

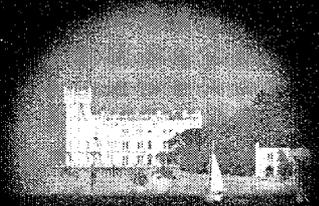


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**JACOBSON GENERATORS,  
FOCK REPRESENTATIONS AND STATISTICS  
OF  $sl(n+1)$**

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THE ABDUS SALAM INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

**JACOBSON GENERATORS, FOCK REPRESENTATIONS  
AND STATISTICS OF  $sl(n+1)$**

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**Abstract**

The properties of  $A$ -statistics, related to the class of simple Lie algebras  $sl(n+1)$ ,  $n \in \mathbb{Z}_+$  (Palev, T.D.: Preprint JINR E17-10550 (1977); hep-th/9705032), are further investigated. The description of each  $sl(n+1)$  is carried out via generators and their relations (see eq. (2.5)), first introduced by Jacobson. The related Fock spaces  $W_p$ ,  $p \in \mathbb{N}$ , are finite-dimensional irreducible  $sl(n+1)$ -modules. The Pauli principle of the underlying statistics is formulated. In addition the paper contains the following new results: (a) the  $A$ -statistics are interpreted as exclusion statistics; (b) within each  $W_p$  operators  $B(p)_1^\pm, \dots, B(p)_n^\pm$ , proportional to the Jacobson generators, are introduced. It is proved that in an appropriate topology (Definition 2)  $\lim_{p \rightarrow \infty} B(p)_i^\pm = B_i^\pm$ , where  $B_i^\pm$  are Bose creation and annihilation operators; (c) it is shown that the local statistics of the degenerated hard-core Bose models and of the related Heisenberg spin models is  $p = 1$   $A$ -statistics.

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# 1 Introduction

During the last two decades quantum statistics became a field of increasing interest among field theorists and condensed matter theorists. Various new statistics were suggested, leading to generalizations or deviations from some of the first principles in quantum physics, such as the Heisenberg commutation relations, the Pauli exclusion principle and the commutativity of space-time.

The literature on the subject is vast, especially in the part related to quantum groups [10, 36, 13, 50, 80]. In a paper entitled “Twisted Second Quantization” [68] Pusz and Woronowicz introduced multimode deformed Bose creation and annihilation operators (CAOs), covariant under the action of the quantum group  $U_q[sl(n)]$  (for  $n$  pairs of them). Another deformation with commuting modes of CAOs was proposed in [48]; the link between them was established in [43]. A third deformation, which for one mode of CAOs was known for many years [8], the so-called quon algebra [23], was defined as an associative algebra, subject to relations  $a_i^- a_j^+ - q a_j^+ a_i^- = \delta_{ij}$ . This generalization (note that no relations among only creation operators or among only annihilation operators are required) was in the origin of a model proposed for a verification of small violations of Bose-Fermi statistics in quantum field theory (QFT) [24]. The quon statistics, which in the classification of Doplicher, Haag and Roberts [9] belongs to the class of “infinite statistics”, was studied by several authors [20] from different points of view (see [25] for further discussions and references).

Recently string theory was also involved in discussions on quantum statistics, the latter related to its prediction that Heisenberg’s uncertainty principle has to be corrected at distances of order of the Planck length  $k_P = 10^{-32}$  cm. Consequently there emerges an absolute minimum uncertainty in the measuring of any length [78]. These predictions motivated several authors to search for model independent arguments, leading to the same conclusions as string theory does (we refer to [16] for a survey on the subject). In particular it has been shown that the above results can be reproduced on a purely kinematical level with appropriate deformations of the Heisenberg commutation relations [49, 40, 33, 1], i.e., of canonical quantum statistics. In all such cases the coordinates do not commute at small distances, a result which is consistent with the spirit of non-commutative geometry [7].

Turning to condensed matter physics we refer to anyons, “particles” with fractional statistics (FS) in two-dimensional (2D) systems [46]. The theoretical studies of this and other noncanonical statistics were strongly pushed forward after the discovery of the fractional quantum Hall effect (FQHE) in two-dimensional electron gases [76]. Its theoretical explanation led Laughlin [45] to the conclusion that there exist quasiparticles carrying fractional electric charges. The statistics of these particles (we write “particles” for the elementary excitations, the “quasiparticles”, when no confusion can arise) also turned out to be fractional statistics [32].

A further breakthrough in the area of quantum statistics was marked with the 1991 paper of

Haldane [29], who proposed a generalized version of the Pauli exclusion principle. For only one kind of identical particles this new statistics, now called (fractional) exclusion statistics (ES), asserts that the change  $\Delta d$  in the dimension  $d$  of the single-particle Hilbert space is defined via the relation

$$\Delta d = -g \cdot \Delta N. \quad (1.1)$$

Here  $\Delta N$  is an *allowed increase* of the number of particles. The constant  $g$  is called an exclusion statistics parameter.

In [81] Wu proposed an “integral form” compatible with Haldane’s definition (1.1) :

$$d(N) = n - g(N - 1). \quad (1.2)$$

In (1.2)  $N - 1$  is the number of particles already accommodated in the system,  $d(N)$  is the dimension of the single-particle space, namely the number of the orbitals, where an additional  $N^{\text{th}}$  particle can be “loaded”, holding the distribution of the initial  $N - 1$  particles fixed;  $n = d(1)$  is the number of orbitals available for the first particle. Eq. (1.2) holds for *all admissible values* of  $N$ .

Contrary to Bose or Fermi statistics the orbitals of an ES may not be filled independently of each other. An essential difference of ES in comparison to fractional statistics is that in general the former is defined for any space dimension. Initially ES was defined for finite-dimensional single-particle Hilbert spaces [29]. The generalization to infinite-dimensional cases is due to Murthy and Shankar [53]. In [81] Wu extended the meaning of species. His definition allows different species indices to refer to particles of the same kind but with different quantum numbers.

In [29] Haldane has shown that when applied to FQHE, ES leads to the same predictions as FS does (see also [37]). The validity of ES was tested on several other examples : spinon excitations in a spin- $\frac{1}{2}$  quantum antiferromagnetic chain (with nearest neighbor-exchange or with inverse-square exchange between all sites) [29]; anyon gas and anyons in a strong magnetic field (confined to the first Landau level) [81, 53, 47]; particles in 1D Luttinger liquid [53]; Calogero-Sutherland models [27].

The discovery of the Yangian  $Y(SU_N)$ -symmetry of  $SU_N$  quantum chains with inverse-square exchange [30] (generalizations of the  $S = \frac{1}{2}$  Haldane-Shastry spin chains [28]) casted a bridge between exclusion statistics and the  $(SU_N)_1$  Wess-Zumino-Witten (rational) conformal field theories, providing a new, alternative, description of these theories (see [31] for a broader review on the subject). Instead of primary chiral fields, the fundamental fields in this picture are “free”  $SU_N$ -spinon fields, namely fields which interact only via statistical interaction [72]. The statistics is encoded in the generalized “commutation” relations between the creation and the annihilation operators, namely between the Fourier modes of the fields. As a result the single particle Fock states are occupied in such a way that the corresponding statistics is an exclusion statistics (we refer to [73] for further details, remarks and additional references).

Despite of the fact that ES is defined for any space dimension, so far it was applied and tested only within 1D and 2D models. In the present paper we show that particular kinds of ES, called *A*-statistics, can exist in spaces with any dimension.

Our approach to quantum statistics is strongly influenced by the ideas of Wigner, outlined in his 1950's work "Do the equations of motion determine the quantum mechanical commutation relations?" [79]. This was the first paper where it was clearly indicated that the canonical quantum statistics may, in principle, be generalized in a logically consistent way. Wigner demonstrated this in the example of a one-dimensional oscillator with a Hamiltonian ( $m = \omega = \hbar = 1$ )  $H = \frac{1}{2}(p^2 + q^2)$ . Abandoning the requirement  $[p, q] = -i$ , Wigner was searching for all operators  $q$  and  $p$ , such that the "classical" equations of motion  $\dot{p} = -q$ ,  $\dot{q} = p$  are identical with the Heisenberg equations  $\dot{p} = -i[p, H]$ ,  $\dot{q} = -i[q, H]$ . Apart from the canonical solution he found infinitely many other solutions. Let  $\sqrt{2}B_1^\pm = q \mp ip$ . It turns out [62] that all these different operators satisfy one and the same triple relation, namely (1.3) below with  $i = j = k = 1$ , (see the end of this Introduction for the notation) :

$$[\{B_i^\xi, B_j^\eta\}, B_k^\varepsilon] = \delta_{ik}(\varepsilon - \xi)B_j^\eta + \delta_{jk}(\varepsilon - \eta)B_i^\xi, \quad i, j, k \in \mathbb{N}, \quad \xi, \eta, \varepsilon = \pm, \pm 1. \quad (1.3)$$

The operators  $B_i^\pm$ ,  $i = 1, 2, \dots$  are para-Bose (pB) operators, discovered by Green [21] three years later as a possible generalization of statistics of tensor fields in QFT. Thus the infinitely many different solutions found by Wigner were in fact the Fock representations of one pair of para-Bose operators.

It is known that the linear span of all operators  $B_i^\xi$ ,  $\{B_j^\eta, B_k^\varepsilon\}$  is a Lie superalgebra [58] isomorphic to the orthosymplectic Lie superalgebra  $osp(1/2n)$  for  $i, j, k = 1, \dots, n$  and  $\xi, \eta, \varepsilon = \pm$  [15]. The para-Bose operators constitute a basis in the odd subspace of this superalgebra and generate it. Consequently the representation theory of  $n$  pairs of pB operators is completely equivalent to the representation theory of  $osp(1/2n)$ . Hence Wigner found all Fock representations of  $osp(1/2)$  long before Lie superalgebras (and supersymmetry) became of interest in physics and even before they were introduced in mathematics.

Similarly, any  $n$  pairs of para-Fermi CAOs  $F_1^\pm, F_2^\pm, \dots, F_n^\pm$  [21], defined by relations

$$[[F_i^\xi, F_j^\eta], F_k^\varepsilon] = \frac{1}{2}\delta_{jk}(\varepsilon - \eta)^2 F_i^\xi - \frac{1}{2}\delta_{ik}(\varepsilon - \xi)^2 F_j^\eta, \quad i, j, k \in \mathbb{N}, \quad \xi, \eta, \varepsilon = \pm, \pm 1, \quad (1.4)$$

generate the Lie algebra  $so(2n + 1)$  [39, 70]. The key observation here is that both  $so(2n + 1)$  and  $osp(1/2n)$  belong to class *B* of the basic Lie superalgebras in the classification of Kac [38]. Hence parastatistics (and in particular Bose and Fermi statistics) appear as particular Fock representations of Lie superalgebras from one and the same class, the Lie superalgebras of class *B*. In this sense Green's parastatistics could be called *B*-(para)statistics.

The clarification of the mathematical structure, hidden in parastatistics, provides a natural background for further searches of new quantum statistics. One such possibility is to consider deformations of parastatistics, namely deformations of  $so(2n + 1)$  and  $osp(1/2n)$  in the sense of quantum groups. We refer to [65] for discussions and results along this line.

In another approach, initiated in [59], it was shown that to each infinite class  $A$ ,  $B$ ,  $C$  and  $D$  of simple Lie algebras there corresponds quantum statistics. Examples from classes  $A$  and  $B$  of proper Lie superalgebras are also available. We have in mind Wigner quantum systems (WQSs) [62]. Some such systems possess quite unconventional physical features. As an example we mention the  $(n+1)$ -particle WQS, based on the Lie superalgebra  $sl(1/3n)$  from class  $A$  [64]. This WQS exhibits a quark-like structure : the composite system occupies a small volume  $V$  around the centre of mass and no particles can be extracted out of  $V$ . Moreover the geometry within  $V$  is noncommutative. Another example is the  $osp(3/2)$  WQS from class  $B$  [63]. It leads to a picture where two spinless point particles, “curling” around each other, produce an orbital (internal angular) momentum  $1/2$ , a result which cannot be obtained in canonical quantum mechanics.

The present paper is also in the frame of quantum statistics. We study further the (microscopic) properties of  $A$ -statistics, introduced in [59] (see also [60]), namely the statistics of Lie algebras  $A_n \equiv sl(n+1)$ ,  $n = 1, 2, \dots$ . Since Refs. [59] and [60] are not available as journal publications, we review the main issues of  $A$ -statistics in Sections 2 and 3, omitting most of the proofs.

We begin (Section 2) by recalling how the Lie algebra  $sl(n+1)$  can be described via generators  $a_1^\pm, \dots, a_n^\pm$  and triple relations, see (2.5). To the best of our knowledge such generators were introduced for the first time by Jacobson [35] in the more general context of Lie triple systems. For this reason we refer to  $a_1^\pm, \dots, a_n^\pm$  as Jacobson generators (JGs). The latter provide an alternative to the Chevalley description of  $sl(n+1)$ .

The Fock modules of the Jacobson generators, extended also to  $gl(n+1)$ -modules, are defined and classified in Section 3. It is shown how they can be selected out of all irreducible  $gl(n+1)$ -modules on the ground of natural physical requirements, see Definition 1. All Fock modules  $W_p$  are finite-dimensional and are labelled by one positive integer  $p \in \mathbb{N}$ . Within  $W_p$  each generator  $a_i^+$  (resp.  $a_i^-$ ) is interpreted as an operator creating (resp. annihilating) a “particle” in a state  $i$ . The Pauli principle for  $A$ -statistics is also formulated (Corollary 3).

In Section 4 we argue that  $A$ -statistics can be interpreted as a particular kind of exclusion statistics [29].

Next, in Section 5, representation dependent creation and annihilation operators  $B(p)_i^\pm = a_i^\pm / \sqrt{p}$  ( $i = 1, \dots, n$ ) in  $W_p$  are defined. We prove that in an appropriate topology  $\lim_{p \rightarrow \infty} B(p)_i^\pm = B_i^\pm$ , where  $B_1^\pm, \dots, B_n^\pm$  are Bose creation and annihilation operators. The operators  $B(p)_1^\pm, \dots, B(p)_n^\pm$  possess also other Bose-like properties. For these reasons  $B(p)_1^\pm, \dots, B(p)_n^\pm$  are referred to as quasi-Bose operators (of order  $p$ ), the representations of  $sl(n+1)$  and  $gl(n+1)$  in  $W_p$  as quasiboson representations and the statistics as quasi-Bose statistics.

The Jacobson CAOs  $a_1^\pm, \dots, a_n^\pm$  are “bosonized” in Section 6. These operators are expressed via  $n$  pairs of Bose CAOs  $B_1^\pm, \dots, B_n^\pm$ . The related realization of  $gl(n+1)$  in  $W_p$  turns to be the known Holstein-Primakoff realization [56].

In Section 7 we point out that the  $p = 1$  quasi-Bose operators (coinciding in this case with the  $p = 1$  representation of the JGs) can also be of more general interest. On the example of a two-leg  $S = 1/2$  Heisenberg spin ladder we show that the Bose realization of the Hamiltonian [19, 75] together with the restrictions selecting the physical subspace actually means that the Bose operators related to each site have to be replaced by quasi-Bose operators of order  $p = 1$ . This conclusion is of a more general nature. It holds for any hard-core Bose model [14], since the  $p = 1$  quasibosons are hard-core bosons (Proposition 5).

The final Section 8 is devoted to some conclusions and discussions.

Throughout the paper we use the following abbreviations and notation (some of them standard) :

JGs – Jacobson generators;

CAOs – creation and annihilation operators;

UEA – universal enveloping algebra;

$\mathbb{N}$  – all positive integers;

$\mathbb{Z}_+$  – all non-negative integers;

$[a, b] = ab - ba, \quad \{a, b\} = ab + ba;$

$\oplus, \dot{\oplus}$  – direct sum of linear spaces and Lie algebras, respectively.

## 2 Jacobson generators of $sl(n + 1)$

The  $sl(n + 1)$ -statistics, including  $n = \infty$ , was introduced in [59] (see also [60]) as an alternative way for quantization of spinor fields in quantum field theory. Refs. [59] and [60] are not available as journal publications. Therefore here and in Section 3 we outline the main features of this statistics in somewhat more details.

In order to define the Jacobson generators, it is convenient to consider  $A_n \equiv sl(n + 1)$  as a subalgebra of the Lie algebra  $gl(n + 1)$ . The universal enveloping algebra  $U[gl(n + 1)]$  of the latter can be defined as an associative algebra with unity of the Weyl generators  $\{e_{ij} | i, j = 0, 1, \dots, n\}$  subject to the relations

$$[e_{ij}, e_{kl}] = \delta_{jk}e_{il} - \delta_{il}e_{kj}. \quad (2.1)$$

Then  $gl(n + 1)$  is a subalgebra of  $U[gl(n + 1)]$ , considered as a Lie algebra, with generators  $e_{ij}$ ,  $i, j = 0, 1, \dots, n$  and commutation relations (2.1).

The Cartan subalgebra  $H'$  of  $gl(n + 1)$  has a basis  $h_i \equiv e_{ii}$ ,  $i = 0, 1, \dots, n$ . Let  $h^0, h^1, \dots, h^n$  be the dual basis,  $h^i(h_j) = \delta_{ij}$ . The root vectors of both  $gl(n + 1)$  and  $sl(n + 1)$  are  $e_{ij}$ ,  $i \neq j = 0, 1, \dots, n$ . The root of each  $e_{ij}$  is  $h^i - h^j$ . Then

$$sl(n + 1) = \text{span}\{e_{ij}, e_{ii} - e_{jj} | i \neq j = 0, 1, \dots, n\}. \quad (2.2)$$

The Jacobson generators (JGs) of  $sl(n+1)$  are part of the Weyl generators, namely

$$a_i^+ = e_{i0}, \quad a_i^- = e_{0i}, \quad i = 1, \dots, n. \quad (2.3)$$

The correspondence with their roots reads

$$a_i^\pm \leftrightarrow \mp(h^0 - h^i), \quad i = 1, \dots, n, \quad (2.4)$$

and therefore the JGs  $a_i^+$  ( $a_i^-$ ) are negative (positive) root vectors with respect to the natural ordering  $h^0, h^1, \dots, h^n$ . Since any other root is a sum of the roots of  $a_j^-$  and  $a_i^+$ , namely

$$h^i - h^j = (h^0 - h^j) - (h^0 - h^i), \quad i \neq j = 1, \dots, n,$$

the JGs (2.3) generate  $A_n$  in the sense of a Lie algebra.

From (2.1) and (2.3) one derives the triple relations

$$\begin{aligned} (a) \quad & [[a_i^+, a_j^-], a_k^+] = \delta_{kj}a_i^+ + \delta_{ij}a_k^+, \\ (b) \quad & [[a_i^+, a_j^-], a_k^-] = -\delta_{ki}a_j^- - \delta_{ij}a_k^-, \\ (c) \quad & [a_i^+, a_j^+] = [a_i^-, a_j^-] = 0. \end{aligned} \quad (2.5)$$

On the contrary, setting  $e_{ij} - \delta_{ij}e_{00} = [a_i^+, a_j^-]$ , one derives from (2.5) the commutation relation between all  $sl(n+1)$  generators  $e_{ij}, e_{ii} - e_{jj}, i \neq j = 0, 1, \dots, n$ . The description of  $A_n$  by means of the generators (2.3) and the relations (2.5) was already given in a paper by Jacobson [35]; therefore, the elements  $a_i^\pm$  are referred to as Jacobson generators of  $A_n$ .

The above description of  $sl(n+1)$  via generators and relations is a particular case of describing Lie algebras via Lie triple systems (LTSs), initiated by Jacobson [35] and further developed to the  $\mathbb{Z}_2$ -graded case by Okubo [57]. Let us be more concrete. By definition [35] a Lie triple system  $\mathcal{L}$  is a subspace of an associative algebra  $U$ , so that  $\mathcal{L}$  is closed under the ternary operation  $\omega : \mathcal{L} \otimes \mathcal{L} \otimes \mathcal{L} \rightarrow \mathcal{L}$  defined as  $\omega(a \otimes b \otimes c) = [[a, b], c], a, b, c \in \mathcal{L}$ . The definition of a Lie supertriple system (equivalent to the definition in [57]) is similar. The difference is that  $\mathcal{L}$  is a  $\mathbb{Z}_2$ -graded subspace of an associative superalgebra  $U$  and the commutators in the definition of  $\omega$  are replaced by supercommutators.

The JGs of  $sl(n+1)$  are closely related to the above definition. More precisely, let  $\mathcal{L}_{sl}$  be the linear span of the generators (2.3) and  $U_{sl}$  be the associative unital algebra of the JGs subject to the relations (2.5). Then  $\mathcal{L}_{sl}$  is a subspace of  $U_{sl}$ . Moreover  $\omega : \mathcal{L}_{sl} \otimes \mathcal{L}_{sl} \otimes \mathcal{L}_{sl} \rightarrow \mathcal{L}_{sl}$  as a consequence of (2.5). Hence  $\mathcal{L}_{sl}$  is a Lie triple system with a basis consisting of the JGs (2.3) and  $U_{sl}$  is the UEA of  $sl(n+1)$ . Similarly, the linear span  $\mathcal{L}_{pf}$  of para-Fermi CAOs  $F_1^\pm, F_2^\pm, \dots, F_n^\pm$  together with the associative algebra  $U_{pf}$  of these operators (subject to the relations (1.4)) is another example of a LTS. Hence the para-Fermi operators  $F_1^\pm, \dots, F_n^\pm$  could be called JGs of  $so(2n+1)$ . In the same spirit the para-Bose operators  $B_1^\pm, \dots, B_n^\pm$  are JGs of  $osp(1/2n)$ .

From a purely algebraic point of view the Jacobson generators provide an alternative to the Chevalley description of  $sl(n+1)$ ,  $so(2n+1)$  and  $osp(1/2n)$ . The JGs of  $so(2n+1)$  and

$osp(1/2n)$  however (contrary to the Chevalley generators) have a direct physical significance. These operators extend the canonical Fermi and Bose statistics to the more general parastatistics. Below we proceed to show that the JGs of  $sl(n+1)$  also introduce a new quantum statistics, different from Bose and Fermi statistics and their generalization – parastatistics. This statistics is intrinsically related to class  $A$  of simple Lie algebras in the same way as the para-Fermi statistics is related to class  $B$  of simple Lie algebras.

Typically the “commutation relations” between the creation and the annihilation operators (or the related position and momentum operators in case of finite degrees of freedom) are derived from (more precisely, are required to be consistent with) *the main quantization equation*

$$[H, a_i^\pm] = \pm \varepsilon_i a_i^\pm, \quad (2.6)$$

where  $H$  is the Hamiltonian and  $i$  replaces all indices that may appear (momentum, spin, charge, etc.). In quantum field theory (2.6) expresses the translation invariance of the field (in infinitesimal form). In quantum mechanics the same equation appears as a compatibility condition (in the sense of Wigner [79]) between the Heisenberg equations of motion and the classical equations, if the system has a classical analogue (for more details see [62, 55]). There are certainly several other conditions to be satisfied (Galilean or relativistic invariance, causality, etc.; we refer to [64] for discussions in case of noncanonical quantum mechanics). The possibility for choosing different statistics essentially depends upon the way one represents the Hamiltonian  $H$ . We are going to illustrate this on the example of para-Fermi statistics.

Consider a nonrelativistic free field locked in a finite volume. In the case of a Fermi field the Hamiltonian  $\hat{H}$  is written in a normal-product form

$$\hat{H} = \sum_i \varepsilon_i f_i^+ f_i^-, \quad (2.7)$$

so that the energy of the vacuum is zero. Here  $f_i^+$  ( $f_i^-$ ) are Fermi creation (annihilation) operators :  $\{f_i^\xi, f_j^\eta\} = \frac{1}{4}(\xi - \eta)^2 \delta_{ij}$ ,  $\xi, \eta = \pm$  or  $\pm 1$ . Then (2.6) holds,

$$[\hat{H}, f_i^\pm] = \pm \varepsilon_i f_i^\pm, \quad (2.8)$$

and each  $f_i^\xi$  can be interpreted as an operator creating ( $\xi = +$ ) or annihilating ( $\xi = -$ ) a particle, i.e. a fermion with energy  $\varepsilon_i$ . Eq. (2.8) is not fulfilled however, if the Fermi operators in (2.7) are replaced by para-Fermi operators (1.4) : for  $H = \sum_i \varepsilon_i F_i^+ F_i^-$  the equation

$$[H, F_i^\pm] = \pm \varepsilon_i F_i^\pm \quad (2.9)$$

does not hold. Why? In order to answer this question using proper Lie algebraic language assume that the sum in (2.7) is finite (finite number of Fermi oscillators),

$$\hat{H} = \sum_{i=1}^n \varepsilon_i f_i^+ f_i^-. \quad (2.10)$$

This is only an intermediate step. The considerations below remain valid for  $n = \infty$ . Recall now that any  $n$  pairs of Fermi CAOs generate a particular Fermi representation of the Lie algebra  $so(2n+1) \equiv B_n$ , whereas the para-Fermi operators  $F_1^\pm, \dots, F_n^\pm$  are (representation independent) generators of  $so(2n+1)$  [39, 70]. Eq. (2.8) is not preserved, when passing to other representations of  $B_n$ , because  $H$  is not an element from  $B_n$  and hence  $[H, F_i^\pm]$  in the LHS of (2.9) is not a representation independent commutator. This observation suggests also the answer : one has to rewrite (2.10) in a representation independent form. In order to achieve this, represent (2.10) in the following identical form :

$$\hat{H} = \frac{1}{2} \sum_{i=1}^n \varepsilon_i ([f_i^+, f_i^-] + \{f_i^+, f_i^-\}). \quad (2.11)$$

Consider the Lie algebra generated from  $f_1^\pm, \dots, f_n^\pm$  and  $\{f_i^+, f_i^-\}$ . Since  $\{f_i^+, f_i^-\} = 1$ , we obtain a representation of the Lie algebra  $B_n \dot{\oplus} I$ , where  $I$  is the one-dimensional center. Now  $\hat{H} \in B_n \dot{\oplus} I$  and therefore the commutation relations (2.8) hold for any other representation of  $B_n \dot{\oplus} I$ . In other words, if we substitute  $f_i^\pm \rightarrow F_i^\pm$  and  $\{f_i^+, f_i^-\} \rightarrow \hat{p}$  in (2.11), i.e. set

$$H = \frac{1}{2} \sum_{i=1}^n \varepsilon_i ([F_i^+, F_i^-] + \hat{p}), \quad (2.12)$$

where  $\hat{p}$  is a generator of the center  $I$ , then the quantization condition (2.8) will be fulfilled for any representation of  $B_n \dot{\oplus} I$  and in particular for the para-Fermi operators (1.4) :  $[H, F_i^\pm] = \pm \varepsilon_i F_i^\pm$ . The requirement  $\hat{p}|0\rangle = p|0\rangle$ ,  $p \in \mathbb{N}$  (and  $F_i^- F_j^+ |0\rangle = \delta_{ij} p |0\rangle$ ,  $F_i^- |0\rangle = 0$ ), leads to a representation with an order of the (para)statistics  $p$  [22]. Then the energy of the vacuum is also zero.

We shall now apply a similar approach for the algebra  $A_n$ . Let  $E_{ij}$ ,  $i, j = 0, 1, \dots, n$ , be a square matrix of order  $(n+1)$  with 1 on the intersection of the  $i^{\text{th}}$  row and the  $j^{\text{th}}$  column and zeros elsewhere, i.e.,

$$(E_{ij})_{kl} = \delta_{ik} \delta_{jl}, \quad i, j = 0, 1, \dots, n. \quad (2.13)$$

The map  $\pi : e_{ij} \rightarrow E_{ij}$ ,  $i, j = 0, 1, \dots, n$ , gives a representation of  $gl(n+1)$  (usually referred to as the defining representation). Its restriction to  $A_n$  gives a representation of  $A_n$ . The operators  $A_i^+ = E_{i0}$ ,  $A_i^- = E_{0i}$ ,  $i = 1, 2, \dots, n$  satisfy the triple relations (2.5). Set

$$\hat{H} = \sum_{i=1}^n \varepsilon_i A_i^+ A_i^-. \quad (2.14)$$

Then

$$[\hat{H}, A_i^\pm] = \pm \varepsilon_i A_i^\pm. \quad (2.15)$$

Hence  $A_i^\xi$  can be interpreted as an operator creating ( $\xi = +$ ) or annihilating ( $\xi = -$ ) a particle (quasiparticle, excitation) with energy  $\varepsilon_i$  for any  $i = 1, \dots, n$ . The representation  $\pi$  is an analog of the Fermi representation of para-Fermi statistics.

The commutation relations (2.15) do not hold for other representations of  $A_n$ . In order to extend the class of admissible representations we rewrite the Hamiltonian (2.14), like in the Fermi case, in the following identical form

$$\hat{H} = \sum_{i=1}^n \varepsilon_i ([A_i^+, A_i^-] + E_{00}). \quad (2.16)$$

The Lie algebra generated from the operators  $A_1^\pm, \dots, A_n^\pm$  and  $E_{00}$  is  $gl(n+1) = A_n \dot{\oplus} I$  (in the representation  $\pi$ ). Since  $\hat{H} \in gl(n+1)$  (in this representation), (2.15) also holds for any other representation of  $gl(n+1)$ . In other words the Hamiltonian

$$H = \sum_{i=1}^n \varepsilon_i ([a_i^+, a_i^-] + e_{00}) = \sum_{i=1}^n \varepsilon_i ([a_i^+, a_i^-] + h_0) \quad (2.17)$$

satisfies (2.6) for any other representation of  $gl(n+1)$ .

One may argue that expression (2.17) is not satisfactory, because the Hamiltonian  $H$  is not a function of the Jacobson generators only. Below, in Corollary 1, we show that within every irreducible representation  $H$  can be written as a function of the JGs. Here we note that  $[a_i^+, a_i^-] + e_{00} = h_i$  and therefore the Hamiltonian (2.17) can be represented manifestly as an element from the Cartan subalgebra of  $gl(n+1)$  :

$$H = \sum_{i=1}^n \varepsilon_i h_i. \quad (2.18)$$

### 3 Fock representations of $sl(n+1)$

We proceed to outline those representations of the Jacobson generators, which possess the main features of Fock space representations in ordinary quantum theory. In order to distinguish between the abstract generators and their representations, the JGs  $a_1^\pm, \dots, a_n^\pm$ , considered as operators in a certain  $A_n$ -module  $W$ , are called (Jacobson) creation and annihilation operators of  $A_n$  (abbreviated also as Jacobson CAOs of  $A_n$ ,  $A_n$ -CAOs,  $A$ -CAOs or simply CAOs).

**Definition 1** Let  $a_1^\xi, \dots, a_n^\xi$  be Jacobson creation ( $\xi = +$ ) and annihilation ( $\xi = -$ ) operators. The  $A_n$ -module  $W$  is said to be a Fock space of the algebra  $A_n$  if it is a Hilbert space, so that the following conditions hold :

1. Hermiticity condition ( $A^*$  denotes the operator conjugate to  $A$ )

$$(a_i^+)^* = a_i^-, \quad i = 1, \dots, n. \quad (3.1)$$

2. Existence of vacuum. There exists a vacuum vector  $|0\rangle \in W$  such that

$$a_i^- |0\rangle = 0, \quad i = 1, \dots, n. \quad (3.2)$$

3. Irreducibility. The representation space  $W$  is spanned on vectors

$$a_{i_1}^+ a_{i_2}^+ \cdots a_{i_m}^+ |0\rangle, \quad m \in \mathbb{Z}_+. \quad (3.3)$$

The Fock space of  $A_n$  is also said to be an  $A_n$ -module of Fock, a Fock module of the  $A$ -operators or simply a Fock space.

Assume that  $W$  is a Fock space. Condition (3.1) asserts that any Fock representation is unitarizable with respect to this star operation, considered as an antilinear antiinvolution on  $A_n$ . It is known that all such representations are realized in direct sums of finite-dimensional irreducible  $A_n$ -modules. Then (3.3) yields that any Fock module is a finite-dimensional irreducible  $A_n$ -module.

We list a few propositions, proofs of which can be found in [59, 60].

**Proposition 1** *The  $A_n$ -module  $W$  is a Fock space if and only if it is an irreducible finite-dimensional module with a highest weight  $\Lambda$  such that*

$$a_i^- a_j^+ x_\Lambda = 0 \quad i \neq j = 1, \dots, n. \quad (3.4)$$

*The vacuum  $|0\rangle$  is unique (up to a multiplicative constant) and can be identified with the highest weight vector  $x_\Lambda$  in  $W$   $|0\rangle = x_\Lambda$ .*

Recall that the Hamiltonian  $H$ , see (2.18), does not belong to  $A_n$ . It is an element from  $gl(n+1)$ . In order to define  $H$  as an operator in  $W$ , we extend each Fock module to an irreducible  $gl(n+1)$ -module. To this end we define the action of the  $gl(n+1)$  central element (also  $gl(n+1)$  Casimir operator)  $h_0 + h_1 + \dots + h_n$  in  $W$ , setting

$$(h_0 + h_1 + \dots + h_n)x = px \quad \forall x \in W, \quad (3.5)$$

where  $p$  can be any number.

The next proposition classifies the Fock spaces. Unless otherwise stated, the roots and the weights are represented by their coordinates in the basis  $h^0, h^1, \dots, h^n$ , i.e.,  $\lambda = \sum_{i=0}^n l_i h^i \equiv (l_0, l_1, \dots, l_n)$ .

**Proposition 2** *The irreducible  $gl(n+1)$ -module  $W_p$  is a Fock space, so that the energy of the vacuum is zero ( $H|0\rangle = 0$ ), if and only if its highest weight (namely the weight of  $|0\rangle$ ) is  $\Lambda = ph^0 \equiv (p, 0, \dots, 0)$ , i.e., if*

$$h_0|0\rangle = p|0\rangle, \quad h_i|0\rangle = 0, \quad i = 1, \dots, n, \quad (3.6)$$

where  $p$  is an arbitrary positive integer.

From (2.3) and (3.5)  $h_0 + h_1 + \dots + h_n = p$ ,  $h_0 - h_i = [a_i^-, a_i^+]$ ,  $i = 1, \dots, n$ , which yields

$$h_0 = \frac{1}{n+1} \left( p + \sum_{i=1}^n [a_i^-, a_i^+] \right), \quad h_i = \frac{1}{n+1} \left( p + n[a_i^+, a_i^-] - \sum_{k \neq i=1}^n [a_k^+, a_k^-] \right) \quad (3.7)$$

The last result shows that within any Fock module the Weyl generators  $e_{ij}$  can be expressed as functions of  $a_1^\pm, \dots, a_n^\pm$ . In view of this we say that  $a_1^\pm, \dots, a_n^\pm$  are Jacobson CAOs of both  $sl(n+1)$  and of  $gl(n+1)$ .

An immediate consequence of (2.17) and (3.7) is the following

**Corollary 1** *Within every Fock module  $W_p$  the Hamiltonian (2.17) can be expressed entirely via the Jacobson creation and annihilation operators :*

$$H = \frac{1}{n+1} \sum_{i=1}^n \varepsilon_i \left( p + n[a_i^+, a_i^-] - \sum_{k \neq i=1}^n [a_k^+, a_k^-] \right). \quad (3.8)$$

From (3.4), (3.6) and (3.7) one concludes :

**Corollary 2** *The Fock module  $W_p$  with a highest weight  $\Lambda = (p, 0, \dots, 0)$  is completely defined by the relations*

$$a_i^- a_j^+ |0\rangle = \delta_{ij} p |0\rangle, \quad a_k^- |0\rangle = 0, \quad p \in \mathbb{N}, \quad i, j, k = 1, \dots, n. \quad (3.9)$$

The above two conditions are the same as in the case of Green's parastatistics of order  $p$  [21]. Therefore  $p$  is referred to as an order of  $A_n$ -statistics (or  $A$ -statistics). The conclusion is that like in parastatistics the Fock spaces are labelled by a positive integer  $p \in \mathbb{N}$ . The representations corresponding to different orders of statistics have different highest weights and are therefore inequivalent.

Taking into account the second relation  $a_k^- |0\rangle = 0$  in (3.9), one can also define the Fock module  $W_p$  by means of the relations

$$[a_i^- a_j^+] |0\rangle = \delta_{ij} p |0\rangle, \quad a_k^- |0\rangle = 0, \quad p \in \mathbb{N}, \quad i, j, k = 1, \dots, n. \quad (3.10)$$

In view of this  $A$ -statistics and its Fock representations can be formulated in a somewhat more mathematical terminology. The latter is based on the observation that the linear span of all generators  $[a_i^- a_j^+]$ ,  $a_i^-$ ,  $i, j = 1, \dots, n$ , is a subalgebra  $\mathcal{A}$  of  $gl(n+1)$  (which contains as subalgebra also  $gl(n) = \text{span}\{[a_i^- a_j^+] | i, j = 1, \dots, n\}$ ). Equations (3.10) define one-dimensional representations of  $\mathcal{A}$ , spanned on the vacuum  $|0\rangle$ . Therefore the Fock modules  $W_p$  can be defined as those irreducible finite-dimensional  $gl(n+1)$ -modules, which are induced from trivial one-dimensional modules of  $\mathcal{A}$  via eqs. (3.10). Then  $p$  labels the different, inequivalent one-dimensional modules of  $\mathcal{A}$ .

On the other hand one can define  $A$ -statistics by means of the triple relations (2.5). Then eqs. (3.9) define completely the Fock modules  $W_p$ . All calculations can be carried out without even mentioning the underlying Lie algebraic structure of  $A$ -statistics (which is usually the case for parastatistics).

Let  $W_p$  be a Fock space with order of statistics  $p$ . From (3.3) and the fact that the creation operators commute with each other one concludes that  $W_p$  is a linear span of vectors  $(a_1^+)^{l_1}(a_2^+)^{l_2}\cdots(a_n^+)^{l_n}|0\rangle$ ,  $l_1, \dots, l_n \in \mathbb{Z}_+$ . The correspondence weight  $\leftrightarrow$  weight vector is one to one :

$$(a_1^+)^{l_1}(a_2^+)^{l_2}\cdots(a_n^+)^{l_n}|0\rangle \leftrightarrow (p - \sum_{k=1}^n l_k, l_1, l_2, \dots, l_n), \quad (3.11)$$

i.e. all weight subspaces are one-dimensional.

**Proposition 3** *Let  $W_p$  be an  $A_n$ -module of Fock with order of statistics  $p$ . The vector*

$$(a_1^+)^{l_1}(a_2^+)^{l_2}\cdots(a_n^+)^{l_n}|0\rangle \quad (3.12)$$

*is not zero if and only if*

$$l_1 + l_2 + \cdots + l_n \leq p. \quad (3.13)$$

The proof is a consequence of the properties of the roots in any finite-dimensional irreducible  $A_n$ -module  $W$ . If  $\Lambda = (L_0, L_1, \dots, L_n)$  is the highest weight in  $W$ , then for any other weight  $\lambda = (l_0, l_1, \dots, l_n)$  the following inequality holds :

$$l_{i_0} + l_{i_1} + \cdots + l_{i_m} \leq L_0 + L_1 + \cdots + L_m, \quad (3.14)$$

where  $i_0 \neq i_1 \neq \dots \neq i_m = 0, 1, \dots, n$  and  $m = 0, 1, \dots, n$ . Equation (3.14) is an equality for  $m = n$ . If  $W_p$  is a Fock space,  $L_0 + L_1 + \dots + L_m = p$ .

Proposition 3 can be proved also by a direct, but rather long computation. One verifies that the infinite-dimensional module  $\hat{W}_p$  spanned on all vectors (3.12) with  $l_1, \dots, l_n$  being arbitrary non-negative integers contains an invariant subspace  $V_p$  spanned on all vectors (3.12) with  $l_1 + l_2 + \dots + l_n > p$ . Then  $W_p$  is the factor module  $\hat{W}_p/V_p$  and all vectors (3.12), subject to (3.13) are (representatives of) the basis vectors in  $W_p = \hat{W}_p/V_p$ .

We proceed to recall how one defines a metric in  $W_p$ , so that it is a Hilbert space and the hermiticity condition (3.1) holds. Consider the vectors

$$(a_1^+)^{l_1}(a_2^+)^{l_2}\cdots(a_n^+)^{l_n}|0\rangle, \quad l_1 + l_2 + \cdots + l_n \leq p \quad (3.15)$$

from  $W_p$ . All such vectors have different weights. Consequently they are linearly independent and can be considered as a basis in  $W_p$ . Define a Hermitian form  $(\ , \ )$  on  $W_p$  in the usual way

(for quantum theory), postulating (in addition to  $a_i^-|0\rangle = 0$ , see (3.2)) :

$$\begin{aligned}
(a) \quad & (|0\rangle, |0\rangle) \equiv \langle 0|0\rangle = 1, \\
(b) \quad & \langle 0|a_i^+ = 0, \quad i = 1, \dots, n, \\
(c) \quad & \left( (a_1^+)^{m_1} (a_2^+)^{m_2} \dots (a_n^+)^{m_n} |0\rangle, (a_1^+)^{l_1} (a_2^+)^{l_2} \dots (a_n^+)^{l_n} |0\rangle \right) = \\
& \langle 0|(a_n^-)^{m_n} \dots (a_2^-)^{m_2} (a_1^-)^{m_1} (a_1^+)^{l_1} (a_2^+)^{l_2} \dots (a_n^+)^{l_n} |0\rangle.
\end{aligned} \tag{3.16}$$

With respect to this form the vectors (3.15) are orthogonal. Moreover,

$$\left( (a_1^+)^{l_1} (a_2^+)^{l_2} \dots (a_n^+)^{l_n} |0\rangle, (a_1^+)^{l_1} (a_2^+)^{l_2} \dots (a_n^+)^{l_n} |0\rangle \right) = \frac{p!}{(p - \sum_{j=1}^n l_j)!} \prod_{i=1}^n l_i! > 0. \tag{3.17}$$

Therefore all vectors

$$|p; l_1, \dots, l_n\rangle = \sqrt{\frac{(p - \sum_{j=1}^n l_j)!}{p!}} \frac{(a_1^+)^{l_1} \dots (a_n^+)^{l_n}}{\sqrt{l_1! l_2! \dots l_n!}} |0\rangle, \quad l_1 + l_2 + \dots + l_n \leq p \tag{3.18}$$

constitute an orthonormal basis in  $W_p$ , i.e.  $(\ , \ )$  is a scalar product. Then by construction the hermiticity condition (3.1) holds too.

The transformation of the basis (3.18) under the action of the Jacobson CAOs reads :

$$a_i^+ |p; l_1, \dots, l_i, \dots, l_n\rangle = \sqrt{(l_i + 1)(p - \sum_{j=1}^n l_j)} |p; l_1, \dots, l_{i-1}, l_i + 1, l_{i+1}, \dots, l_n\rangle, \tag{3.19}$$

$$a_i^- |p; l_1, \dots, l_i, \dots, l_n\rangle = \sqrt{l_i(p - \sum_{j=1}^n l_j + 1)} |p; l_1, \dots, l_{i-1}, l_i - 1, l_{i+1}, \dots, l_n\rangle. \tag{3.20}$$

Moreover,

$$h_0 |p; l_1, l_2, \dots, l_n\rangle = (p - \sum_{i=1}^n l_i) |p; l_1, l_2, \dots, l_n\rangle, \tag{3.21}$$

$$h_i |p; l_1, l_2, \dots, l_n\rangle = l_i |p; l_1, l_2, \dots, l_n\rangle, \quad i = 1, \dots, n. \tag{3.22}$$

Let us consider in some more detail the  $p = 1$  representation. Denote by  $b_i^\pm$  the Jacobson CAOs  $a_i^\pm$  in this representation. In this particular case the representation space  $W_1$  is  $(n + 1)$ -dimensional with a basis

$$|1; l_1, \dots, l_n\rangle, \quad l_1 + \dots + l_n \leq 1, \tag{3.23}$$

i.e. at most one of the labels  $l_1, \dots, l_n$  in  $|1; l_1, \dots, l_n\rangle$  is equal to 1 and all other are zeros.

Then (3.19)-(3.20) reduces to

$$\begin{aligned}
b_i^+ |1; l_1, \dots, l_{i-1}, l_i, l_{i+1}, \dots, l_n\rangle &= (1 - l_i) |1; l_1, \dots, l_{i-1}, l_i + 1, l_{i+1}, \dots, l_n\rangle, \\
b_i^- |1; l_1, \dots, l_{i-1}, l_i, l_{i+1}, \dots, l_n\rangle &= l_i |1; l_1, \dots, l_{i-1}, l_i - 1, l_{i+1}, \dots, l_n\rangle.
\end{aligned} \tag{3.24}$$

The matrix elements of  $b_i^+$  and  $b_i^-$ , in the basis ordered as  $|1; 0, 0, 0, \dots, 0\rangle, |1; 1, 0, 0, \dots, 0\rangle, |1; 0, 1, 0, \dots, 0\rangle, |1; 0, 0, 1, \dots, 0\rangle, \dots, |1; 0, 0, 0, \dots, 1\rangle$  are the same as those of the Weyl generators  $E_{i0}$  and  $E_{0i}$  in the defining  $(n + 1)$ -dimensional matrix representation, see (2.13). Hence

the  $p = 1$  representation is the same as the defining representation and one can think of the operators  $b_i^\pm$  as of matrices,

$$E_{i0} = b_i^+, \quad E_{0i} = b_i^-, \quad i = 1, \dots, n. \quad (3.25)$$

From here and (3.7) (with  $p = 1$ ) one can express also the rest of the Weyl generators (2.13) via  $p = 1$  Jacobson creation and annihilation operators :

$$E_{00} = \frac{1}{n+1} \left(1 - \sum_{i=1}^n [b_i^+, b_i^-]\right), \quad E_{ij} = [b_i^+, b_j^-] + \frac{\delta_{ij}}{n+1} \left(1 - \sum_{k=1}^n [b_k^+, b_k^-]\right), \quad i, j = 1, \dots, n. \quad (3.26)$$

## 4 The Pauli principle for $A$ -statistics

The results obtained so far justify the terminology used. Equations (2.18) and (3.6) yield

$$H|p; l_1, \dots, l_i, \dots, l_n\rangle = \sum_{i=1}^n l_i \varepsilon_i |p; l_1, \dots, l_i, \dots, l_n\rangle. \quad (4.1)$$

Therefore the state  $|p; l_1, \dots, l_i, \dots, l_n\rangle$  can be interpreted as a many-particle state with  $l_1$  particles on the first orbital,  $l_2$  particles on the second orbital, etc. For reasons that will become clear soon, we refer to these particles as quasibosons (of order  $p$ ). The operator  $h_i$ ,  $i = 1, \dots, n$ , see (3.22), is the number operator for the quasibosons on the  $i^{\text{th}}$  orbital, whereas  $\hat{N} = h_1 + \dots + h_n$  counts all quasibosons, accommodated in the state  $|p; l_1, \dots, l_i, \dots, l_n\rangle$ .

Since, see (3.19),

$$a_i^+ |p; l_1, \dots, l_i, \dots, l_n\rangle \sim |p; l_1, \dots, l_{i-1}, l_i + 1, l_{i+1}, \dots, l_n\rangle, \quad \text{if } \sum_{i=1}^n l_i < p, \quad (4.2)$$

the operator  $a_i^+$  creates a quasiboson on the  $i^{\text{th}}$  orbital, a particle with energy  $\varepsilon_i$ , if the state contains less than  $p$  quasibosons. On the other hand,  $a_i^+ |p; l_1, \dots, l_{i-1}, l_i, l_{i+1}, \dots, l_n\rangle = 0$ , if  $\sum_{i=1}^n l_i = p$ , i.e. no more than  $p$  quasibosons can be accommodated. Similarly, if  $l_i > 0$ ,  $a_i^-$  "kills" a quasiboson with energy  $\varepsilon_i$ . Therefore, reformulating Proposition 3, one obtains :

**Corollary 3 (Pauli principle for  $A$ -statistics)** *Let  $W_p$  be a Fock space of  $A_n$ , corresponding to an order of statistics  $p$ . Within  $W_p$  all states containing no more than  $p$  quasibosons, namely all states*

$$|p; l_1, \dots, l_i, \dots, l_n\rangle \quad \text{with } 0 \leq \sum_{i=1}^n l_i \leq p, \quad (4.3)$$

*are allowed. There are no states accommodating more than  $p$  particles.*

Let us consider, as an example,  $A$ -statistics of order  $p = 4$  with  $n = 6$  orbitals (for instance with 6 different energy levels). From (4.3), it follows that there is no restriction on the number of quasibosons to be accommodated on a certain orbital as long as the total number of particles in any configuration does not exceed  $p$ . Hence, the following three states or configurations are



## 5 Quasi-Bose creation and annihilation operators

In the present section we show first approximately and then in a strict sense that  $A$ -statistics can be viewed as a good finite-dimensional approximation to Bose statistics for large values of order of statistics  $p$ . The terminology *finite-dimensional approximation* comes to remind that the Fock spaces  $W_p$  of  $A$ -statistics are finite-dimensional linear spaces, whereas any Bose Fock space is infinite-dimensional.

Introduce new, representation dependent, creation and annihilation operators

$$B(p)_i^\pm = \frac{a_i^\pm}{\sqrt{p}}, \quad i = 1, \dots, n, \quad p \in \mathbb{N}, \quad (5.1)$$

in  $W_p$ . The transformations following from (3.19)-(3.20) read :

$$B(p)_i^+ |p; l_1, \dots, l_i, \dots, l_n\rangle = \sqrt{(l_i + 1) \left(1 - \frac{\sum_{k=1}^n l_k}{p}\right)} |p; l_1, \dots, l_i + 1, \dots, l_n\rangle, \quad (5.2)$$

$$B(p)_i^- |p; l_1, \dots, l_i, \dots, l_n\rangle = \sqrt{l_i \left(1 + \frac{1 - \sum_{k=1}^n l_k}{p}\right)} |p; l_1, \dots, l_i - 1, \dots, l_n\rangle. \quad (5.3)$$

Consider the above equations for values of the order of statistics  $p$ , which are much greater than the number of accommodated quasibosons, namely  $l_1 + l_2 + \dots + l_n \ll p$ . In this approximation one obtains :

$$\begin{aligned} B(p)_i^- |p; l_1, \dots, l_{i-1}, l_i, l_{i+1}, \dots, l_n\rangle &\simeq \sqrt{l_i} |p; l_1, \dots, l_{i-1}, l_i - 1, l_{i+1}, \dots, l_n\rangle, \\ B(p)_i^+ |p; l_1, \dots, l_{i-1}, l_i, l_{i+1}, \dots, l_n\rangle &\simeq \sqrt{l_i + 1} |p; l_1, \dots, l_{i-1}, l_i + 1, l_{i+1}, \dots, l_n\rangle, \end{aligned} \quad (5.4)$$

which yields (an approximation to) the Bose commutation relations :

$$[B(p)_i^+, B(p)_j^+] = [B(p)_i^-, B(p)_j^-] = 0, \quad (\text{exact commutators}), \quad (5.5)$$

$$[B(p)_i^-, B(p)_j^+] \simeq \delta_{ij}, \quad \text{if } l_1 + l_2 + \dots + l_n \ll p. \quad (5.6)$$

Since for  $l_1 + l_2 + \dots + l_n \equiv \sum_k l_k \ll p$

$$\frac{(p - \sum_k l_k)!}{p!} p^{\sum_k l_k} = \frac{p}{p - \sum_k l_k + 1} \frac{p}{p - \sum_k l_k + 2} \dots \frac{p}{p} \simeq 1,$$

in a first approximation (3.18) reduces also to the well known expressions for the orthonormed basis in a Fock space of  $n$  pairs of Bose creation and annihilation operators :

$$|p; l_1, \dots, l_n\rangle = \frac{(B(p)_1^+)^{l_1} \dots (B(p)_n^+)^{l_n}}{\sqrt{l_1! l_2! \dots l_n!}} |0\rangle. \quad (5.7)$$

The conclusion is that the representations of  $B(p)_i^\pm$  in (finite-dimensional) state spaces  $W_p$  with large values of  $p$ , restricted to states with a small amount  $l_1 + l_2 + \dots + l_n \ll p$  of accommodated quasibosons, provide a good approximation to Bose creation and annihilation operators [59, 60]. This is another reason to refer to the operators  $B(p)_i^\pm$  as *quasi-Bose creation and annihilation operators (of order  $p$ )*.

In the remaining part of this section we will prove that in the limit  $p \rightarrow \infty$  the quasi-Bose operators reduce to Bose creation and annihilation operators. To this end we proceed to introduce first an appropriate topology.

Let  $W$  be a Hilbert space with an orthonormed basis

$$|l_1, \dots, l_i, \dots, l_n\rangle \equiv |L\rangle, \quad \forall l_1, \dots, l_n \in \mathbb{Z}_+. \quad (5.8)$$

Whenever possible we write  $|L\rangle$  as an abbreviation for  $|l_1, \dots, l_i, \dots, l_n\rangle$  and denote by  $|L\rangle_{\pm i}$  a vector obtained from  $|L\rangle$  by replacing  $l_i$  with  $l_i \pm 1$ , namely

$$|L\rangle_{\pm i} = |l_1, \dots, l_{i-1}, l_i \pm 1, l_{i+1}, \dots, l_n\rangle. \quad (5.9)$$

The space  $W$  consists of all vectors

$$\Phi = \sum_{l_1=0}^{\infty} \cdots \sum_{l_n=0}^{\infty} c(l_1, \dots, l_n) |l_1, \dots, l_n\rangle \equiv \sum_L c(L) |L\rangle, \quad (5.10)$$

where  $c(l_1, \dots, l_n) \equiv c(L)$  are complex numbers such that

$$\sum_{l_1=0}^{\infty} \cdots \sum_{l_n=0}^{\infty} |c(l_1, \dots, l_n)|^2 \equiv \sum_{l_1, \dots, l_n=0}^{\infty} |c(l_1, \dots, l_n)|^2 \equiv \sum_L |c(L)|^2 < \infty, \quad (5.11)$$

and this is in fact the square of the Hilbert space norm  $(|\Phi|_0)^2$  of  $\Phi$ .

Embed the  $sl(n+1)$ -module  $W_p$  in  $W$  via an identification of the basis vectors

$$|p; l_1, \dots, l_i, \dots, l_n\rangle \equiv |l_1, \dots, l_i, \dots, l_n\rangle \equiv |L\rangle \quad \forall l_1 + \dots + l_n \leq p. \quad (5.12)$$

In order to turn the entire space  $W$  into an  $sl(n+1)$ -module, so that the restriction on  $W_p \subset W$  coincides with (5.2)-(5.3), we set :

$$B(p)_i^+ \Phi = \sum_{l_1 + \dots + l_n \leq p} c(L) \sqrt{(l_i + 1) \left(1 - \frac{\sum_{k=1}^n l_k}{p}\right)} |L\rangle_i, \quad (5.13)$$

$$B(p)_i^- \Phi = \sum_{l_1 + \dots + l_n \leq p} c(L) \sqrt{l_i \left(1 + \frac{1 - \sum_{k=1}^n l_k}{p}\right)} |L\rangle_{-i}, \quad (5.14)$$

where  $\Phi$  is any vector (5.10) from  $W$  and  $\sum_{l_1 + \dots + l_n \leq p}$  is a sum over all possible  $l_1, \dots, l_n \in \mathbb{Z}_+$  such that  $l_1 + \dots + l_n \leq p$ . Note that the sums in (5.13)-(5.14) are finite.

The transformation of the basis, following from (5.13)-(5.14), reads :

$$B(p)_i^+ |L\rangle = \sqrt{(l_i + 1) \left(1 - \frac{\sum_{k=1}^n l_k}{p}\right)} |L\rangle_i, \quad \forall L \text{ such that } \sum_{k=1}^n l_k \leq p, \quad (5.15)$$

$$B(p)_i^- |L\rangle = \sqrt{l_i \left(1 + \frac{1 - \sum_{k=1}^n l_k}{p}\right)} |L\rangle_{-i}, \quad \forall L \text{ such that } \sum_{k=1}^n l_k \leq p, \quad (5.16)$$

$$B(p)_i^{\pm} |L\rangle = 0, \quad \forall L \text{ such that } \sum_{k=1}^n l_k > p. \quad (5.17)$$

The relations (5.15)-(5.16) are the same as (5.2)-(5.3) (via the identification (5.12)).

Since the quasi-Bose operators  $B(p)_i^\pm$  take values in a finite-dimensional subspace of  $W$ , see (5.13)-(5.14), they are bounded and hence continuous linear operators in  $W$ . In view of this, see (5.10),  $B(p)_i^\pm \Phi = B(p)_i^\pm \sum_L c(L)|L\rangle = \sum_L c(L)B(p)_i^\pm |L\rangle$  and therefore (5.13)-(5.14) are a consequence of (5.15)-(5.17).

Next we proceed to define  $n$  pairs of Bose operators  $B_i^\pm$ ,  $i = 1, \dots, n$ , in  $W$ . It is known that such operators cannot be realized as bounded operators in  $W$  (so that the corresponding position and momentum operators are selfadjoint operators in  $W$ ; see, for instance, [69] or [12]). Therefore care has to be taken about the common domain of definition  $\Omega$  of the Bose operators. Following [2] we set  $\Omega$  to be a dense subspace of  $W$  (with respect to the Hilbert space topology), consisting of all vectors (5.10) for which the series

$$(|\Phi\rangle_r)^2 = \sum_{l_1, \dots, l_n=0}^{\infty} (1 + \sum_{k=1}^n l_k)^r |c(l_1, \dots, l_n)|^2 \quad (5.18)$$

is convergent for any  $r = 0, 1, 2, \dots$ . Then the relations

$$B_i^- |L\rangle = \sqrt{l_i} |L\rangle_{-i}, \quad B_i^+ |L\rangle = \sqrt{l_i + 1} |L\rangle_i, \quad (5.19)$$

define a representation of  $n$  pairs of bosons  $B_1^\pm, \dots, B_n^\pm$ , namely of operators, which satisfy the relations

$$[B_i^-, B_j^+] = \delta_{ij}, \quad [B_i^+, B_j^+] = [B_i^-, B_j^-] = 0, \quad i, j = 1, \dots, n, \quad (5.20)$$

in  $\Omega$  (with  $\Omega$  being a common domain of definition for all them). In terms of these operators

$$(|\Phi\rangle_r)^2 = (\Phi, (1 + \sum_{k=1}^n B_k^+ B_k^-)^r \Phi). \quad (5.21)$$

The norms  $|\Phi\rangle_r$ ,  $r = 0, 1, 2, \dots$ , turn  $\Omega$  into a countably normed topological space (which can be viewed also as a metric space [18]). All balls

$$B(\Phi_0; r, \epsilon) = \{\Phi \in \Omega \mid |\Phi - \Phi_0\rangle_r < \epsilon\}, \quad \forall \Phi_0 \in \Omega, \quad \forall r \in \mathbb{Z}_+, \quad \forall \epsilon > 0, \quad (5.22)$$

constitute a basis of open sets in the countably normed topological space  $\Omega$ , whereas the balls (5.22) with a fixed  $r$  yield a basis in  $\Omega$ , viewed as a  $|\cdot\rangle_r$ -normed topological space. Clearly any  $|\cdot\rangle_r$ -normed topology (*r-normed topology*) is weaker than the countably normed topology (*cn-topology*).

From now on we restrict the domain of definition of all quasi-Bose operators (5.1) to be  $\Omega$ . The fact that each quasi-Bose operator maps  $\Omega$  into a finite-dimensional subspace of  $\Omega$ , see (5.13)-(5.14), indicates that each such operator is a bounded and hence a continuous linear operator with respect to the  $r$ -normed topology for any  $r \in \mathbb{Z}_+$ . A similar property however does not hold for the Bose creation and annihilation operators (5.19). These operators are not continuous with respect to any of the  $r$ -normed topologies in  $\Omega$ . Therefore, if  $\sum_{i=1}^{\infty} \Phi_i = \Phi$

converges in the sense of a certain  $r$ -normed topology, for instance in the Hilbert space topology ( $r = 0$ ), one cannot in general use relations like

$$B_i^\pm \sum_{i=1}^{\infty} \Phi_i = \sum_{i=1}^{\infty} B_i^\pm \Phi_i. \quad (5.23)$$

One of the advantages of the  $cn$ -topology is that it avoids the above difficulties. Here are some of the properties of this topology, which will be relevant for the rest of the exposition [2] :

- $\Omega$  is stable under the action of any polynomial of Bose operators,

$$P(B_1^\pm, \dots, B_n^\pm)\Omega \subset \Omega; \quad (5.24)$$

- Any polynomial of Bose CAOs is a continuous linear operator in  $\Omega$  with respect to the  $cn$ -topology;

$$(5.25)$$

- The scalar product in  $\Omega$  is continuous with respect to the convergence defined by the  $cn$ -topology.

$$(5.26)$$

As a consequence, (5.23) holds for any series  $\sum_{i=1}^{\infty} \Phi_i$  which converges in the  $cn$ -topology; moreover (5.26) yields  $(\sum_{i=1}^{\infty} \Phi_i, \Psi) = \sum_{i=1}^{\infty} (\Phi_i, \Psi)$ . The relevance of the  $cn$ -topology however goes far beyond the above considerations. This topology, called nuclear topology, is of prime importance in the theory of generalized functions [18, 17], and their applications in quantum theory (see, for instance [2]).

Let  $\mathcal{P}$  be the set of all linear operators in  $\Omega$  defined everywhere in  $\Omega$ , which are continuous in the  $cn$ -topology. With respect to the usual operations between operators  $\mathcal{P}$  is an associative algebra [18]. According to (5.25) the Bose operators belong to  $\mathcal{P}$ . The quasi-Bose operators (5.1) (with domain of definition restricted to  $\Omega$ ) also belong to  $\mathcal{P}$ . Indeed  $B(p)_i^\pm$  are bounded and hence continuous operators in  $\Omega$  with respect to any  $r$ -normed topology. Let  $B(\Phi_0; r, \epsilon)$  be an arbitrary open ball in the  $cn$ -topology, see (5.22).  $B(\Phi_0; r, \epsilon)$  is an open ball also in the  $r$ -normed topology. Therefore the inverse image  $O = [B(p)_i^\pm]^{-1}B(\Phi_0; r, \epsilon)$  of  $B(\Phi_0; r, \epsilon)$  is an open set in the  $r$ -normed topology. Since the latter is weaker than the  $cn$ -topology,  $O$  is an open set also in the  $cn$ -topology. Thus, the inverse image  $O = [B(p)_i^\pm]^{-1}B(\Phi_0; r, \epsilon)$  of any open ball (i.e. of any open set from the basis) in the  $cn$ -topology is an open set with respect to the same topology. Therefore  $B(p)_i^\pm$  is a continuous operator in the  $cn$ -topology.

Introduce a topology on  $\mathcal{P}$  in a way similar to the strong topology in the algebra  $\mathcal{B}(\mathcal{H})$  of all bounded linear operators on a Hilbert space  $\mathcal{H}$  [54]. Let  $\Phi_1, \dots, \Phi_s$  be  $s$  different elements from  $\Omega$  and  $\epsilon$  be a positive number. A strong neighborhood  $U(A_0; \Phi_1, \dots, \Phi_s; \epsilon)$  of the operator  $A_0 \in \mathcal{P}$  is (defined as) the set of all operators  $A \in \mathcal{P}$ , which satisfy the inequalities

$$|(A - A_0)\Phi_k|_0 < \epsilon, \quad \forall k = 1, \dots, s. \quad (5.27)$$

**Definition 2** A strong topology on  $\mathcal{P}$  is the topology with a basis of open sets consisting of all possible strong neighborhoods  $U(A_0; \Phi_1, \dots, \Phi_s; \epsilon)$  (namely the collection of strong neighborhoods, corresponding to any  $A_0 \in \mathcal{P}$ , to any  $\epsilon > 0$ , to any  $s \in \mathbb{N}$  and to any sequence  $\Phi_1, \dots, \Phi_s$  of different elements from  $\Omega$ ).

**Proposition 4** In the strong topology

$$\lim_{p \rightarrow \infty} B(p)_i^\pm = B_i^\pm, \quad i = 1, \dots, n. \quad (5.28)$$

*Proof.* In order to prove that (5.28) holds it is sufficient to show that every strong neighborhood  $U(B_i^\pm; \Phi_1, \dots, \Phi_s; \epsilon)$  of  $B_i^\pm$  contains all elements of the sequence  $B(1)_i^\pm, B(2)_i^\pm, \dots$  apart from a finite number of them. Since  $U(B_i^\pm; \Phi_1, \dots, \Phi_s; \epsilon) = \bigcap_{k=1}^s U(B_i^\pm; \Phi_k; \epsilon)$ , it is sufficient to show that for any neighborhood  $U(B_i^\pm; \Phi; \epsilon)$  there exists an integer  $N$  such that  $B(p)_i^\pm \in U(B_i^\pm; \Phi; \epsilon)$  for any  $p > N$  or, which is the same, see (5.27), that

$$|(B(p)_i^\pm - B_i^\pm)\Phi|_0 < \epsilon, \quad \forall p > N. \quad (5.29)$$

The above equation has to hold for any  $\Phi$  and any  $\epsilon$ . In general  $N$  depends on  $\Phi$  and  $\epsilon$ ,  $N = N(\Phi, \epsilon)$ .

The fact that  $B_i^+ - B(p)_i^+$  is a continuous linear operator in  $\Omega$  is essential since relations like (5.23) can be used. The latter together with (5.13)-(5.14) and (5.19) yields :

$$\begin{aligned} (B_i^+ - B(p)_i^+)\Phi &= \sum_{l_1 + \dots + l_n < p} c(L)(\sqrt{l_i + 1} \left(1 - \sqrt{1 - \frac{\sum_k l_k}{p}}\right) |L\rangle_i \\ &+ \sum_{l_1 + \dots + l_n \geq p} c(L)(\sqrt{l_i + 1} |L\rangle_i. \end{aligned} \quad (5.30)$$

The continuity of the scalar product with respect to the  $cn$ -topology and the fact that all terms in the RHS of (5.30) are orthogonal to each other yield :

$$\begin{aligned} (|(B_i^+ - B(p)_i^+)\Phi|_0)^2 &= \sum_{l_1 + \dots + l_n < p} |c(L)|^2 (l_i + 1) \left(1 - \sqrt{1 - \frac{\sum_k l_k}{p}}\right)^2 \\ &+ \sum_{l_1 + \dots + l_n \geq p} |c(L)|^2 (l_i + 1). \end{aligned}$$

Let  $\epsilon > 0$ . Select  $p_0 \in \mathbb{N}$  to be fixed. For any  $p > p_0$

$$\begin{aligned} (|(B_i^+ - B(p)_i^+)\Phi|_0)^2 &= \sum_{l_1 + \dots + l_n \leq p_0} |c(L)|^2 (l_i + 1) \left(1 - \sqrt{1 - \frac{\sum_k l_k}{p}}\right)^2 \\ &+ \sum_{p_0 < l_1 + \dots + l_n < p} |c(L)|^2 (l_i + 1) \left(1 - \sqrt{1 - \frac{\sum_k l_k}{p}}\right)^2 + \sum_{l_1 + \dots + l_n \geq p} |(1 + l_i)c(L)|^2 \\ &< \sum_{l_1 + \dots + l_n \leq p_0} |c(L)|^2 (l_i + 1) \left(1 - \sqrt{1 - \frac{\sum_k l_k}{p}}\right)^2 + \sum_{l_1 + \dots + l_n > p_0} |(1 + l_i)c(L)|^2. \end{aligned} \quad (5.31)$$

Since the partial sums of  $\sum_{l_1, \dots, l_n=0}^{\infty} (1+l_i) |c(L)|^2$  constitute an increasing sequence of positive numbers, which is restricted from above,  $\sum_{l_1, \dots, l_n=0}^{\infty} (1+l_i) |c(L)|^2 \leq |\Phi|_1$ , the series  $\sum_{l_1, \dots, l_n=0}^{\infty} (1+l_i) |c(L)|^2$  converges. Choose  $p_0$  such that  $\sum_{l_1+\dots+l_n > p_0} (1+l_i) |c(L)|^2 < \frac{\epsilon^2}{2}$ . Then for any  $p > p_0$

$$\begin{aligned} (|(B_i^+ - B(p)_i^+) \Phi|_0)^2 &< \sum_{l_1+\dots+l_n \leq p_0} |c(L)|^2 (l_i+1) \left(1 - \sqrt{1 - \frac{\sum_k l_k}{p}}\right)^2 + \frac{\epsilon^2}{2} \\ &< \sum_{l_1+\dots+l_n \leq p_0} |c(L)|^2 (l_i+1) \left(1 - \sqrt{1 - \frac{p_0}{p}}\right)^2 + \frac{\epsilon^2}{2} < d \left(1 - \sqrt{1 - \frac{p_0}{p}}\right)^2 + \frac{\epsilon^2}{2}, \end{aligned} \quad (5.32)$$

where  $d = \sum_{l_1+\dots+l_n \leq p_0} |c(L)|^2 (l_i+1)$  is a constant. Clearly there exists  $N \in \mathbb{N}$  such that  $d \left(1 - \sqrt{1 - \frac{p_0}{p}}\right)^2 < \frac{\epsilon^2}{2}$  for any  $p > N$ . Hence for every  $\epsilon > 0$  there exists a positive integer  $N$  such that  $|(B_i^+ - B(p)_i^+) \Phi|_0 < \epsilon$ ,  $\forall p > N$ , i.e. (5.29) holds.

In a similar way one proves that  $\lim_{p \rightarrow \infty} B(p)_i^- = B_i^-$ . This completes the proof.  $\blacksquare$

## 6 Bosonization of $A$ -statistics

A simple comparison of (3.19)-(3.20) with (5.19) suggests that the Jacobson CAOs of any order  $p$  can be bosonized, namely that they can be expressed as functions of Bose CAOs  $B_1^\pm, \dots, B_n^\pm$ , see (5.20). Indeed, taking into account that  $B_i^+ B_i^- \equiv N_i$  is a number operator for bosons in a state  $i$ ,

$$N_i |L\rangle \equiv N_i |l_1, \dots, l_i, \dots, l_n\rangle = l_i |l_1, \dots, l_i, \dots, l_n\rangle, \quad i = 1, \dots, n, \quad (6.1)$$

one rewrites (3.19) as :

$$a_i^+ |L\rangle = \sqrt{(l_i+1)(p - \sum_{k=1}^n N_k + 1)} |L\rangle_i.$$

In view of (5.19) the latter can also be represented as

$$a_i^+ |L\rangle = \sqrt{p+1 - \sum_{k=1}^n N_k} B_i^+ |L\rangle = B_i^+ \sqrt{p - \sum_{k=1}^n B_k^+ B_k^-} |L\rangle. \quad (6.2)$$

Since (6.2) holds for any  $|L\rangle$ ,

$$a_i^+ = B_i^+ \sqrt{p - \sum_{k=1}^n B_k^+ B_k^-}, \quad i = 1, \dots, n. \quad (6.3)$$

In a similar way one derives from (3.20) :

$$a_i^- = \sqrt{p - \sum_{k=1}^n B_k^+ B_k^-} B_i^-, \quad i = 1, \dots, n. \quad (6.4)$$

Evidently also, see (3.21),

$$h_0 = p - \sum_{k=1}^n B_k^+ B_k^-. \quad (6.5)$$

Note that the entire Fock space  $W$  is reducible with respect to the Jacobson CAOs. Its finite-dimensional “physical” subspace  $W_p$ , see (5.12), is a simple (= irreducible)  $gl(n+1)$ -module and within this module  $(a_i^+)^* = a_i^-$  holds.

After simple calculations and taking into account that  $a_i^+ = e_{i0}$ ,  $a_i^- = e_{0i}$ ,  $i = 1, \dots, n$ , see (2.3), one can express all Weyl generators  $\{e_{ij}|i, j = 0, 1, \dots, n\}$  of  $gl(n+1)$  via  $n$  pairs of Bose operators :

$$\begin{aligned}
(a) \quad e_{ij} &= B_i^+ B_j^-, \quad i, j = 1, \dots, n, \\
(b) \quad e_{i0} &= B_i^+ \sqrt{p - \sum_{k=1}^n B_k^+ B_k^-}, \quad e_{0i} = \sqrt{p - \sum_{k=1}^n B_k^+ B_k^-} B_i^-, \quad i = 1, \dots, n, \\
(c) \quad e_{00} &= p - \sum_{k=1}^n B_k^+ B_k^-,
\end{aligned} \tag{6.6}$$

where, we recall,  $p$  is any positive integer,  $p \in \mathbb{N}$ .

The above bosonization of  $gl(n+1)$  is not unknown. Up to a choice of notation it is the same as the so-called Holstein-Primakoff (H-P) realization of  $gl(n+1)$  [56], initially introduced for  $sl(2)$  [34, 11]. Note that (6.6a) alone gives the known Jordan-Schwinger realization of  $gl(n)$  via  $n$  pairs of Bose operators.

## 7 Other applications : a two-leg $S = 1/2$ quantum Heisenberg ladder

In the present section we show that the Jacobson CAOs may also be of more general interest. We demonstrate this on the example of a two-leg  $S = 1/2$  Heisenberg spin ladder [19, 75], where the Jacobson CAOs of order  $p = 1$  appear in a natural way. The considerations below hold however for several other Heisenberg spin models (examples include lattice models with dimerization [71, 4, 5], two-layer Heisenberg models [6, 42, 74]) and more generally for any hard-core Bose model [14] with degenerated orbitals per site (as for instance in [82, 44]).

The Hamiltonian of the model reads:

$$\hat{H} = \sum_i (J \hat{S}_i^+ \hat{S}_{i+1}^+ + J \hat{S}_i^- \hat{S}_{i+1}^- + J_{\perp} \hat{S}_i^+ \hat{S}_i^-). \tag{7.1}$$

Here  $\hat{S}_i^{\pm} \equiv (\hat{S}_{1i}^{\pm}, \hat{S}_{2i}^{\pm}, \hat{S}_{3i}^{\pm})$  are two commuting spin-1/2 vector operators “sitting” on site  $i$  of the chain  $\pm$  and the Hamiltonian is a scalar with respect to the total spin operator  $\hat{S} = \sum_i (\hat{S}_i^+ + \hat{S}_i^-)$  :

$$[\hat{S}_{\alpha i}^{\pm}, \hat{S}_{\beta i}^{\pm}] = i \sum_{\gamma} \epsilon_{\alpha\beta\gamma} \hat{S}_{\gamma i}^{\pm}, \quad [\hat{S}_{\alpha i}^+, \hat{S}_{\beta j}^-] = 0, \quad [\hat{H}, \hat{S}] = 0. \tag{7.2}$$

Every local state space  $W_i$  related to site  $i$  is 4-dimensional with a basis  $|\uparrow, \uparrow\rangle, |\uparrow, \downarrow\rangle, |\downarrow, \uparrow\rangle, |\downarrow, \downarrow\rangle$  and  $W = W_1 \otimes W_2 \otimes \dots \otimes W_N$  is the global state space of the system (in the case of a ladder with  $N$  sites). The notation is standard : if  $A$  is any operator in  $W_i$ , then the corresponding to it operator in  $W$  is denoted as  $A_i$ , where  $A_i \equiv id_1 \otimes \dots \otimes id_{i-1} \otimes A \otimes id_{i+1} \otimes \dots \otimes id_N$ .

If the system is in a disordered phase ( $J_{\perp} \gg J$ ) its state is well described with the bond operator representation of spin operators [3, 71], which is a particular kind of bosonization :

$$\hat{S}_{\alpha i}^{\pm} = \frac{1}{2}(\pm B_{\alpha i}^{-} \pm B_{\alpha i}^{+} - i\epsilon_{\alpha\beta\gamma} B_{\beta i}^{+} B_{\gamma i}^{-}), \quad \alpha, \beta, \gamma = 1, 2, 3. \quad (7.3)$$

Here  $B_{1i}^{\pm}, B_{2i}^{\pm}, B_{3i}^{\pm}$  are three pairs of Bose CAOs related to site  $i$  and the vectors  $|0\rangle_i, B_{1i}^{+}|0\rangle_i, B_{2i}^{+}|0\rangle_i, B_{3i}^{+}|0\rangle_i$  constitute another basis in  $W_i$ .

The treatment of the model in terms of bosonic operators is advantageous because of the simpler commutation rules of Bose statistics. It rises however certain problems. As mentioned above, any local state space  $W_i$  is 4-dimensional, whereas the local Bose Fock space  $\Phi_i$  is infinite-dimensional. Moreover  $W_i$  is not invariant in  $\Phi_i$  with respect to the Bose CAOs (and, as a result, with respect to the local spin operators (7.3)). The physical state space  $W$  is not an invariant subspace of the global Fock space  $\Phi = \Phi_1 \otimes \Phi_2 \otimes \dots \otimes \Phi_N$  with respect to the Hamiltonian (7.1).

Various approaches have been proposed in order to overcome the problem. Following [71], additional scalar bosons  $s_i^{\pm}$  were introduced in [19]. Then the physical states are those which satisfy an additional constraint  $s_i^{+} s_i + \sum_{\alpha} B_{i\alpha}^{+} B_{i\alpha} = 1$ . Another way is to keep the realization (7.3) but to introduce “by hands” a fictitious infinite on-site repulsion between the “bosons” [42] (first proposed in [14] for a nondegenerate case). This forbids configurations with two or more bosons accommodated on one and the same site. The latter leads to the “hard-core” condition  $B_{\alpha i}^{\pm} B_{\beta i}^{\pm} = 0$ , i.e. the hard-core bosons are not quite bosons, since they satisfy fermionic-like conditions.

A third approach was worked out in [3] (see also [4, 5, 6]). It proposes the Bose operators  $B_{\alpha i}^{\pm}$  in (7.3) to be replaced throughout by new operators  $b_{\alpha i}^{\pm}$  as follows :

$$B_{\alpha i}^{+} \rightarrow b_{\alpha i}^{+} = B_{\alpha i}^{+} \sqrt{1 - \sum_{\beta=1}^3 B_{\beta i}^{+} B_{\beta i}^{-}}, \quad B_{\alpha i}^{-} \rightarrow b_{\alpha i}^{-} = \sqrt{1 - \sum_{\beta=1}^3 B_{\beta i}^{+} B_{\beta i}^{-}} B_{\alpha i}^{-}. \quad (7.4)$$

A simple comparison with (6.3), (6.4) indicates that

- The Bose operators related to site  $i$ , i.e.  $B_{1i}^{\pm}, B_{2i}^{\pm}, B_{3i}^{\pm}$ , are replaced by  $p = 1$  Jacobson CAOs (or, which is the same, by  $p = 1$  quasi-Bose operators),

$$B(1)_{\alpha i}^{\pm} \equiv b_{\alpha i}^{\pm}, \quad \alpha = 1, 2, 3, \quad (7.5)$$

in their Holstein-Primakov realization. Consequently (Proposition 3) the hard-core condition  $b_{\alpha i}^{+} b_{\beta i}^{+} = 0$  holds;

- The Jacobson CAOs from different sites commute :

$$[b_{\alpha i}^{\xi}, b_{\beta j}^{\eta}] = 0, \quad \text{if } i \neq j \text{ for any } \xi, \eta = \pm \text{ and } \alpha, \beta = 1, 2, 3. \quad (7.6)$$

It is essential that the substitution (7.4) does not change the commutation relations (7.2) between the new spin operators

$$S_{\alpha i}^{\pm} = \frac{1}{2}(\pm b_{\alpha i}^{-} \pm b_{\alpha i}^{+} - i\epsilon_{\alpha\beta\gamma} b_{\beta i}^{+} b_{\gamma i}^{-}), \quad \alpha, \beta, \gamma = 1, 2, 3, \quad (7.7)$$

and the corresponding new Hamiltonian

$$H = \sum_i (JS_i^+ S_{i+1}^+ + JS_i^- S_{i+1}^- + J_\perp S_i^+ S_i^-). \quad (7.8)$$

Moreover each local state space  $W_i$  is an invariant subspace of  $\Phi_i$  with respect to the Jacobson CAOs and hence with respect to any function of them (in particular with respect to the spin operators (7.7)). The Hamiltonian (7.8) is also a well defined operator in  $W$ .

The conclusion is that replacing throughout the model the Bose operators with  $p = 1$  Jacobson CAOs  $b_{\alpha i}^\pm$ , which commute at different sites (see (7.6)), one obtains directly the physical state space and the correct expressions for the spin operators and the Hamiltonian. There is no need to introduce either a fictitious infinite-dimensional repulsion or additional relations. All these requirements are already encoded in the properties of the Jacobson CAOs.

Let us point out that the above results can be also derived from the following proposition, which is of independent interest.

**Proposition 5** *Let  $B_\alpha^\pm$ ,  $\alpha = 1, \dots, n$ , be  $n$  pairs of Bose CAOs with a Fock space  $\mathcal{F}$  and a basis (5.8). Denote by  $\mathcal{F}_1$  the subspace of  $\mathcal{F}$  linearly spanned on the vacuum and all “single-particle” states,*

$$\mathcal{F}_1 = \text{span}\{|l_1, \dots, l_n\rangle \mid l_1 + \dots + l_n \leq 1\}. \quad (7.9)$$

Let  $\mathcal{P}$  be a projection operator of  $\mathcal{F}$  onto  $\mathcal{F}_1$  :

$$\mathcal{P}|l_1, \dots, l_n\rangle = \begin{cases} |l_1, \dots, l_n\rangle, & \text{if } l_1 + \dots + l_n \leq 1; \\ 0, & \text{if } l_1 + \dots + l_n > 1. \end{cases} \quad (7.10)$$

Then the operators  $\mathcal{P}B_\alpha^\pm\mathcal{P}$ ,  $\alpha = 1, \dots, n$ , considered as operators in  $\mathcal{F}_1$ , are  $p = 1$  Jacobson CAOs,

$$\mathcal{P}B_\alpha^\pm\mathcal{P} = B(1)_\alpha^\pm \equiv b_\alpha^\pm, \quad \alpha = 1, \dots, n. \quad (7.11)$$

*Proof.* One verifies directly that (2.5) and (3.24) hold. ■

Coming back to the two-leg spin ladder model, introduce a projection operator  $\mathcal{P}_w = \mathcal{P}_1 \otimes \mathcal{P}_2 \otimes \dots \otimes \mathcal{P}_N$  of  $\Phi$  onto  $W$ , where each  $\mathcal{P}_i$  projects  $\Phi_i$  onto  $W_i$  according to (7.10) with  $n = 3$ . The projector  $\mathcal{P}_w$  provides an alternative way for writing down the expressions for the spin operators (7.7) and the Hamiltonian (7.8). Instead of using the substitution (7.4), one can set:

$$H = \mathcal{P}_w \hat{H} \mathcal{P}_w, \quad S_{\alpha i}^\pm = \mathcal{P}_i \hat{S}_{\alpha i}^\pm \mathcal{P}_i, \quad i = 1, \dots, N. \quad (7.12)$$

The operator  $\mathcal{P}_w$  is a Bose analogue of the Gutzwiller projection operators [26], extensively used in the  $t$ - $J$  models in order to exclude the double occupation of fermions at each site (see, for instance [77] where a similar problem, a  $t$ - $J$  two-leg ladder is investigated).

## 8 Concluding remarks

From a mathematical point of view the JGs  $a_1^\pm, \dots, a_n^\pm$  provide a new description of the Lie algebra  $sl(n+1)$  in terms of generators and relations (2.5), based on the concept of Lie triple systems. For the same reason any  $n$  pairs of parafermions (resp. parabosons) can be called Jacobson generators of the orthogonal Lie algebra  $so(2n+1)$  (resp. of the orthosymplectic Lie superalgebra  $osp(1/2n)$ ). The JGs provide an alternative to the Chevalley descriptions of these Lie (super)algebras.

From a physical point of view the interest in the JGs of  $sl(n+1)$  stems from the observation that they indicate the possible existence of a new quantum statistics. Indeed, we have seen that within each Fock space  $W_p$  the operator  $a_i^+$  (resp.  $a_i^-$ ) can be interpreted as an operator creating (resp. annihilating) a particle, a quasiboson in a state  $i$  (in particular with an energy  $\varepsilon_i$ ).

In many respects the quasibosons behave as bosons. Similar as for bosons, the quasibosons can be distributed along the orbitals in an arbitrary way as far as the number of accommodated particles  $M$  does not exceed  $p$ . The number of different states of  $M \leq p$  quasibosons is the same as for bosons (the  $M$ -particle subspaces of quasibosons and bosons have one and the same dimension). There is however one essential difference : quasiboson systems of order  $p$  can accommodate at most  $p$  particles.

In order to use a proper Lie algebraic language we have restricted our considerations to finite-dimensional Lie algebras. In other words, we were studying systems with a finite number  $n$  of orbitals. Such systems certainly do exist. Examples are the local state spaces of spin systems (in particular the example considered in Section 7),  $su(n)$  lattice models etc. Nevertheless it is natural to ask whether  $A$ -statistics can be extended to incorporate infinitely many orbitals as this is usual in quantum theory. The answer to this question is positive and it is in fact evident from the results we have obtained so far. First of all the description of  $sl(n+1)$  via generators (2.3) and relations (2.5) is well defined for  $n = \infty$ , namely for  $sl(\infty)$ . Secondly, any Fock module  $W_p$  as given in Corollary 2 and in particular equations (3.9) are also well defined for  $n = \infty$ . In this case any  $W_p$  is an irreducible  $sl(\infty)$  module, generated out of the vacuum by means of the Jacobson creation operators. Therefore each state  $|p; l_1, \dots, l_i, \dots\rangle$  contains no more than a finite number of nonzero entries  $l_i$ . Moreover due to Proposition 3 the physical state space is a linear span of all vectors  $|p; l_1, \dots, l_i, \dots\rangle$  with

$$l_1 + l_2 + \dots + l_i + \dots \leq p. \quad (8.1)$$

All such states constitute an (orthonormal) basis in  $W_p$ . They transform according to the same relations (3.19)-(3.20) with  $n = \infty$ . It is straightforward to verify that any  $sl(\infty)$  module  $W_p$  is a Fock space in the sense of Definition 1. Finally, the Pauli principle (Corollary 3) remains valid also for  $n = \infty$  : despite of the infinitely many available orbitals, the infinitely many places to be occupied by the quasibosons, the system cannot accommodate more than  $p$  particles.

The indices labelling the CAOs in QFT do not constitute a countable set. For instance the Bose CAOs  $a(\mathbf{p})^\pm$  of a scalar field are labelled by the momentum of the particle, which takes values in  $\mathbb{R}^3$ . Also in this case the above considerations remain valid, but now, as in the canonical Bose case, the CAOs are operator valued distributions (and  $\delta_{ij}$  is  $\delta(i - j)$ ). More generally, the indices labelling the CAOs can take values in spaces of any dimension. Therefore  $A$ -statistics is not restricted to  $1D$  or  $2D$  spaces only.

We should point out that within  $A$ -statistics the main quantization equation (2.6) does not determine uniquely the creation and annihilation operators. The Jacobson generators (2.3) yield one possible solution of (2.6). For another possible choice (a causal  $A$ -statistics), we refer to [61].

The quasi-Bose operators  $B(p)_1^\pm, \dots, B(p)_n^\pm$ , introduced in Section 5 can be used as an approximation, in fact a good approximation, to Bose statistics for values of the order of statistics  $p$ , which is much bigger than the number of accommodated particles. An additional advantage of the quasi-Bose CAOs of any order  $p$  is that they are bounded linear operators, defined everywhere in the Fock space  $W_p$ . This property avoids the rather delicate questions of whether the operators under consideration can be defined on a common domain of definition  $\Omega$ , so that any polynomial of them is also well defined in  $\Omega$ .

The “opposite” to  $p \rightarrow \infty$  case, namely the  $p = 1$  Jacobson CAOs (or, which is the same, the  $p = 1$  quasi-Bose operators) turns out to be of interest too. We have illustrated this on a particular example from condensed matter physics. The  $p = 1$  quasiboson representation appears naturally in lattice Bose models with infinitely strong repulsion between the particles, which forbids configurations with more than one particle per site. One can speculate that representations with order of statistics  $p$  could be of interest in pictures where no more than  $p$  particles can be accommodated on each site of the lattice.

For applications of quasiboson representations in nuclear theory we refer to [51]. As indicated there, the  $p = 1$  quasi-Bose operators reduce to Klein-Marshalek algebras [41], which are extensively used in nuclear physics.

One way to enlarge the class of statistics studied here is to deform the relations (2.5) or, which is the same, to deform  $sl(n + 1)$  so that the main quantization equation (2.6) remains unaltered. The possibility for such deformations stems from the observation that the commutation relations between the Cartan elements (the Hamiltonian is a Cartan element, see (2.18)) and the root vectors (the Jacobson generators are root vectors, see (2.4)) remain unaltered upon quantum deformations ( $q$ -deformations). Therefore the problem actually is to express the known  $q$ -deformations of  $sl(n + 1)$  via deformed Jacobson generators. This is the first step. The second step will be to define the Fock representations and to write down the deformed analogue of (3.19)-(3.20). Partial results in this respect were already announced [66, 67].

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