fields and currents providing r.h.s. of Eq. (13) differing from zero in the *purely stationary* case ($\partial\vec{j}/\partial t = 0$), when no displacement current whatever is present in r.h.s. of Eq. (14).

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[3]The proof that (18) is the solution of (19) is easily performed using the vector potentials in the Coulomb gauge $\mathrm{div}\vec{A} = 0$ (see, for example [6], chapter 3, sections 19,20)

It is easy to check that the configuration[4] of the form

$$\begin{aligned} H_1 &= H_2 = 0, \quad H_3 = f(x_1, x_2), \\ j_1 &= \frac{c}{4\pi}\,\partial_2 f, \quad j_2 = -\frac{c}{4\pi}\,\partial_1 f, \quad j_3 = 0 \end{aligned} \tag{21}$$

satisfies the stationary Maxwell equations $\vec{\mathrm{rot}}\vec{H} = (4\pi/c)\,\vec{j}$, $\mathrm{div}\vec{H} = 0$, and the stationarity condition $\mathrm{div}\vec{j} = 0$. Eq. (13) with the choice (16), for configuration (21), looks as

$$\begin{aligned} I &= \int d^3x\, e^{-\alpha\vec{x}^2}\, x_j\big(\,\vec{\mathrm{rot}}\vec{H}\big)_j \\ &= \frac{4\pi}{c}\int d^3x\, e^{-\alpha\vec{x}^2}\,[x_1\partial_2 - x_2\partial_1]\, f(x_1, x_2). \end{aligned} \tag{22}$$

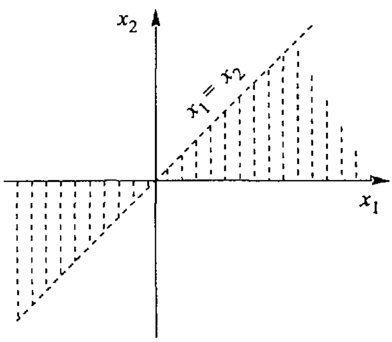

Fig. 1. Cross-section in plane orthogonal to the x_3 — axis, showing the region where $\theta(x_1/x_2 - 1)$ differs from zero. The shaded and blank regions are the regions where $\theta(x_1/x_2 - 1)$ equals one and zero, respectively

Let us now choose function f in the form (see the Figure)

$$f = R\left(\sqrt{x_1^2 + x_2^2}\right)\theta\big(x_1/x_2 - 1\big), \tag{23}$$

where R is an arbitrary function of $\rho \equiv \sqrt{x_1^2 + x_2^2}$, satisfying the condition (see below)

$$\int\limits_0^{\infty} d\rho\,\rho\;\mathrm{e}^{-\alpha\rho^2}\,R(\rho) \neq 0, \tag{24}$$

and, θ is the usual step-function with the derivative $d\theta(x)/dx = \delta(x)$.

Using the equality

$$\frac{\partial}{\partial x_i}\,F[g(x_1, \ldots, x_n)]\Big|_{i=1,\ldots,n} = \frac{dF}{dg}\,\frac{\partial g}{\partial x_i} \equiv F'[g]\frac{\partial g}{\partial x_i}$$

one easily gets

$$\partial_1\theta(x_1/x_2 - 1) = (1/x_2)\,\delta(x_1/x_2 - 1), \quad \partial_2\theta(x_1/x_2 - 1) = -(x_1/x_2^2)\,\delta(x_1/x_2 - 1), \tag{25}$$

and

$$(x_1\partial_2 - x_2\partial_1)\;R\left(\sqrt{x_1^2 + x_2^2}\right) = 0. \tag{26}$$

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[4]The idealized current (21) is not restricted in z direction and, certainly, cannot be created in reality (just as, for example, the well-known idealized stationary current $\vec{j}(x, y, z) = \hat{z}\,V\,\delta(x)\delta(y)$ flowing inside infinitely long and infinitely thin wire placed along z-axis). However, it absolutely does not matter for existence of the contra-example (showing violation of the Gauss theorem under special conditions) whether we can create experimentally the appropriate magnetic fields and currents or not (the integral (12) is a formal mathematical object — just the test object for the Gauss theorem). So, we even could allow $\mathrm{div}\vec{H} \neq 0$ (some unphysical monopole-like fields but with nonzero curls) and maintain only the equation $\vec{\mathrm{rot}}\vec{H} = (4\pi/c)\vec{j}$. Nevertheless, having in mind the following material in this paper, we will provide for both Maxwell equations being valid for a stationary field $\vec{H}$. Notice also, that in the rest of the paper it is shown that it is possible to find the proper current restricted in all the space.

Making use of (23)–(26) in (22), one obtains nonzero result

$$
\begin{aligned}
I &= -\frac{4\pi}{c}\int_{-\infty}^{\infty} dx_3\, e^{-\alpha x_3^2}\int_{-\infty}^{\infty} dx_2\, \times \\
&\times\; e^{-\alpha x_2^2}\int_{-\infty}^{\infty} dx_1\, e^{-\alpha x_1^2} R\left(\sqrt{x_1^2+x_2^2}\right)\left(\frac{x_1^2}{x_2^2}+1\right)\delta\left(\frac{x_1}{x_2}-1\right) \\
&= -\frac{8\pi}{c}\sqrt{\frac{\pi}{\alpha}}\int_{-\infty}^{\infty} dx_2\; e^{-\alpha x_2^2}\int_{-\infty}^{\infty} dx_1\; e^{-\alpha x_1^2} R\left(\sqrt{x_1^2+x_2^2}\right)\left[|x_2|\delta(x_1-x_2)\right] \\
&= -\frac{8\pi}{c}\sqrt{\frac{\pi}{\alpha}}\int_{0}^{\infty} d\rho\,\rho\; e^{-\alpha\rho^2} R(\rho)\neq 0, \qquad (27)
\end{aligned}
$$

where we use the usual properties of the δ-function

$$\delta\left[f(x)\right] = \left(1/|f^{'}(x_0)|\right)\,\delta(x-x_0), \quad f(x)\,\delta(x-x_0) = f(x_0)\,\delta(x-x_0)$$

in the second line, integrating with respect to x_1, and use the fact that the integrand is the even function of x_2 in the third line.

So, we conclude, eventually, that the purely stationary configuration (21),(23), satisfying the full set of the Maxwell equations, yields the nonzero result for the basic integral (12).

Thus, we see that in the general case the integral (12) differs from zero. On the other hand, however, if we started with application of the Gauss theorem to this integral and chose ρ in the form (16) then, evidently, we would obtain zero.

So, what is the matter? Let us stress that to obtain (13) we use nothing but the field strength antisymmetry and such absolutely legal operations as differentiation and use of the Maxwell equations. Thus, equation (13) seems to be valid without any doubts. Then why does the Gauss theorem fail in this case?

To understand this apparent contradiction one must remember that the Gauss theorem (11) is proved in mathematics *only for continuous* integrands (see, for example, [7]) and, therefore, the Gauss theorem (11) *may be not valid*[5] for the discontinuous integrands, like the integrands we just deal with in two considered examples.

Let us now stress that the second considered example is especially interesting because, on the one hand, it is directly connected with the rest material of the paper, and, on the other hand, the arguments may be present that the nonzero result in this case ought to be regarded as a *topological phenomenon.* Indeed, first of all, one can see that it is not sufficient to have just a θ-like discontinuity of the magnetic field for the Gauss theorem violation[6]. The nonzero

[5]However, and it seems to be even amazing now, despite this circumstance the Gauss theorem in its classical formulation (11) occurs very stable and, as a rule, holds even for discontinuous integrands — see below.

[6]On the contrary, this is rare enough situation (Gauss theorem occurs really very stable!). The identity $x_i F_{ij} x_j \equiv 0$ plays the crucial role in (12). For example, if you change $x_j F_{ij}$ in the test integral (12) to $n_j F_{ij}$, where $\vec{n}$ is a constant vector, then you obtain zero result.

result is a consequence of the nontrivial space-field composition (the test object (12) already implicitly contains the such composition — the presence of the term x_jF_{ij} in the integrand plays the crucial role for the nonzero result (see footnote 6)). So, to obtain nonzero result one must either consider the object similar to (12), where the magnetic field enters in a nontrivial manner, or look for some very special configuration of the magnetic field, satisfying the proper restriction (like the orthogonality condition $x_1H_2 - x_2H_1 = 0$ in the rest of this paper, playing the same role as the identity $x_iF_{ij}x_j = 0$ for the test integral (12)). Moreover, even if the respective orthogonality condition is satisfied, it is not still sufficient for a nonzero result and one must look for still more specific magnetic field configurations. So, for example, the replacement of $\theta(x_1/x_2 - 1)$ by $\theta(x_1 - x_2)$ in (23) (or in (41) – see below) would immediately give rise to zero. Thus, only very rare configurations of the magnetic field yield nonzero result (27). On the other hand, however, it is well known, that namely the presence of the magnetic fields of the special configurations (remember Dirack string, Aharonov–Bohm solenoid, etc.) can make the space nonsimply-connected (topologically nontrivial), and this is just what happens in the considered case.

There exists still one important argument confirming this statement. Indeed, in one-dimensional (topologically trivial) space, the analog of the integral (12), where under the symbol of the derivative in the integrand stays θ-function and some well decreasing at infinity function like (16), looks as $I_{(d=1)} = \int_{-\infty}^{\infty} dx\, d/dx[\theta(x)\exp(-\alpha x^2)f(x)]$, where $f(x)$ is a continuous function satisfying the condition $\lim_{|x|\to\infty} f(x)\exp(-\alpha x^2) = 0$. Then, using the usual properties of the θ- and δ-functions, one obtains zero result:

$$I_{(d=1)} = \int\limits_{-\infty}^{\infty} dx\, \delta(x) d/dx[\exp(-\alpha x^2)f(x)] + \int\limits_{0}^{\infty} dx\, d/dx[\exp(-\alpha x^2)f(x)] = 0.$$

Notice, that this circumstance, even apart from the Gauss theorem (11), could lead one to the wrong conclusion that the initial three-dimensional test integral (12), at the choice (16),(21),(23), also equals zero. The incorrect logical chain would look as: «At the choice (16),(21),(23) there is nothing more dangerous than the discontinuous θ-function in the square brackets of (12), and the rest is a well decreasing at infinity continuous function. Therefore, this case just reduces to the above considered one-dimensional case (the last equation), and one obtains zero for every of the three terms in r.h.s. of (12), containing ∂_1, ∂_2 and ∂_3 in the integrands, respectively». However, we know that the correct calculation leads to the nonzero result (27) just in this case. Thus, we conclude that there exist the selected situations, when even the presence under the full derivative symbol of such «innocent» in the trivial one-dimensional space quantity as θ-function, can make three-dimensional space multiconnected and give rise to a nonzero result.

Let us now return to equation (10) for $\Delta \vec{F}^{magn}$. It is obvious that to perform a rigorous calculation one must know the exact form of the wave functions entering in (10). However, the task of solving the Schrödinger equation for a charged particle in a magnetic field was till now exactly solved only in the case of a homogeneous magnetic field (see, for example, [7]) and we have to look for some approximation to deal with the inhomogeneous magnetic field. Fortunately, such a wide spread in quantum mechanics approximation as the *instantaneous approximation* (see, for example, [8]) turns out to be quite adequate for our aim — finding the physical situations where $\Delta\vec{F}^{magn}$ would differ from zero. Let us briefly remind the essence of this approximation. Let $\psi(t)$ be the wave function to be found at the time moment t, if the wave function $\psi(t_0)$ at the time moment t_0 is known. The Hamiltonian of the system $H(t)$ explicitly depends on time and changes its value from $H(t_0)$ at the moment t_0 to $H(t)$ at the moment t during the time interval $\Delta t = t - t_0$. Then $\psi(t) = U(t, t_0)\psi(t_0)$, where the evolution operator $U(t, t_0)$ is uniquely determined and satisfies the integral equation

$$U(t, t_0) = 1 + \frac{1}{i\hbar}\int_{t_0}^{t} d\tau\, H(\tau)U(t, t_0). \tag{28}$$

In the limit of «very fast» transition $\Delta t \to 0$ the second term in r.h.s of (28) disappears and thus $U(t, t_0) \to 1$ and $\psi(t) \sim \psi(t_0)$, i.e., we can consider the dynamical state of the system during such «very fast» transition to be constant. The words «very fast transition» mathematically mean that the transition time Δt must satisfy the criterion of validity of the instantaneous approximation

$$\Delta t < \hbar/\Delta\bar{H}, \tag{29}$$

where in terms of the dimensionless variable $s = \tau - t_0/\Delta t$ and of the operators

$$H(s) = H(\tau) \equiv H(t_0 + s\Delta t), \quad \bar{H} = \int_0^1 ds\, H(s) = \frac{1}{\Delta t}\int_{t_0}^{t} d\tau\, H(\tau),$$

the root-mean-square deviation $\Delta\bar{H}$ takes the form $(\Delta\bar{H})^2 = \langle\psi_{(0)}|\bar{H}^2|\psi_{(0)}\rangle - \langle\psi_{(0)}|H|\psi_{(0)}\rangle^2$.

Thus, we can now reformulate our task as follows. Let no magnetic field be present at the initial time moment t_0 and, thus, the charged particle state $|\psi_{(0)}\rangle \equiv |\psi(t_0)\rangle$ at the moment t_0 corresponds to a purely potential interaction $q\varphi + W$, where W is the potential of some self-consistent field corresponding to the concrete many-particle system when the many-particle task is reduced to the one-particle one in the Hartree-Fock method. Suppose now that we can create such experimental conditions that the intensive strong-inhomogeneous magnetic field switches on very fast, namely, so fast that the field $\vec{H}$ and the respective current density achieve their *stationary* (or *quasi-stationary* — see footnote 11 below) configuration during the time $\Delta t_1 \equiv t_1 - t_0$ which is less than the time interval $\Delta t_2 \equiv t_2 - t_1$ satisfying the criterion of validity of the instantaneous approximation (29). Thus, during the time interval $t_1 < t < t_2$ the instantaneous approximation is still valid (i.e. we deal with the «frozen» wave functions $\psi_{(0)}$) and, simultaneously, the magnetic field and the respective current take their stationary values (i.e. the displacement current $(1/c)\partial\vec{E}/\partial t$ is already absent[7]). Then, in correspondence with the instantaneous approximation, we can rewrite the expression for $\Delta\vec{F}^{magn}$ (10) in the form[8]

$$\Delta F_i^{magn}\Big|_{t_1<t<t_2} = -i\frac{q\hbar}{2mc}\int d^3x\, \partial_j \left[\psi^*_{(0)} F_{ij}\psi_{(0)}\right]. \tag{30}$$

[7] or gives the small corrections in the quasi-stationary case — see footnote 11 below.

[8] Here we consider the initial states normalized to 1.

Therefore, our goal, now, is to find the such physical situations, i.e. such initial quantum-mechanical states «prepared» by the experimenter and such stationary configurations of the magnetic fields and currents, that the expression for ΔF_i^{magn} (30) turns out to be different from zero.

Let us consider as an example the simplest quantum-mechanical system — hydrogen atom in an external magnetic field (actually, the procedure given below suits any central-symmetrical initial states of a charged particle). As the initial states we consider only completely spherically symmetrical electron bound states[9], i.e. the states with zero angular momentum $l = 0$:

$$\psi_{(0)} = \psi_{n00}^{(H)} = R_{n00}(r)Y_{00}(\theta,\phi) = \frac{1}{2\sqrt{\pi}}R_{n0}(r), \tag{31}$$

$$|\psi_{(0)}|^2 \equiv \Phi(r). \tag{32}$$

Let us, now, note that for the effect to occur it is quite sufficient, if even only one of the components of $\Delta\vec{F}^{magn}$, for example ΔF_3^{magn} turns out to be different from zero. Making use of Eqs. (30), (32) and the identity $\partial_i\Phi(r) = x_i\big(\Phi^{'}(r)/r\big)$ one easily gets

$$\Delta F_3^{magn} = -i\frac{e\hbar}{2mc}\left[I_1 + I_2\right], \tag{33}$$

where

$$I_1 = \int d^3x\, \frac{\Phi^{'}(r)}{r}\,(x_jF_{3j}), \tag{34}$$

$$I_2 = \int d^3x\,\Phi(r)\,(\partial_jF_{3j}) \equiv \int d^3x\,\Phi(r)\Big(\,\mathrm{r\vec{o}t}\vec{H}\Big)_3. \tag{35}$$

Thus (remember the paradox), one can see that to provide for the value ΔF_3^{magn} to differ from zero it suffices to find such stationary magnetic field configurations for which the integral I_2 would not equal zero and, simultaneously, the orthogonality condition $x_jF_{3j} = 0$ would be valid. It is easy to verify that the choice

$$H_1 = x_1f_1 + x_2f_2, \quad H_2 = x_2f_1 + x_2\frac{x_2}{x_1}f_2, \tag{36}$$

where f_1 and f_2 are arbitrary functions of $\vec{x}$, satisfies the orthogonality condition, and our task, now, is to look for the functions f_1 and f_2 providing a nonzero value of the integral I_2 (35). Using the arbitrariness in the choice of f_1 and f_2 we can simplify the task and set one of the functions, for example, f_2, equal to zero in the general expression (36) for the appropriate magnetic field configuration. Then we obtain

$$H_1 = x_1f, \quad H_2 = x_2f, \tag{37}$$

$$I_1 = \int d^3x\,\frac{\Phi^{'}(r)}{r}\,(x_1H_2 - x_2H_1) \equiv 0 \tag{38}$$

[9]For example, the ground state reads $\psi_{(0)} = \pi^{-1/2}\,e^{-r}$.

and

$$\Delta F_3^{magn} = -i\frac{e\hbar}{2mc} I_2 = -i\frac{e\hbar}{2mc} \int d^3x\, \Phi(r)\, (x_2\partial_1 - x_1\partial_2)\, f. \tag{39}$$

The order of action is now the following. We first have to find a proper function f providing for r.h.s. of Eq. (39) to differ from zero. If we manage to do this and to find the such function f and, therefore, using (37) to find the respective H_1 and H_2 components of the magnetic field, then we would easily reproduce the remaining third component[10] solving the Maxwell equation $\mathrm{div}\vec{H} = 0$ with respect to H_3:

$$H_3 = -\int_0^{x_3} dx_3\ (\partial_1 H_1 + \partial_2 H_2)\,. \tag{40}$$

Then, one easily restores the respective current density $\vec{j}$ from the stationary (or quasi-stationary — see footnote 11 below) Maxwell equation $\mathrm{r\vec{o}t}\vec{H} = (4\pi/c)\,\vec{j}$.

Thus, let us choose the function f entering (39) in the form

$$f(\vec{x}) = \frac{4\pi}{c} F\big(\sqrt{x_1^2 + x_2^2}, x_3\big)\, \theta(x_1/x_2 - 1), \tag{41}$$

where F is an *arbitrary* function of $\rho \equiv \sqrt{x_1^2 + x_2^2}$ and $x_3 \equiv z$ variables, satisfying the condition (see below)

$$\int_0^{\infty} d\rho\rho \int_{-\infty}^{\infty} dz\, \Phi\big(\sqrt{\rho^2 + z^2}\big)\, F(\rho, z) \neq 0. \tag{42}$$

It is very important that, using the arbitrariness in the choice, one can take as an F in (37), (41) the function well decreasing in the ρ and z directions, which makes all the components of the respective magnetic field and current density *completely restricted*[11] in the space.

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[10]Note that H_3 is uniquely fixed by this way. Otherwise, we would have $\vec{H} \neq \mathrm{r\vec{o}t}\vec{A}$ and thus $\vec{H}$ would not be the physical (experimentally created) magnetic field.

[11]Besides, it is very important to stress that it is quite enough for a nonzero result to create not purely stationary configuration (37), (41) but the quasi-stationary one (and it gives us the additional experimental possibilities). Let us remind that for an arbitrary system of slowly moving charges, the condition of the quasi-stationarity reads (see [6], chapter 3, section 19) $V \approx L/T \ll c$, where L is the average dimension of the system, T is the characteristic period of movement and $V \sim L/T$ is interpreted as the average velocity of the charges in the system So, one can allow the function F in (41) to depend also on time as on a parameter $F = F(\sqrt{x_1^2 + x_2^2}, x_3|t)$, having in mind that the quasi-stationarity condition must be satisfied. Then, in the spirit of the quasi-stationary approximation [6], one just omits the displacement current $(1/c)\partial\vec{E}/\partial t$ and the term $-(1/c)\partial\vec{H}/\partial t$ in the respective Maxwell equations for $\mathrm{r\vec{o}t}\vec{H}$ and $\mathrm{r\vec{o}t}\vec{E}$, neglecting the terms of highest orders in powers of a small dimensionless parameter V/c (which just transforms to the parameter $v/c \ll 1$ in the particular case [5,6] of the quasi-stationary approximation — alone charge moving with a constant nonrelativistic velocity $\vec{v}(t) = \vec{v}$, considered above, in the paradox formulation). Thus, up to corrections of highest orders in powers of V/c, one again deals with the equation $(\mathrm{r\vec{o}t}\vec{H})_3 = (x_2\partial_1 - x_1\partial_2) f(\vec{x}|t) = (4\pi/c) j_3(\vec{x}|t)$, where $f = F(\sqrt{x_1^2 + x_2^2}, x_3|t)\, \theta(x_1/x_2 - 1)$ depends on t only as on a parameter, and, therefore, again arrives at nonzero result (44) for ΔF_3^{magn}, because the corrections in powers of a small dimensionless parameter cannot cancel the main contribution.

So, using the obvious identity

$$(x_1\partial_2 - x_2\partial_1)\ F\left(\sqrt{x_1^2+x_2^2}, x_3\right) = 0, \tag{43}$$

the relations (25), and making use of the usual properties of the δ-function integrating with respect to x_1, one literally repeats the chain of operations leading to Eq. (27) of the paradox formulation, and gets from (39), (41) the nonzero result for ΔF_3^{magn}:

$$\begin{aligned} \Delta F_3^{magn} &= -i\hbar\frac{4\pi e}{mc^2}\int\limits_{-\infty}^{\infty} dx_3 \int\limits_{-\infty}^{\infty} dx_2\,|x_2|\,\Phi\left(\sqrt{2x_2^2+x_3^2}\right)\,F\left(\sqrt{2x_2^2}, x_3\right) \\ &= -i\hbar\frac{4\pi e}{mc^2}\int\limits_{-\infty}^{\infty} dz \int\limits_{0}^{\infty} d\rho\,\rho\,\Phi\left(\sqrt{\rho^2+z^2}\right)\,F(\rho,z) \neq 0, \end{aligned} \tag{44}$$

where Φ is the squared wave function (32) and F is some well decreasing function of $\rho \equiv \sqrt{x_1^2+x_2^2}$ and z variables, satisfying the condition (42).

While the components H_1 and H_2 are given by (37), where function f has the form (41), the H_3 component of the magnetic field is restored from (40). Using (25) it is easy to show that

$$(x_1\partial_2 + x_2\partial_1)\,\theta(x_/x_2 - 1) = 0 \tag{45}$$

and

$$(x_1\partial_1 + x_2\partial_2)\,F\big(\sqrt{x_1^2+x_2^2}, z\big) = \rho\,\frac{\partial F(\rho,z)}{\partial\rho}. \tag{46}$$

Substituting (37), (41) to (40), and making use of (45), (46), one obtains for H_3 the expression

$$H_3 = \theta(x_1/x_2 - 1)\,\left(2 + \rho\frac{\partial}{\partial\rho}\right)\int\limits_0^z dx_3\,F(\rho, x_3), \tag{47}$$

where $\rho \equiv \sqrt{x_1^2+x_2^2}$.

At last, knowing all the components of the magnetic field $\vec{H}$, one easy restores the respective stationary (or quasi-stationary) current density $\vec{j}$ (that, evidently has the δ–like singularity in the plane $x_1 = x_2$), making use of the stationary (or quasi-stationary – see footnote 11) Maxwell equation $\vec{\mathrm{rot}}H = (4\pi/c)\,\vec{j}$.

Thus, we have found one of the possible configurations of the fields and currents resulting (at least, at the moment when the magnetic field switches on) in the quantum-mechanical, purely imaginary addition $\Delta\vec{F}^{magn}$ (44) to the usual Lorentz force differing from zero.

Certainly, this is a very unusual result. What does it mean and what would happen with a particle under such (very special) conditions providing for $\Delta\vec{F}^{magn} \neq 0$? It is reasonable to suppose that, besides the obvious and rather science-fiction scenario, there may exist some much less exotic and hither to unknown explanation of the phenomenon (perhaps this is just an indication of some instability[12] of a charged particle in such strong, inhomogeneous, «instantly» switched on magnetic fields of certain special configurations?). In any case, it seems to us that the results presented here (for example, violation of the Gauss theorem in some, very rare, topologically–nontrivial physical situations) can shed new light on our understanding of the role of nontrivial space topology and the problem of hermiticity in quantum mechanics.

The authors are grateful to V.G. Kadyshevsky, B.M. Barbashov, Yu.A. Budagov, G.B. Pontecorvo, I.L. Solovtsov, W. Kallies, V. Ter-Antonyan, G.S. Pogosyan and especially E.A. Kuraev for helpful discussions. We also thank G. Trilling, D. Jackson and G. Afanasiev for acquaintance with the manuscript at the different stages of its writing.

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Received on May 30, 2000

[12] Here it is relevant to remind that in the case of a usual, purely potential interaction the appearance of an imaginary part of the energy in the Schrödinger equation points to an instability of a composite particle and determines its decay width. Certainly, this is very indirect analogy, because in this paper we consider a stable, in the usual treatment, quantum-mechanical particle (like the proton or electron) put in very special external conditions.