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DISCRETE PLANCK SPECTRA

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Abstract

The Planck radiation spectrum of ideal cubic and spherical cavities, in the region of small adiabatic invariance, $\gamma = TV^{1/3}$, is shown to be discrete and strongly dependent on the cavity geometry and temperature. This behavior is the consequence of the random distribution of the state weights in the cubic cavity and of the random overlapping of the successive multiplet components, for the spherical cavity. The total energy (obtained by summing up the exact contributions of the eigenvalues and their weights, for low values of the adiabatic invariance) does not obey any longer Stefan-Boltzmann law. The new law includes a corrective factor depending on γ and imposes a faster decrease of the total energy to zero, for $\gamma \to 0$. We have defined the double quantized regime both for cubic and spherical cavities by the superior and inferior limits put on the principal quantum numbers or the adiabatic invariance. The total energy of the double quantized cavities shows large differences from the classical calculations over unexpected large intervals, which are measurable and put in evidence important macroscopic quantum effects.

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Dedicated to the 100^{th} Scientific Anniversary of Max Planck (Quantum Radiation Law)

1. Introduction

The ideal classical cavity may be defined as a closed surface with a perfectly smooth and unitary reflection interior wall, where the discrete absorption and emission of quanta by the atoms are leading to the thermal equilibrium [1,2]. The quantum counterpart of this classical definition is the concept of an infinite potential well, ensuring a vanishing probability for the photon presence outside its surface.

The quantum version of the photon confinement is actually an eigenvalue problem, the discrete spectrum of the photon energies being a direct consequence of the volume finiteness and of the shape of limiting surface. For a free particle of vanishing rest mass, the relativistic energy equation (with the corresponding quantum operators) can lead to a Schrödinger-Helmholtz equation. The cavity introduces a Dirichlet boundary condition. The history of these types of problems is very rich [3-7] and a good account of it was given by Gutierrez and Yanez [5].

If we refer definitely to the black-body radiation, the effect of the geometrical confinement upon the frequency spectrum of the radiation stored inside the cavity may be assigned to an additional quantizing, beyond that considered by Planck (referred to the discrete absorption and emission of quanta by the atoms of the cavity in view of reaching the thermal equilibrium). In this case, not only the energy exchanged between atoms and radiation is quantified (Planck's quanta), but also the radiation energy, through the agency of the discrete spatial directions of the allowed wave-vectors (as a result of the radiation confinement). We named this quantum device as *double quantized cavity* (DQC)[8-11]. In physics literature, a good number of papers are describing the state statistics in quantum devices under the name of *quantum billiards* [12-15] and are relating them to the relatively new field of quantum chaos [15]. From this point of view, our study may be associated to a three-dimensional quantum billiard in a special double quantized regime.

The effect of the additional energy quantizing is controlled by the dimensionless factor $h\nu/kT$ (which, in turn, is inversely proportional to the adiabatic invariant $\gamma = TV^{1/3}$). For $(h\nu/kT) \leq 1$ (this implying for instance small temperatures, in the proximity of about 1° K, and small volumes, in the proximity of about 1 cm^3), the Planck spectrum of the black-body radiation presents a discrete pattern (of lines with irregular intensities), strongly depending on the cavity geometry. The total energy does not obey any longer the Stefan-Boltzmann law, but a new law, which includes a corrective factor depending on γ and imposes a faster decrease to zero, for $\gamma \to 0$. DQC are defined by the special regime: $\gamma_{min} \leq \gamma \leq \gamma_{max}$.

A complete study of the geometrical confinement effect is elaborated for cubic and spherical DQCs. The main results of this study show that, in spite of some additional complexity in the eigenvalue problem, the behavior of the double quantized spherical cavity (DQSC) resembles that of the double-quantized cubic cavity (DQCC). Some differences are also put in evidence: e.g. the quantum regime is defined for DQCC, by $0.1 \leq \gamma \leq 1$ and for DQSC, by $0.1 \leq \gamma \leq 65$.

1. Schrödinger-Helmholtz eigenvalue problem in a reflecting cavity

For a free particle of rest mass m_0, the relativistic energy equation is written as

$$E^2 = (m_0 c^2)^2 + (c\vec{p})^2 \tag{1}$$

A free photon is such a particle, but with vanishing rest mass. Its energy equation is:

$$\vec{K}^2 - \frac{\vec{p}^2}{\hbar^2} = 0, \quad |\vec{K}| = K = \frac{E}{\hbar c} = \frac{2\pi}{\lambda} \tag{2}$$

Avoiding the spin effects (irrelevant for a free photon) and associating quantum operators to mechanical quantities, $p \to (\hbar/i)\nabla$, a Schrœdinger-Helmholz (S-H) type equation is derived from eq. (2):

$$(\nabla^2 + K^2)\psi(\vec{r}) = 0, \tag{3}$$

Second quantizing condition is a boundary one, particularly a Dirichlet one:

$$\psi_S = 0 \ , \tag{4}$$

for a finite-sized domain with defined geometry.

In physics literature, the problem of distribution of the eigenvalues of Eq.(3), which are the **cavity states (levels),** is often treated in terms of the **state density**, ρ_ε, which is defined as the number of states (eigenvalues) lying around the energy ε in a (frequency) unit interval [2]. The mean state density is:

$$D_\varepsilon = V^{-1}\,(d\rho_\varepsilon / d\nu)\ , \tag{5}$$

where V is the cavity volume.

For cavities with relatively small sizes and at relatively small temperatures (we shall define later what means "relatively small"), we have to face the random distribution of the eigenvalue intervals and/or degeneracy. One can expect that the **selection rules** imposed by the boundary conditions and eigenvalue ortho-normalization will lead to **allowed states** and **forbidden states (antiresonances),** i.e. a discrete and irregular spectrum of the S-H operator.

The weight (degeneracy) of state with a quantum number N can be defined as:

$$g(N) = \Delta\rho_\varepsilon(N) / \Delta N. \tag{6}$$

Thus, one can write:

$$D_\varepsilon = \frac{1}{V}\cdot\frac{d\rho_\varepsilon}{dN}\bigg/\frac{d\nu}{dN} = \frac{1}{V}\cdot g(N)\cdot\left(\frac{d\nu}{dN}\right)^{-1}. \tag{7}$$

Let us admit that the energy is quantified and it depends (asymptotically or exactly) on a single "effective" quantum number, N:

$$\upsilon = \upsilon_0 f(N). \tag{8}$$

According to Planck's model, the energy distribution (spectrum) of this cavity (with volume V at the absolute temperature T) is:

$$dE_P = 8\pi \frac{V\upsilon^2}{c^3} \cdot \frac{h\upsilon}{\exp[h\upsilon / kT] - 1} \cdot d\upsilon = 8\pi \frac{V}{c^3} \frac{h\upsilon_0^4 f^3(N) f'(N)}{\exp[h\upsilon_0 f(N) / kT] - 1} dN \tag{9}$$

Alternatively, one can write Eq.(9) under the form :

$$\frac{dE_P}{dN} = \frac{2g(N) h\upsilon_0 f(N)}{\exp\left[\frac{h\upsilon_0}{kT} f(N)\right] - 1} \tag{10}$$

Comparing (9) and (10), we get:

$$g(N) = 4\pi \left(\frac{\upsilon_0}{c} V^{1/3} \right)^3 f^2(N) f'(N). \tag{11}$$

Let us assume now that the weights, $g(N)$, do not depend on the volume V, but on the cavity shape only. In this case, the state frequencies are

$$\upsilon_0 = a_1 \frac{c}{V^{1/3}}, \quad \upsilon = a_1 \frac{c}{V^{1/3}} f(N) \qquad (a_1 \text{ constant}) \tag{12}$$

and the degeneracy factor becomes

$$g_{asy}(N) = 4\pi a_1^3 f^2(N) f'(N). \tag{13}$$

It was demonstrated by Weyl that the number of states per unit volume, in the asymptotic limit (short wavelengths), is independent of the shape of the surface enclosing the volume [3, 5]. The number of eigenvalues (states) lying in the range dk about k (state density) in classical cavities is:

$$d\rho_k = \frac{V}{2\pi^2} k^2 dk \tag{14}$$

With $k = \frac{2\pi}{\lambda} = \frac{2\pi}{c}\upsilon = \frac{2\pi}{hc} \cdot \varepsilon$, the density becomes:

$$d\rho_\varepsilon = \frac{V}{2\pi^2} \left(\frac{\varepsilon}{\hbar c} \right)^2 \frac{d\varepsilon}{\hbar c} \tag{15}$$

If we take, as it was assumed in Eq.(8),

$$\varepsilon = h\upsilon_0 f(N) \ , \qquad d\varepsilon = h\upsilon_0 f'(N)dN \ , \tag{16}$$

one can find immediately

$$d\rho_\varepsilon = \frac{V}{2\pi^2}\left(\frac{h\upsilon_0}{\hbar c}\right)^2 f^2(N)\left(\frac{h\upsilon_0}{\hbar c}\right) f'(N)dN \tag{17}$$

and the weight, in the asymptotic limit, is:

$$\frac{d\rho_\varepsilon}{dN} = g_{asy}(N) = \frac{V}{2\pi^2}\left(\frac{2\pi\upsilon_0}{c}\right)^3 f^2(N)f'(N) = 4\pi\left(\frac{\upsilon_0}{c}V^{1/3}\right)^3 f^2(N)f'(N). \tag{18}$$

Accounting (12), this expression (derived from Weyl's general result for the state density, irrespective of the cavity shape) coincides with Eq. (13). This coincidence is a proof of the fact that this expression can provide only the asymptotic weight, for the states with high quantum numbers (N) in these cavities. Clearly, for high quantum numbers, the ratio of the real and asymptotic weights tends to unity:

$$\zeta = g(N)/g_{asy}(N) \to 1, \quad N \to \infty \ . \tag{19}$$

Going back to the Planck distribution law, we can write Eq.(10) as:

$$\frac{dE_P}{dN} = \frac{2g(N)a_1 \frac{hc}{V^{1/3}} f(N)}{\exp\left\{a_1 \frac{hc}{k}\frac{f(N)}{TV^{1/3}}\right\} - 1} \tag{20}$$

and remark that the discrete states in the cavity are closer as the cavity volume is larger leading to a continuous Planck spectrum in the limit $V \to \infty$. Moreover, the discrete Planck spectrum and its transition to the continuous one depends on the *cavity adiabatic invariant*, defined by $\gamma = TV^{1/3}$.

Particularly, when $\gamma \to \infty$, Eq.(20) leads to Rayleigh – Jeans energy distribution and when $\gamma \to 0$, Eq.(20) leads to a discrete Wien energy distribution.

In order to study the exact eigenvalue distribution for cavities with perfectly reflecting walls and with small quantum numbers, we have to introduce some common boundary conditions for the S-H operator. For cavities with rough or strongly absorbing walls, the discrete behavior we are studying will probably be absent.

2. Discrete Planck spectra in cubic and spherical reflecting cavities with small number of states

2.1. Cubic cavity

S-H equation (3) can be separated on the independent variables and the corresponding energy eigenfunctions have the form:

$$\psi_k(x_k) = A_k \sin\left[2\pi \frac{\alpha_k x_k}{\hbar} + \delta_k\right] \qquad (k = 1,2,3) \tag{21}$$

Imposing Dirichlet boundary condition at the reflecting walls of the cube with size L,

$$\psi_k\left(\pm\frac{L}{2}\right) = 0 \quad \rightarrow \quad \sin\left[\pm\frac{\pi\alpha_k L}{\hbar} + \delta_k\right] = 0\,, \tag{22}$$

one can find:

$$\pm\frac{\pi\alpha_k L}{\hbar} + \delta_k = n_k\pi\,; \quad \alpha_k = \pm\frac{\hbar}{L}n_k\,, \quad \delta_k = 0\,, \tag{23}$$

where the quantum numbers, n_k, are integers and zero.
The normalization condition for ψ leads to the amplitudes of the eigenfunctions:

$$A_k = \sqrt{\frac{2}{L}}\,; \qquad 2\int_{-1/2}^{+1/2} \sin^2(2\pi n_k\xi)d\xi = 1 \tag{24}$$

and to the allowed energies in the cavity:

$$E^2 = c^2\frac{\hbar^2}{L^2}\left(n_1^2 + n_2^2 + n_3^2\right); \qquad E = \hbar\frac{c}{L}\sqrt{n_1^2 + n_2^2 + n_3^2} = \hbar\frac{c}{L}\sqrt{N} \tag{25}$$

Taking $V = L^3$ and $E_N = h\upsilon = a_1\dfrac{hc}{L}f(N)$, (26)

and comparing E_N from Eq.(25) with the homologous expression from Eq.(8) we can identify:

$$a_1 = 1, \qquad f(N) = \sqrt{N}, \qquad g_{asy}(N) = 2\pi\sqrt{N} \tag{27}$$

The exact weights, which can be calculated state by state using the quantum numbers from Eq. (25), are randomly fluctuating around the asymptotic function from Eq. (27).

The frequencies of the cavity states, ν_N, form a discrete spectrum defined by the spatial quantisation rule:

$$n_1^2 + n_2^2 + n_3^2 = N \tag{28}$$

The allowed triplets of integers of the Diophantine equation (28) are all numbers which *do not lead* to state numbers of the form (Gauss solution) [8, 9]:

$$N(p, l) = 4^p\ (8l + 7), \quad (p \text{ and } l \text{ positive integers}) . \qquad (29)$$

We have observed that the number of degenerate states in the cavity is strongly and randomly fluctuating. The degeneracy occurs due to the discrete spatial orientations of the state wave-vectors with the same quantum number (N). In Fig.1, the exact calculation of weights, $g(N)$, as resulted from Eq.(28), is represented for a number of $N \sim 200$ states of the cubic cavity. The average of the distribution $g(N)$ follows the asymptotic trend found in Eq.(27).

There are combinations of integers, which did not satisfy the Eq.(28) leading to "antiresonances" in the spectrum [8]. The antiresonance frequencies can be identified in Fig.1 as points on the N-axis ($g(N) = 0$). It is interesting to point out that (1/6) of the cavity spectrum is emptied by antiresonances.

The mean state-density of the cubic cavity with degeneracy fluctuations (for two polarizations) can be calculated with Eq.(7):

$$D_\varepsilon = 2 \cdot \frac{1}{V} \cdot 2\pi\sqrt{N} \cdot \left[\frac{g(N)}{2\pi\sqrt{N}}\right] \cdot \frac{L}{c} 2\sqrt{N} = \frac{8\pi}{c^3}\upsilon^2 \cdot \zeta(N) = (D_\varepsilon)_{asy} \cdot \zeta(N) \qquad (30)$$

The mean state-density from (30) is different from the asymptotic (classical) one by *the quantum degeneracy factor*

$$\zeta(N) = \frac{g(N)}{g_{asy}(N)} = \frac{g(N)}{2\pi\sqrt{N}} = \frac{hc}{2\pi L} \cdot \frac{g(N)}{E_N} = C_1 \frac{hc}{V^{1/3}} \frac{g(N)}{E_N} \ , \qquad (31)$$

which includes the spatial quantization effects and *strongly fluctuates around one, for small N* (Fig.1). We have checked that the factor $\zeta(N)$ is randomly fluctuating around the value 1 by the calculation of the average number of states on *constant frequency intervals* and the result from Fig.1 is very convincing: although the degeneracy fluctuations are large for a cavity with a small number of states (and must be taken into account), the average of D_ε tends to $(D_\varepsilon)_{asy}$ very rapidly, so that for a number of states larger than ≈ 100, the classical mean state-density formula can be safely used.

We define *double quantized cubic cavities* (DQCC) as cavities with a small number of states, more precisely, with a special upper limit on the highest significant state number in the cavity: $N_T < 100$. In this case, the energy density of the cubic cavity radiation *does depend upon the cavity size (volume)*, i.e. upon the boundary conditions (which is a non-classical effect).

According to Bose statistics, the Planck radiation law for a cubic cavity can be written with the mean state-density from Eq.(30) as:

$$u(\upsilon,T)d\upsilon = D_\varepsilon \frac{h\upsilon}{\exp(h\upsilon / kT) - 1} d\upsilon = \frac{8\pi h}{c^3} \frac{\upsilon^3}{\exp(h\upsilon / kT) - 1} \zeta(N) d\upsilon \qquad (32)$$

or $L^3 u(N,\gamma) / h = 4Ng(N)\left[\exp(\alpha\sqrt{N} / \gamma) - 1\right]^{-1}$, (33)

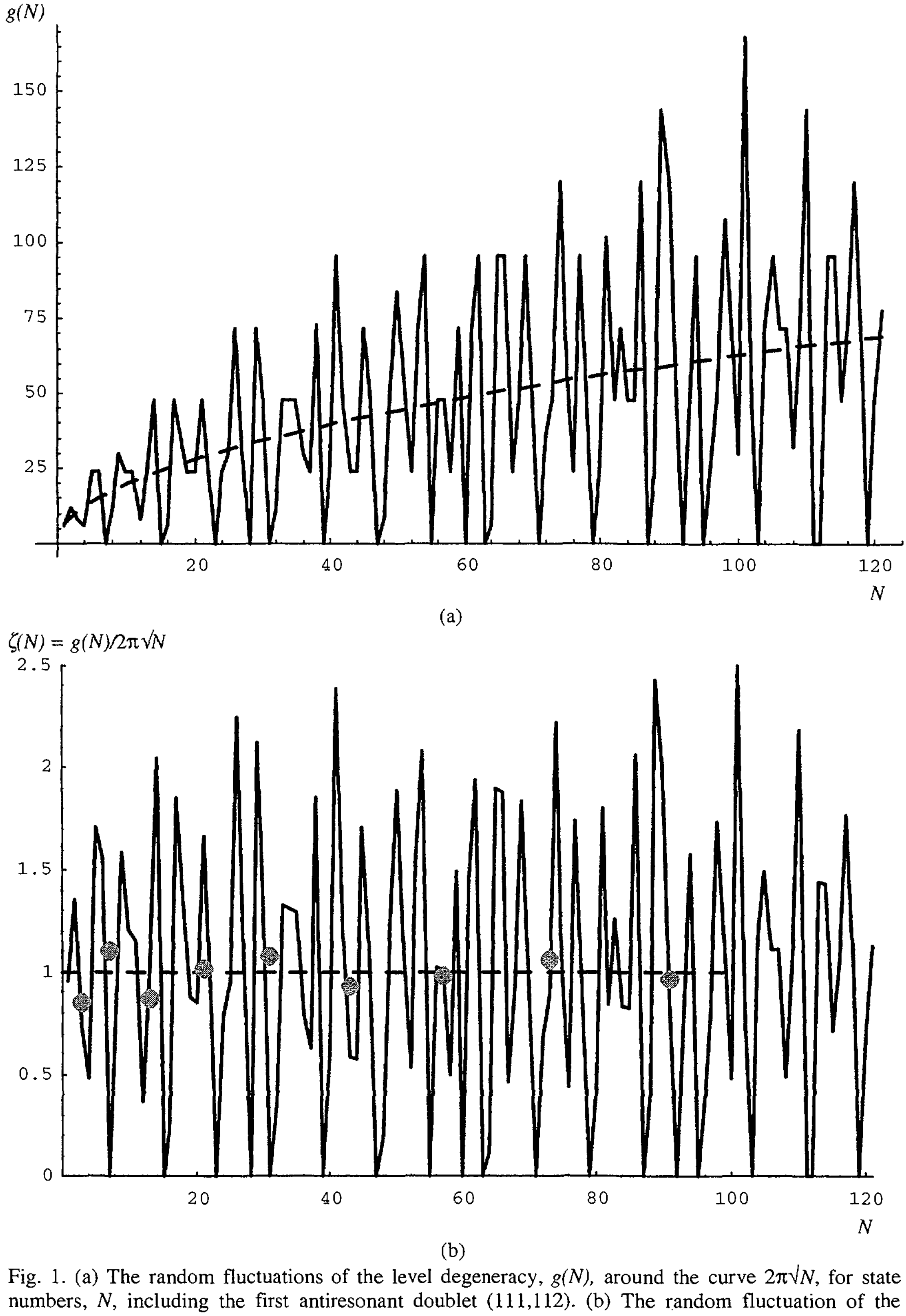
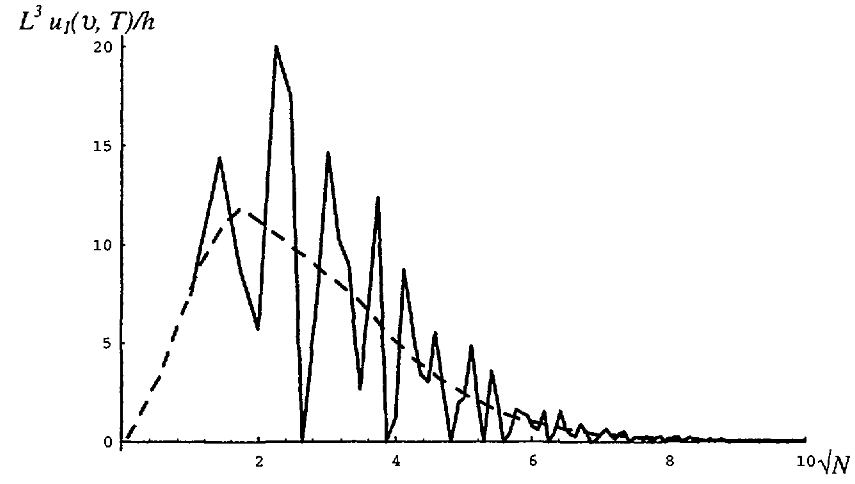
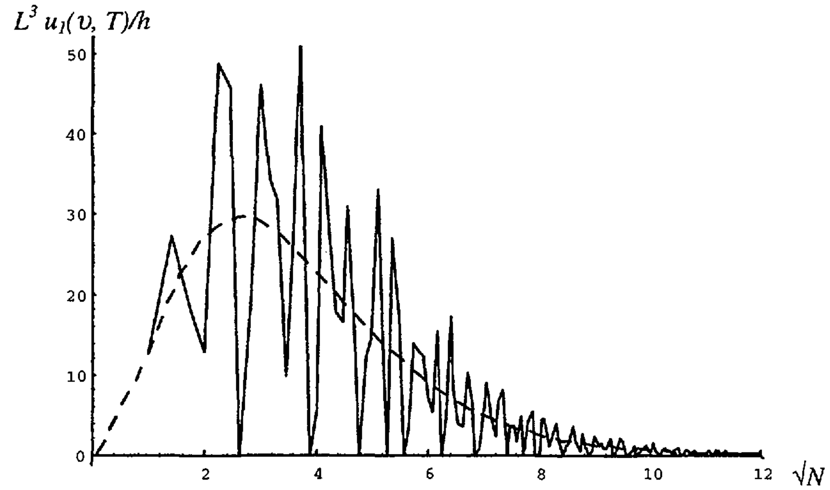


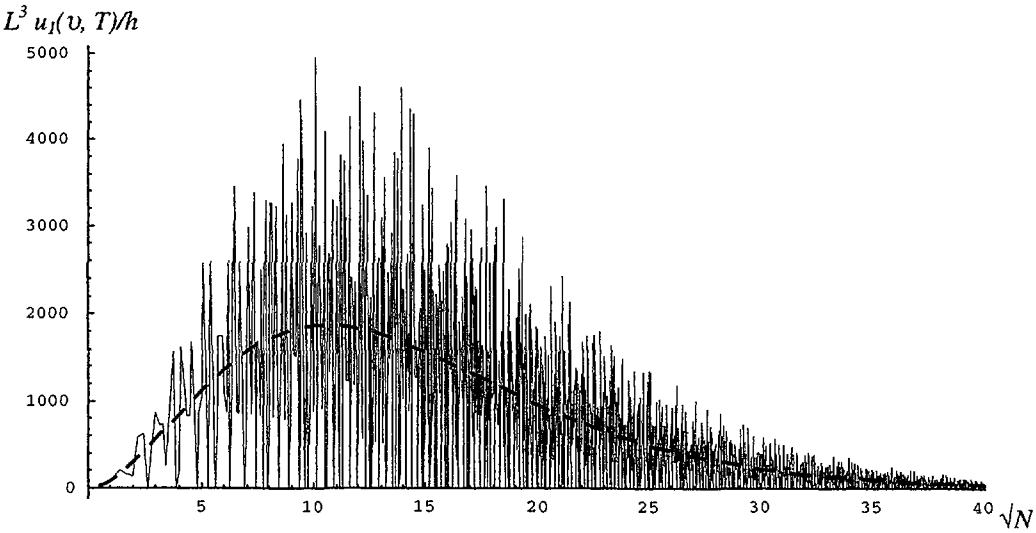
Fig. 1. (a) The random fluctuations of the level degeneracy, $g(N)$, around the curve $2\pi\sqrt{N}$, for state numbers, N, including the first antiresonant doublet (111,112). (b) The random fluctuation of the weight factor, $\zeta(N)$, around the unit value (graph with jointed points); the dots represent the calculated average number of modes on constant frequency intervals and show that the classical (asymptotic) mode density can be reached when $N>110$, by the averaging of the actual mode density.



(a) (T =1K)



(b) (T = 1.35 K)



(c) (T = 5.38 K)

Fig. 2. Some conventional Planck spectra (dashed lines) and discrete Planck spectra (solid lines), for L = 1cm and three low temperatures: (a) T=1K; (b) T=1.35K; (c) T=5.38K.

The Planck spectrum is discrete for a small number of states in the cubic cavity, as shown in Eq.(33) and in Fig. 2. From the graphs of Fig. 2, one observes that the quantum effects may occur in cubic cavities with macroscopic (but small) sizes, at temperatures around the liquid helium one. An interesting feature of DQCC occurs for $LT = 1.35$: the state, N_M, which would hold in the classical Planck spectrum the maximum total energy density, holds in the discrete Planck spectrum zero total energy density (the first antiresonance).

We can introduce a reasonable superior limit of the number of states in the cavity, N_T, which bring a significant contribution to the Planck radiation. Observing that, at high frequencies, in Eq.(32), the exponential term dominates and $\zeta(q)$ goes to 1, the total energy density can be brought to the form: $u(q) = Aq^3 \exp(-q)$, with $q = hc\sqrt{N}/kLT$ and $A =$ constant. If $u(q)/A \leq 10^{-3}$, i.e. $q \geq 15$, the higher levels bring a negligible effect and one can truncate the Planck distribution at *the highest significant level number (HSL)* in the cavity:

$$N_T = 108.69\ (LT)^2 \quad [\mathrm{CGS}] \tag{34}$$

In the particular case: $L = 1$cm and $T = 1$K, Eq.(33) leads to: $N_T \approx 109$.

The Wien displacement law gives the state with the maximum total energy density:

$$N_M \approx 3.8454\ (LT)^2 \quad [\mathrm{CGS}]. \tag{35}$$

Thus, the ratio between HSL and the maximum (peak) state numbers is: $N_T/N_M \approx 28$. In this case, the ratio between the corresponding frequencies is $\upsilon_T / \upsilon_M \approx 5.3$ and one can assert that the significant bandwidth of the black-body radiation is $B \approx 5.3\ \nu_M$.

2.2. Spherical cavity

One can solve S-H equation (3) in spherical coordinates and the energy eigenfunctions have the form:

$$\psi(\bar{r}) = U(\varphi,\vartheta)F(r),$$
$$\frac{d^2F}{dr^2} + \frac{2}{r}\frac{dF}{dr} + \left[K^2 - \frac{l(l+1)}{r^2}\right]F(r) = 0, \qquad l = 0, 1, 2, \ldots \tag{36}$$

The Dirichlet boundary condition (at the reflecting walls) imposes:

$$F(R) = 0 \tag{37}$$

Together with the normalization condition for ψ, the boundary condition completely determines the eigenenergies and eigenfunctions for radiation:

$$U(\varphi,\vartheta) = \sqrt{4\pi}Y_{lm}(\vartheta,\varphi)$$

$$Y_{lm}(\vartheta,\varphi) = \sqrt{\frac{(2l+1)}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos\vartheta)e^{im\varphi}$$

$$\int_0^{2\pi} d\varphi \int_0^{\pi} \sin\vartheta d\vartheta Y^*_{l'm'}(\vartheta,\varphi)Y_{lm}(\vartheta,\varphi) = \delta_{ll'}\delta_{mm'} \qquad (38)$$

$$F(r) = F_{Nl}(r), \qquad F_{Nl}(R) = 0$$

$$F_{Nl}(r) = \frac{1}{R\sqrt{2\pi r}}\frac{J_{l+1/2}(K_{Nl}r)}{|J_{l+3/2}(K_{Nl}R)|}$$

$$\int_0^R F_{Nl}(r)F_{N'l}(r)4\pi r^2 dr = \delta_{NN'},$$

The following selection rules hold:

$n = 0, 1, 2, ...,$ (radial quantum number),
$l = 0, 1, 2, ...,$ (angular quantum number),
$m = 0, \pm 1, \pm 2, ... \pm l,$ (magnetic quantum number), (39)
$N = 2n + l = 1, 2, 3, ...$.; n and l cannot be zero simultaneously. The names of the three quantum numbers are taken by analogy with those of the hydrogen atom.

These selection rules define an additional quantisation of the cavity, leading to allowed states defined by two quantum numbers (N, l), forbidden states (antiresonances, N_a, l_a) and degeneracy. Thus, boundary condition (37) and ortho-normalization introduce a rich discrete behavior in the spherical cavity, which could lead to differences between the Planck radiation spectra of the cavities with small and large number of states. We shall call the spherical cavities with a small number of states as *double quantized spherical cavities* (DQSC).

The asymptotic eigenfunctions have the form:

$$F_{Nl} \rightarrow F^0_{Nl} = (2\pi R)^{-\frac{1}{2}} \cdot \frac{\sin\left(K_{Nl}r - l\frac{\pi}{2}\right)}{r} \qquad (40)$$

$$F^0_{Nl}(R) = 0 \rightarrow K_{Nl} \cdot R = N\frac{\pi}{2}, \qquad N = 2n + l,$$

These eigenfunctions, $F^{(0)}_{Nl}(r)$, make up an ortho-normalized set of functions, over the discrete set of quantum numbers $N = 1, 2, 3, ...\infty$. This feature entitles us not only to adopt the asymptotic classifying of states for the real ones, but also to use asymptotic wave functions for calculating energies and weights.

This adoption fails to hold for DQSC, a case when *a rigorous treatment of the quantizing problem* must be applied. The states of DQSC depend on two quantum numbers, which cannot be grouped into a single one, as in the case of cubic cavity (DQCC). The dependence of the energy states on a single quantum number, N, holds in the asymptotic (classical) limit only. The treatment of the Planck spectrum has special features and an extension of the results of the cubic cavity is not straightforward, though the starting point, S-H equation for photons is same.

The allowed states (N,l) in the spherical cavity, determined by the roots of the radial functions (i.e., by the spatial quantizing condition), sorted in the order of the increasing energies and accompanied by the "magnetic" weights, $g_l = 2l + 1$, are shown in Annex 1, up to $z = 20$. Our calculations are extended to a larger range, $z \leq$

50, i.e., up to more than 300 ordered roots. From Annex 1, we ascertain *a strong variation of the inter-state intervals, which also seems to be randomly distributed in the frequency scale, as a result of effective dependence of energy states on two quantum numbers (N, l).*

According to Eqs.(2) and (40), the asymptotic energy values are:

$$E = h\nu_0 N,\quad N = 1, 2, 3, \ldots \infty\ ,\qquad \nu_0 = c/4R \tag{41}$$

As a state (N, l) has a "magnetic" weight $(2l + 1)$, the supplementary degeneracy in the asymptotic approximation,

$$\sum_{m=-l}^{l} m = 2l + 1\,, \tag{42}$$

leads to a total weight $(1/2)(N+1)(N+2)$, for an energy level N, irrespective of the parity of N [10]. To account for all the states in the phase space, we still need to multiply the previous weight by a factor 2 to take into account the polarization states and by another factor $C = \pi^2/6$ to go from Cartesian to polar coordinates [10]. Thus, *the asymptotic total weight* of the state N in the spherical cavity is :

$$g_{asy}(N) = \frac{\pi^2}{6}(N+1)(N+2) \tag{43}$$

Concerning the energy spectrum problem, we may point out the following specific features:

1) the spectrum is a discrete one, the density of lines, on a frequency unit being inversely proportional to the radius of spherical cavity;
2) the spectrum - analyzed in terms of the radial quantum number, n and orbital quantum number, l, in (n, l) states, having a $(2l +1)$ - magnetic type degeneracy - is a succession of higher and higher degenerated multiplets (this second degeneracy being however not intrinsic to the radiation problem, but rather determined by the error associated to the replacing of the exact wave function F_{Nl} with the asymptotic ones $F_{Nl}^{(0)}$);
3) owing to the peculiarity that (n, l) are combined one to another to make up the principal quantum number, N, in a specific way $2n + l = N$, (different from the homologous way we come across in the Hydrogen atom case $n + l = N$), two adjacent states, inside a multiplet labeled by N, namely (N, l), (N, l'), are placed at a relative distance $|\, l - l'| = 2$. At the same time, N and l always have the same parity, whence two adjacent states (N, l), (N', l) are placed at a relative distance $|N - N'| = 2$.

These are serious arguments to classify the states not in the (n, l) plane, but rather in the (N, l) plane, where the energy states cover the region between the vertical axis ON and the bisectrix of the plane $N0l$, presenting the characteristic aspects of a chess table (Fig. 3). Reading this diagram horizontally, the multiplet structure of the spectrum is revealed: $N = 1$ singlet, $N = 2$ doublet, $N = 3$ doublet, $N = 4$ triplet, $N = 5$ triplet, $N = 6$ quadruplet, $N = 7$ quadruplet etc. On the other hand, all states with the same l, which appear along the corresponding vertical line, have the same degeneracy, $g = 2l + 1$.

By going over from the asymptotic radial wave functions to the exact ones, the following effects occur:

1) the energy state positions are displaced;
2) the degeneracy of the multiplets is removed;

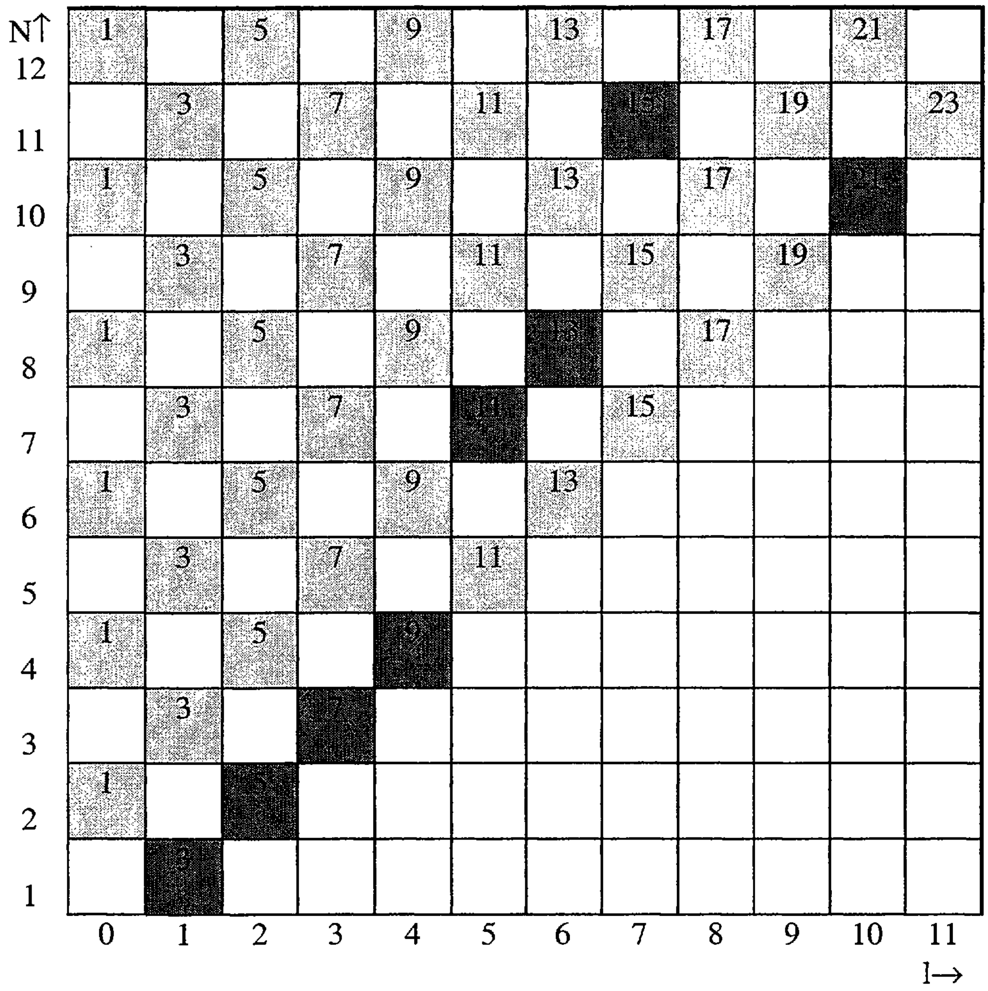


Fig. 3. The diagram of the lowest energy states of Planck discrete radiation in the spherical cavity in the asymptotic approximation. Allowed states are drawn in gray and labeled by the $(2\,l+1)$ - "magnetic" weights. The ground state is the singlet $N = 1$, $(n = 0, l = 1)$. Reading the diagram horizontally, the multiplet structure of the spectrum is revealed. The dark gray cells correspond to the forbidden states (antiresonances). The exact states may occur also under the diagonal ($N = l$, i.e could have $n < 0$).

3) some of the asymptotic states (N, l) became forbidden (the dark grey cells in Fig.3) due to the poles of the solution of a transcendent equation giving the exact eigenvalues [10];
4) the multiplet with a quantum number N is penetrating the following multiplet with the quantum number N+1, in such a way that the components of these multiplets occupy the same frequency interval, their succession being apparently random [for example, the multiplets $(30, l)$ and $(31, l)$ are interlaced and have very close weight centers, i.e. $N_{30} = 30.8670$ and $N_{31} = 30.9227$];
5) there are some states with negative radial quantum number (marked by an asterix in the list in Annex 1 and occurring for $N \geq 10$), which yield the factor $\pi^2/6$ in the total degeneracy;

6) in the discrete spectrum, the forbidden states (above the diagonal of the (*NOl*) quadrant) occur earlier than states with negative radial quantum number and together with the pairs of multiplets structured as mentioned at the point 4), seems to produce the maximum deviations of discrete Planck spectrum from the classical one.

In order to force an analogy with the cubic cavity, we can reduce formally this frequency spectrum to an **equivalent multi-channel spectrum**, depending solely on a single integer parameter, namely *the number, k, of the equidistant frequency channel* of a spectrometric analyzer. We can firstly adopt equidistant channels with a conventionally fixed width $\Delta z = 0.5$.

All the weights, g_j, falling inside a certain channel k are summed up and the *channel weight*, $\bar{g}_k$, is assigned to the fictitious (dimensionless) frequency, z_k (located at the middle of the channel):

$$z_k = \left(k - \frac{1}{2}\right)\Delta z\,,\quad k = 1, 2, 3, ... \tag{44}$$

The multi-channel spectrum of the Planck radiation in DQSC can be written as:

$$\frac{4}{3}\pi\frac{R^3}{h}u(\nu_k) = \frac{\pi^2}{3}\cdot\frac{\bar{g}_k z_k}{\Delta z}\cdot\left[\exp(\alpha_1 z_k/\gamma_1) - 1\right]^{-1} \tag{45}$$

$$\bar{g}_k = \sum_{\Delta}(g_j)_{Ch.k} \rightarrow \frac{4}{\pi^3}z_k^2\Delta z \rightarrow \frac{4}{\pi^3}\left(k - \frac{1}{2}\right)^2(\Delta z)^3 \tag{46}$$

where $\gamma_l = RT = 0.62035(V^{1/3}T)$ is the adiabatic invariant up to the shown factor and $\alpha_l = hc/2\pi k = 0.22899$ [cm.K].

The discrete Planck spectrum of DQSC (45) was plotted against the dimensionless frequency z and compared to the continuous Planck spectrum, for different values of the adiabatic invariant, *RT* (Fig.4):

$$\frac{4}{3}\pi\frac{R^3}{h}u(\nu) = \frac{4}{3\pi}\cdot z^3\left[\exp(\alpha_1 z/\gamma_1) - 1\right]^{-1}. \tag{47}$$

It is essentially a discrete spectrum, exhibiting sudden and random variations of intensity and even *forbidden bands of frequency (antiresonances),* just as in the case of the cubic cavity.

We can truncate the Planck spectrum of DQSC up to a highest significant state, which would bring a contribution smaller than 10^{-2} to the spectrum energy and can be calculated as:

$$z_T \geq 48\,RT\,, \tag{48}$$

This truncation frequency of the Planck spectrum of DQSC can be used as in the case of the equivalent cubic cavity. From Fig.4a,b,c, one can see that this relation holds well: in Fig. 4d, for $RT = 1.5$, the frequency interval covered by the 300 Bessel roots

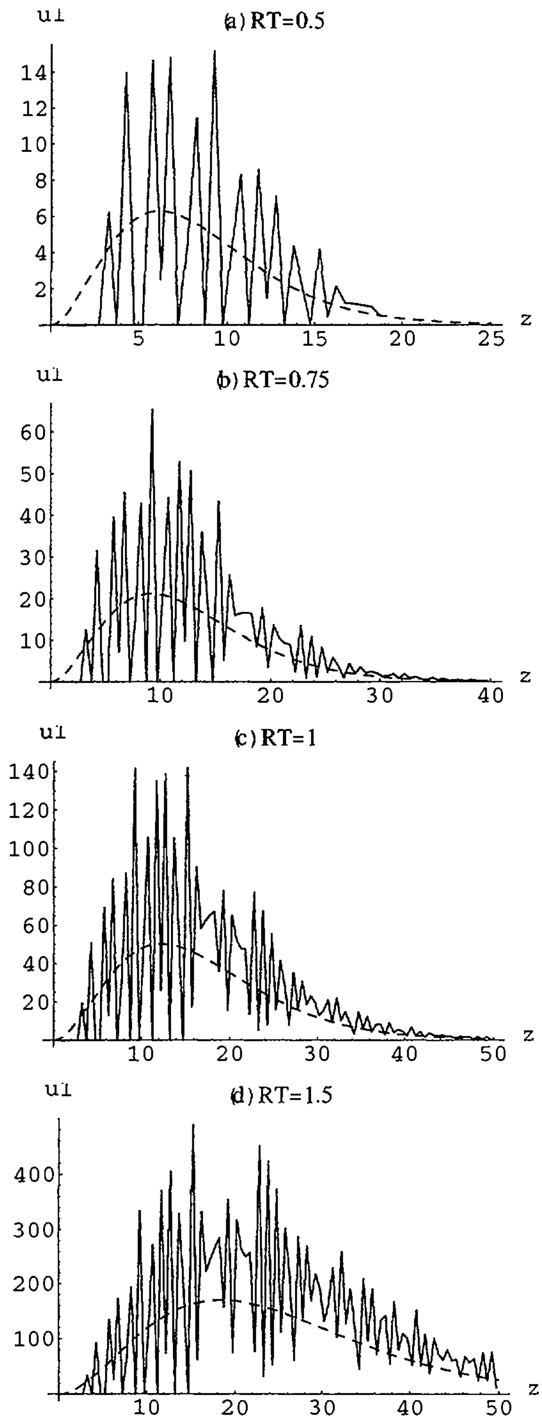


Fig.4. The discrete Planck spectrum of DQSC plotted against the dimensionless frequency z (joined solid lines) and compared to the continuous Planck spectrum (dashed line) for several values of the adiabatic invariant, *RT*: (a) 0.5; (b) 0.75; (c) 1; (d) 1.5 cm.K.

($z_T = 50$) is not long enough to ensure an error of 10^{-2}, which proves again that (48) is correct.

One can remark also, from Fig. 4a, that the maximal condition, still preserving the validity of summing up to the 70th state, could be a sphere of 1 cm diameter kept at an absolute temperature of 1K. This means that we can expect *quantum effects in the spherical black-body radiation energy for macroscopic conditions.*

Assuming that the highest significant state reaches the asymptotic limit, one can take:

$$N_T \approx (2/\pi) z_T \approx 31\, RT\,, \tag{49}$$

which is useful for the asymptotic formalism.

In Fig.4, the low frequency discrete structure of the spectrum is well resolved by the chosen spectrometer bandwidth. At higher frequencies, the individual components are not resolved by this spectrometer (many components are collected in its bandwidth) and an averaging occurs leaving only slow changes. The vicinity of the asymptotic region could also explain this fact.

If we compare the discrete Planck spectrum of DQSC with that obtained from DQCC, it is easy to find some equivalencies. The random degeneracy factor, $\zeta(N)$, defined as the ratio between the exact weight and the asymptotic one,

$$(g_k)_{asy} \approx (\pi^2/6)(N^2/2) = \pi^2 N^2/12\,, \tag{50}$$

is $\zeta(k) = \dfrac{g_k}{\pi^2 N^2/12} \to 1, \quad N \geq N_T \quad .$ (51)

Thus, one can force a formula for the discrete Planck spectrum in the spherical cavity by including the random degeneracy factor, $\zeta(k)$, for a small number of states as for the cubic cavity:

$$\frac{4}{3}\pi\frac{R^3}{h}u(\nu) = \frac{4}{3\pi}\cdot z^3\left[\exp(\alpha_1 z/\gamma_1) - 1\right]^{-1}\cdot\zeta(k)\,. \tag{52}$$

In the previous calculation of the discrete Planck spectrum of DQSC, a key point was the consideration of the equivalent multi-channel spectrum depending solely on the integer number of the equidistant frequency channel, k. In this calculation, the channel width was taken a constant, a solution which introduced a spectrum averaging at high frequencies. Another possiblity is the use of a multi-channel spectrum analyzer with variable channel width $\Delta z = (2\pi R/c)\,\Delta\nu$. In this case, we can write the total energy density of the cavity as:

$$\frac{4}{3}\pi\frac{R^3}{h}u(\nu_k) = 2\pi\frac{R}{hc}\cdot\frac{2g(z_k)\varepsilon_z}{\Delta z}\cdot\left[\exp(\varepsilon_z/kT) - 1\right]^{-1} \tag{53}$$

where $\varepsilon_z \to (hc/2\pi R)\, z$ is the state energy in the cavity. Comparing this expression with the continuous Planck spectrum, one can deduce:

$$\zeta = \frac{2\pi \frac{R}{hc} \cdot \frac{2g(z_k)\varepsilon_z}{\Delta z}}{0.424413\left(2\pi \frac{R}{hc}\right)^3 \varepsilon_z^3} = 4.71239\left(\frac{hc}{2\pi R}\right)^2 \frac{g(z_k)}{\varepsilon_z^2 \Delta z} = a\left(\frac{hc}{V^{1/3}}\right)^2 \frac{g(z_k)}{\varepsilon_z^2 \Delta z} = \frac{g(z_k)}{g(z_k)_{asy}} \quad (54)$$

If $\Delta z \propto (1/\varepsilon_z) \propto (1/z)$, the asymptotic weight will take the form

$$g(z)_{asy} \propto \left(\frac{V^{1/3}}{hc}\right)\varepsilon_z, \quad (55)$$

which is similar to that of Eq.(31) deduced for the cubic cavity. Thus, if we use narrower channels, as the spectrum frequency is higher, not only the resolution is preserved but also the asymptotic weight for DQSC takes the same form as for DQCC.

A multi-channel spectrum analyzer with variable channel width could be built using the mean interval of the two successive components of the interlaced multiplet pairs discussed above. Thus, from the state statistics, one can find the average number of components, $C(z)$, which are distributed in a frequency bandwidth of π, around the frequency z:

$$C(z) = 0.774(z - 0.785) \quad (56)$$

and the average distance between two successive components in the same frequency band,

$$\Delta z = \frac{4.058}{z - 0.785}, \qquad C(z)\cdot\Delta(z) = 3.1416. \quad (57)$$

which can lead to the multi-channel analyzer with the desired variable channel width. This approach could solve the problem of maintaining the analyzer resolution at high frequencies and the asymptotic weights as in Eq. (55).

3. The total energy of the double quantified cubic and spherical cavities

3.1. Cubic cavity

The classical Stefan-Boltzmannn (S-B) law provides the total energy in the cavity [2]:

$$E = V\int_0^\infty u(\upsilon,T)d\upsilon = \left(8\pi^5 k^4 / 15h^3c^3\right)VT^4 = \sigma VT^4 \quad (\alpha = hc/k = 1.4388 \text{ cm.K}) \quad (58)$$

In DQCC, the total energy should be written by summing up the state energies up to the highest significant one (characterized by N_T):

$$E=\sum_{N=1}^{N_T}\frac{2g(N)\cdot h\upsilon}{\exp(h\upsilon/kT)-1}=\frac{4\pi k\alpha}{(LT)^4}VT^4\sum_{N=1}^{N_T}\frac{N}{\exp(\alpha\sqrt{N}/LT)-1}\cdot\zeta(N)\cdot \qquad (59)$$

We can write the total energy from Eq. (59) as in S-B law, with a corrective factor:

$$F\left(\frac{\alpha}{LT}\right)=\frac{\sigma_1}{\sigma}=\frac{15}{4\pi^5}\left(\frac{\alpha}{LT}\right)^4\sum_{N=1}^{N_T=\infty}\frac{\sqrt{N}g(N)}{e^{\frac{\alpha}{LT}\sqrt{N}}-1}=\frac{5.2515\cdot10^{-2}}{(LT)^4}\sum_{N=1}^{109(LT)^2}\frac{\sqrt{N}g(N)}{e^{\frac{\alpha}{LT}\sqrt{N}}-1}\leq 1 \qquad (60)$$

In the asymptotic limit, $g(N)$ goes to $g_{asy}(N)$ (by averaging over many and very close modes), σ_1/σ tends to 1, and one arrives to the conventional formalism. The corrective factor is represented in Fig. 5, in function of the adiabatic invariant. We found out a convenient approximation for σ_1/σ as:

$$F(\alpha/LT)=\sigma_1/\sigma\approx\exp\,[0.06/LT-0.082/(LT)^2] \qquad (61)$$

and $F_{asy}(\alpha/LT)\approx 1-0.06/LT+c_1/(LT)^2$. (62)

The corrected S-B "constant" σ_1/σ is down limited by the lowest cavity mode to the value: $\sigma_1 = 0.00178\ \sigma = 1.346\cdot10^{-17}$ [erg cm^{-3} K^{-4}] and arrives to the asymptotic (upper limit) at the conventionally selected value $LT\approx 1$, for which: $\sigma_1 = 0.9703\ \sigma = 7.340\cdot10^{-15}$ [erg cm^{-3} K^{-4}]. The asymptotic corrective factor from (55b) is in agreement with that provided by the Weyl-Pleijel formula for the density of states [5, 6].

Thus, *the total energy in DQCC has a stronger dependence on temperature than was predicted by Stefan-Boltzmann law* and is dependent also on the adiabatic invariant. As the cavity is emptied of states, its total energy is strongly decreasing according *a new law* derived in Eq.(59).

Calculating the ratio (σ_1/σ) from (60) with the exact degeneracy provided by the Diophantine eq. (28) and with the asymptotic relation $g(N)\approx 2\pi\sqrt{N}$, we found out differences in the order of $\approx 5\cdot10^{-3}$, which are negligible in these calculations. We can put the *total energy density law of DQCC* into the final form:

$$E=\left[\frac{0.32996}{(LT)^4}\sum_{N=1}^{109(LT)^2}\frac{N}{e^{\frac{\alpha}{LT}\sqrt{N}}-1}\right]\cdot\sigma VT^4 \qquad (63)$$

and observe that *the small number of states in cavity (up to N_T, i.e. small $\gamma = LT$) plays the key role in E and not the exact degeneracy.*

We have shown that the positions of the energy density peak and of HSL depend on the product $\gamma_N = LT$. Eq. (60) and figure 5 show that the asymptotic limit can be set for $F(\alpha/LT)\approx 1 \rightarrow \gamma_{Nmax}\approx 1$ [cm.K] $\rightarrow N\approx 100$.

On the other hand, the lowest cavity mode (1,0,0) imposes an inferior limit to the level number at $N_T = 1\approx 109(LT)^2$ (the smallest frequency in the cavity) leading to $\gamma_{Nmin}\approx 0.1$ [cm.K]. Thus, we can define the *double quantization regime* of the cubic cavity in the range:

$$1 \leq N \leq 100 \tag{64}$$

$$0.1 \leq \gamma \leq 1 \qquad [\text{cm.K}]. \tag{65}$$

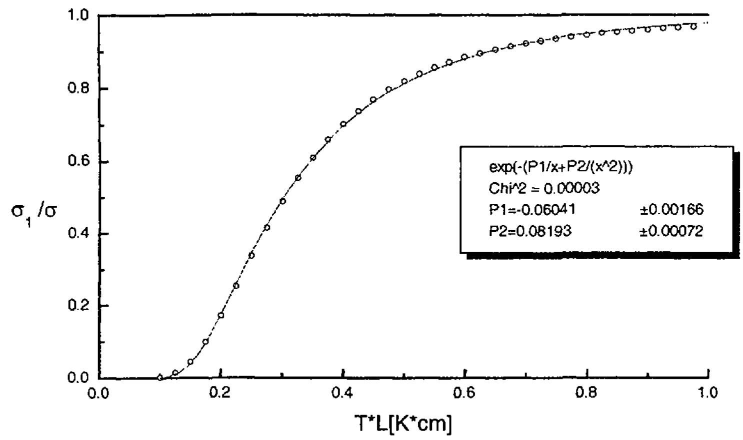


Fig. 5. The corrective factor of the total energy in DQCC in function of the adiabatic constant LT in the interval $LT \in [0.1, 1]$ (with dots). The analytical function fitting the best dependence of σ_1/σ on LT.

The following **reciprocity rule** holds: the cavity size and the temperature are reciprocal parameters in the DQCC, i.e. the same effects (in the thermodynamics of the photon gas) can be obtained either by varying $L = V^{1/3}$ or by varying T, if their product remain constant.

3.2. Spherical cavity

Using the allowed states in a sphere of radius R, we may write down the total energy of the black-body radiation inside the spherical cavity [10] as:

$$E = 2 \cdot \frac{\pi^2}{6} kT(\alpha_1 / \gamma) \sum_{k=1}^{k=\infty} \frac{z_k g_k}{\exp(\alpha_1 z_k / \gamma_1) - 1}, \tag{66}$$

with $\gamma_l = RT = 0.62035(V^{1/3}T)$ and $\alpha_l = hc/2\pi k = 0.22899$ [cm.K]. (67)

Similar to the case of DQCC, we can calculate a corrective factor for the total energy in DQSC:

$$F\left(\frac{\alpha_1}{RT}\right) = \frac{\sigma_1}{\sigma} = \frac{15}{4\pi}\left(\frac{\alpha_1}{RT}\right)^4 \sum_{k=1}^{k_T} \frac{z_k g_k}{\exp(\alpha_1 z_k / RT) - 1} = \frac{3.282067 \cdot 10^{-3}}{(RT)^4} \sum_{k=1}^{k_T} \frac{z_k g_k}{\exp(0.22899 z_k / RT) - 1} \tag{68}$$

It may be proved that: $\lim_{RT\to\infty} F(\alpha_1/RT) = 1$. (69)

The demonstration [10] is based on the asymptotic degeneracy of the N – multiplets. The convergence of the function F toward 1 also resorts to the fact that the ratio of the antiresonance number to the allowed state number approaches zero for $N \to \infty$. The existence of allowed states beyond the (parabolic) barrier of the highest poles and the zigzag diagonal line is essential in reaching the aforementioned convergence.

The summation in Eq.(68) can be taken up to the index, k_T, corresponding to the maximum frequency found in (48), $z_M = 48\,RT$. Thus, for example, in the correction calculation at $RT = 0.5$, the upper limit in the sum can be taken $k_T = 70$.

We can write the *new total energy density law of DQSC* into the final form:

$$E = \left[\frac{15}{4\pi}\left(\frac{\alpha_1}{RT}\right)^4 \sum_{k=1}^{k_T} \frac{z_k g_k}{\exp(\alpha_1 z_k/\gamma_1) - 1}\right] \cdot \sigma T^4 = \\ = \left[\frac{3.282067 \cdot 10^{-3}}{(RT)^4} \sum_{k=1}^{k_T} \frac{z_k g_k}{\exp(0.22899 z_k / RT) - 1}\right]. \tag{70}$$

In Fig.6a, we have plotted the corrective function $F(\alpha_1/RT)$ (with solid line) in the interval $0 \le RT \le 1.5$, which provides acceptable errors, for the limited number of calculated and ordered states (300). One can remark that this "exact" corrective factor imposes a faster decrease of the cavity energy than that predicted by the Stefan-Boltzmann law, as $RT \to 0$. Then, a special feature of DQSC is the growth of the corrective factor over the value corresponding to the Stefan-Boltzmann law. The calculations give a maximum of the corrective factor, $F_M \approx 1.36$, at $RT \approx 1.3$.

Taking into account the accuracy limit of exact calculations of $F(\alpha_1/RT)$ set by the limited number of calculated eigenvalues, we could use the asymptotic formula:

$$E_{asy} = \frac{\pi^2}{6} h\upsilon_0 \sum_{N=1}^{\infty} \frac{N(N+1)(N+2)}{\exp(aN) - 1}, \qquad a = h\upsilon_0 / kT \tag{71}$$

in order to obtain $F_{asy}(\alpha_1/RT)$ and to extend the quantum correction to larger values of RT. The summation in (71) may be performed in a special way to reveal the asymptotic quantum effects:

$$E_{asy} = \frac{\pi^2}{6} h\nu_0 \sum_{j=1}^{\infty} \sum_{N=1}^{\infty} N(N+1)(N+2) e^{-jaN} = \\ = \frac{\pi^2}{6} h\nu_0 \cdot \left\{ \sum_{j=1}^{\infty} \frac{e^{ja}\left(e^{2ja} + 4e^{ja} + 1\right)}{\left(e^{ja} - 1\right)^4} + 3\sum_{j=1}^{\infty} \frac{e^{ja}\left(e^{ja} + 1\right)}{\left(e^{ja} - 1\right)^3} + 2\sum_{j=1}^{\infty} \frac{e^{ja}}{\left(e^{ja} - 1\right)^2} \right\} \tag{72}$$

This degenerated discrete energy spectrum can be approximated by more simple expressions given by the dominant terms, in the asymptotic (classical) limit, $a << 1$:

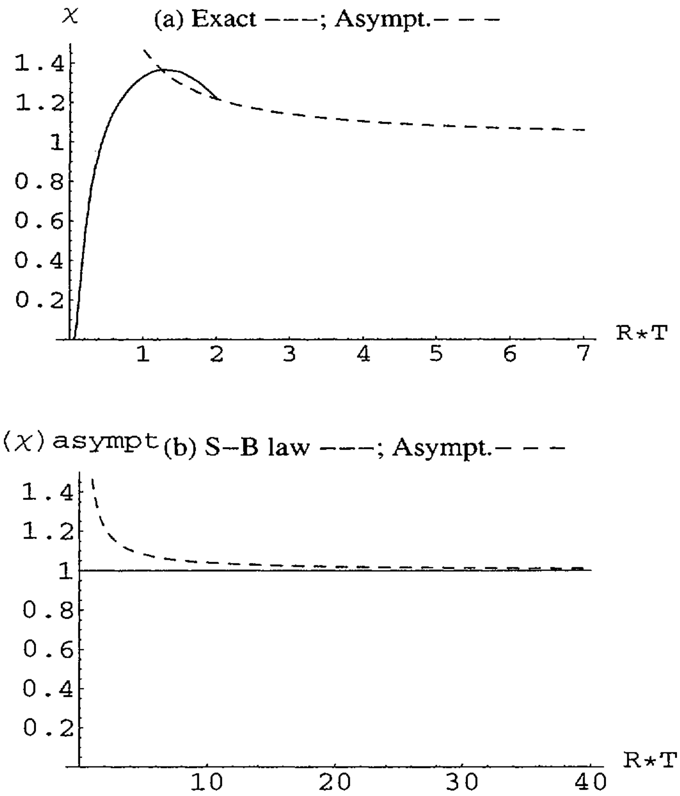


Fig. 6. (a) The corrective factor of the Planck constant, calculated with the exact eigenvalues, up to $RT = 2$ (solid line) and calculated in asymptotic conditions (dashed), with $RT \in [2,7]$. (b) The corrective factor of the asymptotic total energy yielded by the DQSC (dashed) and the Stefan-Boltzmann calculation (solid line) vs. RT, in the range $RT \in [2, 40]$.

$$E_{asy} \approx \frac{\pi^2}{6} h\nu_0 \left\{ \frac{6}{a^4}\sum_{j=1}^{\infty}\frac{1}{j^4} + \frac{6}{a^3}\sum_{j=1}^{\infty}\frac{1}{j^3} + \frac{2}{a^2}\sum_{j=1}^{\infty}\frac{1}{j^2} \right\} =$$

$$= \sigma \left\{ VT^4 + \frac{hc}{k}\cdot\frac{15\zeta(3)}{2\pi^4} ST^3 + \left(\frac{hc}{k}\right)^2 \frac{5}{12\pi} RT^2 \right\}, \tag{73}$$

where $\sigma = \frac{8}{15}\pi^5 \frac{k^4}{h^3 c^3}$, $V = \frac{4}{3}\pi R^3$, $S = 4\pi R^2$, $\zeta(3) = 1.202\ 056\ 9$.

Eq. (73) is compatible with the Weyl-Pleijel formula for the density of states in a cavity [3,5,6]. The asymptotic corrective factor for evaluating the departure of the total energy yielded by the quantified spherical cavity from the classical Stefan-Boltzmann calculation is:

$$F_{asy}(RT) = \sigma_1 / \sigma \approx 1 + 0.39949055(RT)^{-1} + 0.06554584(RT)^{-2}. \tag{74}$$

Thus, the asymptotic total energy of DQCC depends on two correction terms, which depend on the cavity surface and radius.

We have used Eq.(74) to extend the graphical representation of the corrective function $F(\alpha_l/RT)$ for $RT \leq 2$. In Fig. 6a, it is possible to remark that the two graphs, the exact one and the asymptotic one are intersecting in a point close to $RT \approx 2$, which seems to be the lower limit of applicability of the asymptotic formula (74). Another remarkable feature of DQSC is that the corrective factor is significant (larger than 1%), i.e. the asymptotic total energy "feels" the cavity dimension (and shape) and temperature, up to $RT < 40$. This feature is illustrated in Fig. 6b, where the asymptotic corrective factor is plotted against the classical prediction of the Stefan-Boltzmann law; one can observe that the classical limit is reached for very large values of RT.

This asymptotic limit set for $F(\alpha/LT) \approx 1$ leads to $\gamma_{Nmax} \approx 65$ [cm.K] and $N_M \approx$ 1222. On the other hand, the lowest allowed state imposes an inferior limit to the level number at $N_T = 2 \approx 31RT$ leading to $\gamma_{Nmin} \approx$ 0.1 [cm.K]. Thus, we can define the *double quantization regime* of the spherical cavity in the range:

$$2 \leq N \leq 1222 \tag{75}$$

$$0.1 \leq \gamma \leq 65 \quad \text{[cm.K]} \tag{76}$$

which is quantitatively (not qualitatively) different compared with the cubic cavity.

The same *reciprocity rule* (as for DQCC) holds: the cavity size and the temperature are reciprocal parameters in the DQSC, i.e. the same effects (in the thermodynamics of the photon gas) can be obtained either by varying R or by varying T, if their product remain constant.

4. Conclusions

The Planck radiation spectrum of ideal cubic and spherical cavities, in the region of small adiabatic invariance, $\gamma = TV^{1/3}$, is discrete and depends strongly on the cavity geometry and temperature. The complex aspect of the spectrum is the consequence of the random distribution of the state weights in the cubic cavity and of the random overlapping of the successive multiplet components, for the spherical cavity. The discrete Planck spectra show forbidden frequency bands called, by us, antiresonances and clearly modulate the corresponding quantum vacuum.

The total energy (obtained by summing up the exact contributions of the eigenvalues and their weights, for low values of the adiabatic invariants) does not obey any longer the Stefan-Boltzmann law. The new law includes a corrective factor depending on γ and imposes a faster decrease of the total energy to zero, for $\gamma \to 0$. The asymptotic total energies, calculated by us, have surface and radius dependent corrections, which are compatible with the Weyl-Pleijel state density formalism.

We have defined the double quantized regime both for cubic and spherical cavities by the conditions put on the principal quantum numbers: $N_{min} \leq N \leq N_{max}$ or the adiabatic invariants: $\gamma_{min} \leq \gamma \leq \gamma_{max}$. Thus, in this regime, the cavity size and the temperature are reciprocal parameters in the sense that the same effects (in the photon gas) can be obtained either by varying (L, R) or by varying T, if their product remain constant. The limits of the double quantized regime are different for cube and sphere.

The total energy of the double quantized cavities shows large differences from the classical calculations over unexpected large intervals, which can be measured and show important macroscopic quantum effects.

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Annex 1. The allowed states of DQSC in the interval z∈(0, 20)

k	(N, l)	g_k	$z_k = 2\pi\nu_{nl} R/c$
1	(2, 0)	1	3.141 592 7
2	(3, 1)	3	4.493 409 5
3	(4, 2)	5	5.763 459 2
4	(4, 0)	1	6.283 185 3
5	(5, 3)	7	6.987 932 0
6	(5, 1)	3	7.725 251 8
7	(6, 4)	9	8.182 561 6
8	(6, 2)	5	9.095 011 3
9	(5, 5)	11	9.355 812 0
10	(6, 0)	1	9.424 778 0
11	(7, 3)	7	10.417 119 0
12	(6, 6)	13	10.512 836 0
13	(7, 1)	3	10.904 122 0
14	(7, 7)	15	11.657 033 0
15	(8, 4)	9	11.704 908 0
16	(8, 2)	5	12.322 941 0
17	(8, 0)	1	12.566 371 0
18	(8, 8)	17	12.790 718 2
19	(9, 5)	11	12.966 531 0
20	(9, 3)	7	13.698 023 0
21	(9, 9)	19	13.915 823 0
22	(9, 1)	3	14.066 194 0
23	(10, 6)	13	14.207 393 0
24	(10,10)	21	15.033 469 3
25	(10, 4)	9	15.039 665 0
26	(9, 7)	15	15.431 288 0
27	(10, 2)	5	15.514 603 0
28	(10, 0)	1	15.707 963 0
29	(11, 11)	23	16.144 741 0
30	(11, 5)	11	16.354 710 0
31	(10, 8)	17	16.641 003 0
32	(11, 3)	7	16.923 622 0
33	(11, 1)	3	17.220 755 0
34	(10,12)	25	17.250 454 8*
35	(12, 6)	13	17.647 975 0
36	(11, 9)	19	17.838 644 0
37	(12, 4)	9	18.301 256 0
38	(11,13)	27	18.351 261 2*
39	(12, 2)	5	18.689 036 0
40	(12, 0)	1	18.849 556 0
41	(13,7)	15	18.922 999 2
42	(12, 10)	21	19.025 853 5
43	(12,14)	29	19.447 701 5*
44	(13,11)	11	19.653 152 1