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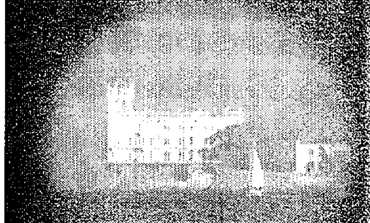

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THE BOOSTS IN THE NONCOMMUTATIVE SPECIAL RELATIVITY

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Abstract

From the quantum analogue of the Iwasawa decomposition of $SL(2, C)$ group and the correspondence between quantum $SL(2, C)$ and Lorentz groups we deduce the different properties of the Hopf algebra representing the boost of particles in noncommutative special relativity. The representation of the boost in the Hilbert space states is investigated and the addition rules of the velocities are established from the coaction. The q-deformed Clebsch-Gordon coefficients descibing the transformed states of the evolution of particles in noncommutative special relativity are introduced and their explicit calculation are given.

MIRAMARE – TRIESTE
January 2001

1 Introduction

The Lorentz group plays a fundamental role in special relativity. It gives the lifetime dilatation of unstable particles in terms of their velocities and relativistic formulas of the energy-momentum four-vector in terms of the mass and the velocities.

The above considerations make especially interesting the study of the noncommutative special relativity in the frame of the quantum Minkowski space-time and its transformations under the quantum Lorentz group to derive measurable observables describing the evolution of particles in noncommutative space-time.

In the past few years, attention has been paid to formulate the particle evolution in quantum Minkowski space times through the construction of the q-analogue of the relativistic plane waves [1] or the Hilbert space representation of the q-deformed Minkowski space-time [2].

Despite all the theoretical interests, the relevance of the quantum Minkowski space-time and its transformations under the quantum Lorentz group to derive measurable observable effects in particle physics has not been discussed very much.

In Ref.[3] the evolution of particles in noncommutative Minkowski space-time has been analysed. From the transformation of particle coordinates at rest, the quantum analogue of the lifetime dilatation of unstable particles and the relativistic formulas of the energy-momentum four vector in terms of the mass and the velocity have been established. The mean results of this work concern the principle of the causality in the noncommutative special relativity and the quantization of the moving particle lifetime which is deduced from the discrete spectrum of the velocity operators. In addition, it is shown that for a particle moving in the noncommutative Minkowski space-time, only the length of the velocity and its projection on the quantized direction can be measured exactly.

In this paper we investigated further the tranformations of the Hilbert space states describing particles moving in the noncommutative space-time to show how the addition rule of the velocities can be deduced from the coaction of the boost generators and how the quantum analogue of Glebsch-Gordon coefficients of the transformed states can be calculated.

This paper is organized as follows: In section 2, we present the quantum analogue of the Iwasawa decomposition of the $SL(2,C)$ group [4]. From this decomposition and the correpondence between the quantum $SL(2,C)$ and Lorentz groups [5], we extract the quantum boost generators and their commutation relations. In section 3 we give a representation of these quantum generators in the Hilbert space states. From the state transformation principle and the coaction on the boost generators we establish the addition rule of the velocities in the noncommutative special relativity and calculate the quantum analogue of the Clebsch-Gordon coefficients of the transformed states.

2 The quantum boost generators

Before we consider the Hopf algebra representing the quantum boost of particles in the noncommutative Minkowski space-time let us briefly recall the correspondence between the generators Λ_N^M $(N,\ M=\ 0,\ 1,\ 2,\ 3)$ of the quantum Lorentz group and those of the quantum $SL(2,C)$ group generated by ${M_\alpha}^\beta$, $(\alpha,\beta=1,2)$ and $({M_\alpha}^\beta)^\star={M_{\dot\alpha}}^{\dot\beta}$ [5].
${M_\alpha}^\beta=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$ corresponds to the representation of classical $SL(2,C)$ group acting on space of undotted spinors and ${M_{\dot\alpha}}^{\dot\beta}$ corresponds to the classical $SL(2,C)$ group acting on space of dotted spinors.
The unimodularity of ${M_\alpha}^\beta$ is expressed by $\varepsilon_{\alpha\beta}{M_\gamma}^\alpha{M_\delta}^\beta=\varepsilon_{\gamma\delta}I_{\mathcal A}$, $\varepsilon^{\gamma\delta}{M_\gamma}^\alpha{M_\delta}^\beta=\varepsilon^{\alpha\beta}I_{\mathcal A}$, and $\varepsilon_{\dot\alpha\dot\beta}{M_{\dot\gamma}}^{\dot\alpha}{M_{\dot\delta}}^{\dot\beta}=\varepsilon_{\dot\gamma\dot\delta}I_{\mathcal A}$, $\varepsilon^{\dot\gamma\dot\delta}{M_{\dot\gamma}}^{\dot\alpha}{M_{\dot\delta}}^{\dot\beta}=\varepsilon^{\dot\alpha\dot\beta}I_{\mathcal A}$ where $I_{\mathcal A}$ is the unity of the $\star$ algebra $\mathcal A$ generated by ${M_\alpha}^\beta$ and the spinor metric $\varepsilon_{\alpha\beta}$ and its inverse $\varepsilon^{\alpha\beta}$ $(\varepsilon_{\alpha\delta}\varepsilon^{\delta\beta}=\delta_\alpha^\beta=\varepsilon^{\beta\delta}\varepsilon_{\delta\alpha})$ satisfy $(\varepsilon_{\alpha\beta})^\star=\varepsilon_{\dot\beta\dot\alpha}$ and $(\varepsilon^{\alpha\beta})^\star=\varepsilon^{\dot\beta\dot\alpha}$. If we consider the case where the quantum $SL(2,C)$ group admits the quantum $SU(2)$ group as a subgroup, the spinor metric $\varepsilon_{\alpha\beta}$ must satisfy the additional condition $\varepsilon_{\alpha\beta}=-\varepsilon^{\dot\beta\dot\alpha}$ and $\varepsilon^{\alpha\beta}=-\varepsilon_{\dot\beta\dot\alpha}$ required by the compatibility of the unitarity, ${M_{(c)\dot\alpha}}^{\dot\beta}=S({M_{(c)\beta}}^\alpha)=\varepsilon_{\beta\rho}{M_{(c)\sigma}}^\rho\varepsilon^{\sigma\alpha}$, with the modularity conditions of the quantum $SU(2)$ group generators. In this case the commutation rules of the quantum $SU(2)$ subgroup are given by those of $SL(2,C)$ group where we impose the unitarity condition. The commutation rules are given by

$$M_\alpha^\rho M_\beta^\sigma R^{\pm\gamma\delta}_{\rho\sigma}=R^{\pm\rho\sigma}_{\alpha\beta}M_\rho^\delta M_\sigma^\delta,\quad and\quad M_{\dot\alpha}^{\dot\rho}M_{\dot\beta}^{\dot\sigma}R^{\pm\dot\gamma\dot\delta}_{\dot\rho\dot\sigma}=R^{\pm\dot\rho\dot\sigma}_{\dot\alpha\dot\beta}M_{\dot\rho}^{\dot\delta}M_{\dot\sigma}^{\dot\delta}. \tag{1}$$

where the R-matrices are given by $R^{\pm\delta\beta}_{\alpha\gamma}=\delta_\alpha^\delta\delta_\gamma^\beta+q^{\pm1}\varepsilon^{\delta\beta}\varepsilon_{\alpha\gamma}$ $(R^{\pm\dot\delta\dot\beta}_{\dot\alpha\dot\gamma}=\delta_{\dot\alpha}^{\dot\delta}\delta_{\dot\gamma}^{\dot\beta}+q^{\pm1}\varepsilon^{\dot\delta\dot\beta}\varepsilon_{\dot\alpha\dot\gamma})$ and the spinor metric is of the form $\varepsilon^{\alpha\beta}=\begin{pmatrix}0 & -q^{-\frac12}\\ q^{\frac12} & 0\end{pmatrix}$ where $q\neq0$ is a real deformation parameter.
The $R^\pm$-matrices satisfy $R^{\pm\delta\beta}_{\alpha\gamma}R^{\mp\rho\sigma}_{\delta\beta}=\delta_\alpha^\rho\delta_\gamma^\sigma$ $(R^{\pm\dot\delta\dot\beta}_{\dot\alpha\dot\gamma}R^{\mp\dot\rho\dot\sigma}_{\dot\delta\dot\beta}=\delta_{\dot\alpha}^{\dot\rho}\delta_{\dot\gamma}^{\dot\sigma})$, the Hecke conditions $(R^\pm+q^{\pm2})(R^\pm-1)=0$ and the Yang-Baxter equations. An explicit calculation gives from (1) the following commutation rules of the quantum $SL(2,C)$ group as:

$$\begin{aligned}\alpha\beta=q\beta\alpha,\quad \alpha\gamma=q\gamma\alpha,\quad \alpha\delta-q\gamma\beta=1,\\ \gamma\delta=q\delta\gamma,\quad \gamma\beta=\beta\gamma,\quad \beta\delta=q\delta\beta,\quad \delta\alpha-q^{-1}\beta\gamma=1.\end{aligned}\tag{2}$$

From the unitarity condition, the quantum $SU(2)$ subgroup reads ${M_{(c)\alpha}}^\beta=\begin{pmatrix}\alpha_c & -q\gamma_c^\star\\ \gamma_c & \alpha_c^\star,\end{pmatrix}$ and from (2) it follows that

$$\alpha_c\alpha_c^\star+q^2\gamma_c\gamma_c^\star=1,\quad \alpha_c^\star\alpha_c+\gamma_c\gamma_c^\star=1,\quad \gamma_c\gamma_c^\star=\gamma_c^\star\gamma_c,\quad \alpha_c\gamma_c^\star=q\gamma_c^\star\alpha_c,\quad \alpha_c\gamma_c=q\gamma_c\alpha_c. \tag{3}$$

It is shown in [5] that the generators Λ_N^M of quantum Lorentz group may be written in terms of those of quantum $SL(2,C)$ group as

$$\Lambda_N^M=\frac{1}{Q}\varepsilon_{\dot\gamma\dot\delta}\overline{\sigma}_N^{\dot\delta\alpha}{M_\alpha}^\sigma\sigma^M_{\sigma\dot\rho}{M_{\dot\beta}}^{\dot\rho}\varepsilon^{\dot\gamma\dot\beta}. \tag{4}$$

They are real, $(\Lambda_N^{\ M})^\star = \Lambda_N^{\ M}$, and generate a Hopf algebra $\mathcal{L}$ endowed with a coaction Δ, a counit ε and an antipode S acting as $\Delta(\Lambda_N^{\ M}) = \Lambda_N^{\ K} \otimes \Lambda_K^{\ M}$, $\varepsilon(\Lambda_N^{\ M}) = \delta_N^M$ and $S(\Lambda_N^{\ M}) = G_{\pm NK}\Lambda_L^{\ K} G_\pm^{LM}$ respectively. $G_\pm^{NM}$ is an invertible and hermitian quantum metric. It may be expressed in terms of the four matrices $\sigma^N_{\ \alpha\dot\beta}$ $(N = 0,1,2,3)$, where $\sigma^n_{\ \alpha\dot\beta}$ $(n = 1,2,3)$ are the usual Pauli matrices and $\sigma^0_{\ \alpha\dot\beta}$ is the identity matrix, as:

$$G_\pm^{IJ} = \frac{1}{Q}Tr(\sigma^I\overline{\sigma}_\pm^J) = \frac{1}{Q}\varepsilon^{\alpha\nu}\sigma^I_{\ \alpha\dot\beta}\overline{\sigma}_\pm^{J\dot\beta\gamma}\varepsilon_{\gamma\nu} = \frac{1}{Q}Tr(\overline{\sigma}_\pm^I\sigma^J) = \frac{1}{Q}\varepsilon_{\dot\nu\dot\gamma}\overline{\sigma}_\pm^{I\dot\gamma\alpha}\sigma^J_{\ \alpha\dot\beta}\varepsilon^{\dot\nu\dot\beta}$$

where $\overline{\sigma}_\pm^{I\dot\alpha\beta} = \varepsilon^{\dot\alpha\dot\lambda}R^{\mp\sigma\dot\rho}_{\ \ \dot\lambda\nu}\varepsilon^{\nu\beta}\sigma^I_{\ \sigma\dot\rho}$. The undotted and dotted spinorial indices are raised and lowered as $\sigma^{I\alpha}_{\ \ \dot\beta} = \sigma^I_{\ \rho\dot\beta}\varepsilon^{\rho\alpha}$ and $\sigma^{I\ \dot\beta}_{\ \alpha} = \varepsilon^{\dot\beta\dot\rho}\sigma^I_{\alpha\dot\rho}$ and the inverse of the metric may be written under the form $G_{\pm IJ} = \frac{1}{Q}Tr(\overline{\sigma}_J\sigma_{\pm I}) = \frac{1}{Q}\varepsilon_{\dot\nu\dot\gamma}\overline{\sigma}_J^{\ \dot\gamma\alpha}\sigma_{\pm I\alpha\dot\beta}\varepsilon^{\dot\nu\dot\beta}$ where $\sigma_{\pm I\alpha\dot\beta} = G_{\pm IJ}\sigma^J_{\ \alpha\dot\beta}$. The form of the antipode of $\Lambda_N^{\ M}$ guarantees the orthogonality condition on the generators of quantum Lorentz group as:

$$G_{\pm NM}\Lambda_L^{\ N}\Lambda_K^{\ M} = G_{\pm LK}I_{\mathcal{L}} \ \ and \ \ G_\pm^{LK}\Lambda_L^{\ N}\Lambda_K^{\ M} = G_\pm^{NM}I_{\mathcal{L}} \tag{5}$$

where $I_{\mathcal{L}}$ is the unity of the Hopf algebra $\mathcal{L}$.

In this framwork, it is also shown that there exist two copies of the quantum Minkowski space-times $\mathcal{M}_\pm$ equipped with metric $G^{\pm IK}$ and real coordinates $X_{\pm I}$ which transform under the left coaction $\Delta_L : \mathcal{M}_\pm \to \mathcal{L} \otimes \mathcal{M}_\pm$ as:

$$\Delta_L(X_{\pm I}) = \Lambda_I^{\ K} \otimes X_{\pm K}. \tag{6}$$

The coordinates X_{+I} transform under the quantum lorentz whose generators $\Lambda_N^{\ M}$ are subject to commutation rules controlled by the $\mathcal{R}^{+NM}_{\ PQ}$, $\Lambda_L^{\ P}\Lambda_K^{\ Q}\mathcal{R}^{+NM}_{\ PQ} = \mathcal{R}^{+PQ}_{\ LK}\Lambda_P^{\ N}\Lambda_Q^{\ M}$, and the coordinates X_{-I} transform under the quantum lorentz whose generators $\Lambda_N^{\ M}$ are subject to commutation rules of the form $\Lambda_L^{\ P}\Lambda_K^{\ Q}\mathcal{R}^{-NM}_{\ PQ} = \mathcal{R}^{-PQ}_{\ LK}\Lambda_P^{\ N}\Lambda_Q^{\ M}$ where the $\mathcal{R}$-matrices of the Lorentz group are constructed out of those of $SL(2,C)$ group as:

$$\mathcal{R}^{\pm NM}_{\ LK} = \frac{1}{Q^2}R^{\mp\alpha\kappa}_{\ \tau\sigma}R^{\mp\mu\beta}_{\ \kappa\nu}R^{\pm\rho\epsilon}_{\ \delta\alpha}R^{\pm\lambda\gamma}_{\ \epsilon\mu}\sigma^M_{\ \gamma}{}^{\dot\beta}\sigma^N_{\ \rho}{}^{\dot\lambda}\overline{\sigma}_{L\dot\tau}^{\ \ \delta}\overline{\sigma}_{K\dot\nu}^{\ \ \sigma}.$$

These $\mathcal{R}$-matrices lead to the symmetrization of the Minkowski metric in the quantum sense as:

$$\mathcal{R}^{\pm NM}_{\ KL}G_\pm^{KL} = G_\pm^{NM} \quad , \quad \mathcal{R}^{\pm NM}_{\ KL}G_{\pm NM} = G_{\pm KL}$$

and satisfy the Yang-Baxter equations and the cubic Hecke conditions

$$(\mathcal{R}^\pm + q^{\pm 2})(\mathcal{R}^\pm + q^{\mp 2})(\mathcal{R}^\pm - 1) = 0.$$

In the following we shall consider the right invariant basis $X_I = X_{+I}$ as a quantum coordinate system of the Minkowski space-time $\mathcal{M} = \mathcal{M}_+$ equipped with the metric $G^{IJ} = G_+^{IJ}$. X_0 represents the time operator and X_i $(i = 1,2,3)$ represent the space coordinate operators.

With this choice, the commutation rules between the undotted and dotted generators of the quantum $SL(2,C)$ group must necessarily be controlled by the R^--matrix, $M_\alpha^\gamma M_{\dot\sigma}^{\dot\delta} R^{-\beta\delta}_{\rho\gamma} = R^{-\gamma\sigma}_{\delta\alpha} M^{\dot\rho}_{\dot\delta} M^\beta_\gamma$ which gives explicitly

$$
\begin{aligned}
\alpha\alpha^\star = \alpha^\star\alpha - (1-q^{-2})\beta\beta^\star \quad &, \quad \alpha\gamma^\star = q^{-1}\gamma^\star\alpha - (1-q^{-2})\beta\delta^\star, \\
\gamma\gamma^\star = \gamma^\star\gamma + (1-q^{-2})(\alpha\alpha^\star - \delta\delta^\star) \quad &, \quad \gamma\delta^\star = q\delta^\star\gamma + q(1-q^{-2})\beta^\star\alpha \\
\beta\beta^\star = \beta^\star\beta, \quad \beta\delta^\star = q^{-1}\delta^\star\beta \quad &, \quad \delta\delta^\star = \delta^\star\delta + (1-q^{-2})\beta^\star\beta, \\
\alpha\beta^\star = q\beta^\star\alpha, \quad \alpha\delta^\star = \delta^\star\alpha \quad &, \quad \gamma\beta^\star = \beta^\star\gamma
\end{aligned}
\tag{7}
$$

Note that if we have considered the coordinates $X_I = X_{-I}$ which correspond to the metric $G^{NM} = G^{-NM}$ and the $\mathcal{R}$-matrix $R^{NM}_{KL} = R^{-NM}_{KL}$ the commutation rules between the undotted and dotted generators of the quantum $SL(2,C)$ group must necessarily be controlled by the R^+-matrix.

To make an explicit calculation of the different commutation rules of the generators of the quantum Lorentz group, we take the following appropriate choice of Pauli hermitian matrices

$$
\sigma^0_{\alpha\dot\beta} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \;,\; \sigma^1_{\alpha\dot\beta} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \;,\; \sigma^2_{\alpha\dot\beta} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \;,\; \sigma^3_{\alpha\dot\beta} = \begin{pmatrix} q & 0 \\ 0 & -q^{-1} \end{pmatrix}.
$$

The advantage of this choice arises from the fact that:

1)- $\overline{\sigma}_0{}^{\dot\alpha\beta} = -\sigma^0_{\alpha\dot\beta} = -\delta^\beta_\alpha$, $\overline{\sigma}_{N\dot\alpha\alpha} = \overline{\sigma}_{N\dot 1 1} + \overline{\sigma}_{N\dot 2 2} = -(q+q^{-1})\delta^0_N = -Q\delta^0_N$ and $\sigma^{N\alpha\dot\alpha} = Q\delta^N_0$. In fact, we shall see that these properties make explicit the restriction of the quantum Lorentz group to the quantum subgroup of the three dimensional space rotations by restricting the quantum $SL(2,C)$ group generators to those of the $SU(2)$ group.

2)- The form of the quantum metric G^{LK} exhibits two independent blocks, one for the time component X_0 and the others for space components X_k $(k = 1,2,3)$. The nonvanishing elements of the metrics are; $G^{00} = -q^{-\frac{3}{2}}$, $G^{11} = G^{22} = G^{33} = q^{\frac{1}{2}}$, $G^{12} = -G^{21} = -iq^{\frac{1}{2}}\frac{q-q^{-1}}{Q}$ and the non vanishing elements of its inverse are $G_{00} = -q^{\frac{3}{2}}$, $G_{11} = G_{22} = q^{-\frac{1}{2}}\frac{Q^2}{4}$, $G_{33} = q^{-\frac{1}{2}}$ and $G_{12} = -G_{21} = iq^{-\frac{1}{2}}\frac{(q-q^{-1})Q}{4}$. In the classical limit $q = 1$, this metric reduces to the classical Minkowski metric with signature $(-,+,+,+)$.

When we restrict the generators of the quantum $SL(2,C)$ group to those of the $SU(2)$ by imposing unitarity condition, we get [3]

$$
\Lambda_N^0 = \frac{1}{Q}\overline{\sigma}_{N\dot\gamma}{}^\alpha M_\alpha{}^\sigma \sigma^0{}_{\sigma\dot\rho} S(M_\rho{}^\beta)\varepsilon^{\dot\gamma\dot\beta} = \frac{1}{Q}\overline{\sigma}_{N\dot\gamma}{}^\alpha\varepsilon^{\dot\gamma\dot\alpha} = -\frac{1}{Q}\overline{\sigma}_{N\dot\gamma\gamma} = \delta^0_N \tag{8}
$$

and

$$
\begin{aligned}
\Lambda_0{}^M = \frac{1}{Q}\varepsilon_{\dot\gamma\dot\delta}\overline{\sigma}_0{}^{\dot\delta\alpha} M_\alpha{}^\sigma \sigma^M_{\sigma\dot\rho} S(M_\rho{}^\beta)\varepsilon^{\dot\gamma\dot\beta} \quad &= \quad -\frac{1}{Q}\varepsilon^{\alpha\gamma} M_\alpha{}^\sigma \sigma^M_{\sigma\dot\rho}\varepsilon_{\rho\delta} M_\gamma{}^\delta = \\
\frac{1}{Q}\varepsilon^{\dot\delta\dot\rho}\sigma^M_{\sigma\dot\rho}\varepsilon^{\sigma\delta} \quad &= \quad \frac{1}{Q}\sigma^{M\delta\dot\delta} = \delta^M_0
\end{aligned}
\tag{9}
$$

which lead us to the restriction of the Minkowski space-time transformations to the orthogonal transformations of the three dimensional space R_3 equipped with the coordinate system X_i,

$(i = 1,2,3)$. These transformations leave invariant the time coordinate X_0. In fact, as a consequence of (8) and (9) we have

$$\Delta_L(X_0) = \overline{\Lambda}_0^{\ 0} \otimes X_0 = I \otimes X_0$$
$$\Delta_L(X_i) = \overline{\Lambda}_i^{\ j} \otimes X_j \tag{10}$$

where $\overline{\Lambda}_i^{\ j} = \frac{1}{Q}\overline{\sigma}_{i\dot\gamma}{}^{\alpha} M_{(c)\alpha}{}^{\sigma} \sigma^j{}_{\sigma\dot\rho} M_{(c)\dot\beta}{}^{\dot\rho} \varepsilon^{\dot\gamma\dot\beta} = \frac{1}{Q}\overline{\sigma}_{i\dot\gamma}{}^{\alpha} M_{(c)\alpha}{}^{\sigma} \sigma^j{}_{\sigma\dot\rho} S(M_{(c)\rho}{}^{\beta})\varepsilon^{\dot\gamma\dot\beta}$ generate a Hopf subalgebra $SO_q(3)$ of $\mathcal{L}$ whose axiomatic structure is derived from those of $\mathcal{L}$ as $\Delta(\overline{\Lambda}_i^{\ j}) = \overline{\Lambda}_i^{\ k} \otimes \overline{\Lambda}_k^{\ j}$, $\varepsilon(\overline{\Lambda}_i^{\ j}) = \delta_i^{\ j}$ and $S(\overline{\Lambda}_i^{\ j}) = G_{iK}\overline{\Lambda}_L^{\ K} G^{\ Lj} = G_{ik}\overline{\Lambda}_l^{\ k} G^{\ lj}$ where $G^{\ ij}$ is the restriction of $G^{\ IJ}$ to the quantum space R_3 satisfying $G^{\ ik}G_{kj} = \delta^i_j = G_{jk}G^{\ ki}$. The form of the antipode of $\overline{\Lambda}_i^{\ j}$ implies the orthogonality properties

$$G^{\ ij}\overline{\Lambda}_i^{\ l}\overline{\Lambda}_j^{\ k} = G^{\ lk} \quad and \quad G_{lk}\overline{\Lambda}_i^{\ l}\overline{\Lambda}_i^{\ k} = G_{ij}.$$

Therefore, $\overline{\Lambda}_i^{\ j} = \frac{1}{Q}\overline{\sigma}_{i\dot\gamma}{}^{\alpha} M_{(c)\alpha}{}^{\sigma} \sigma^j{}_{\sigma\dot\rho} S(M_{(c)\rho}{}^{\beta})\varepsilon^{\dot\gamma\dot\beta}$ establishes a correspondence between the $SU_q(2)$ and $SO_q(3)$ group. In the three dimensional space spanned by the basis Z, $\overline{Z}$ and X_3, the generators $\overline{\Lambda}_i^{\ j} = \frac{1}{Q}\overline{\sigma}_{i\dot\gamma}{}^{\alpha} M_{(c)\alpha}{}^{\sigma} \sigma^j{}_{\sigma\dot\rho} S(M_{(c)\rho}{}^{\beta})\varepsilon^{\dot\gamma\dot\beta}$ give the irreducible three-dimensional representation of $SU_q(2)$ considered in [6] as

$$(d_{1,i}{}^{j}) = \begin{pmatrix} -2q\gamma_c\gamma_c & 2\alpha_c^\star\alpha_c^\star & Q\gamma_c\alpha_c^\star \\ 2\alpha_c\alpha_c & -2q\gamma_c^\star\gamma_c^\star & Q\alpha_c\gamma_c^\star \\ -2\alpha_c\gamma_c & -2\gamma_c^\star\alpha_c^\star & 1 - qQ\gamma_c\gamma_c^\star \end{pmatrix} \in M_3 \otimes C(SU_q(2)) \tag{11}$$

where the indices i, j run over $z, \overline{z}, 3$, $z = 1 + i2$ and $\overline{z} = 1 - i2$.

It is shown in [4] that the quantum $SL(2, C)$ group admits a unique Iwasawa decomposition of the form $M_\alpha{}^{\beta} = M_{(c)\alpha}{}^{\rho} M_{(d)\rho}{}^{\beta}$ where $M_{(c)}$ is a quantum $SU(2)$-matrix and $M_{(d)}$ is a quantum-matrix the left-lower-corner element equal to zero. The matrix elements of $M_{(c)}$ doubly commute with matrix elements of $M_{(d)}$ (two operators a and b doubly commute if $ab = ba$ and $ab^\star = b^\star a$). With our choice of the commutation rules (7) between the undotted and dotted generators of the quantum $SL(2, C)$ group, in order to have nontrivial commutation relations, $M_{(d)}$ must have the right-upper-corner element equal to zero

$$M_{(d)\alpha}{}^{\beta} = \begin{pmatrix} a & 0 \\ n & a^{-1} \end{pmatrix} \tag{12}$$

From (12) and (7), it follows that

$$an = qna, \quad aa^\star = a^\star a, \quad an^\star = q^{-1}n^\star a, \tag{13}$$
$$nn^\star = n^\star n + (1 - q^{-2})(aa^\star - (aa^\star)^{-1}). \tag{14}$$

As $M_{(d)}$ is a subgroup of the quantum $SL(2, C)$ with coaction, counity and antipode acting in the following way: $\Delta(a) = a \otimes a$, and $\Delta(n) = n \otimes a + a^{-1} \otimes n$, $\varepsilon(a) = 1$, $\varepsilon(n) = 0$, $S(a) = a^{-1}$, and $S(n) = -qn$. The Iwasawa decomposition and the correspondence between quantum $SL(2, C)$ and Lorentz group (4) permits us to extract out of the latter the $SO_q(3)$ subgroup left by restriction to $M_{(d)}$ with the quantum boost.

More precisely by replacing the matrix elements of M by those of $M_{(d)}$ into the relation (4), we get the following generators of the quantum boost as:

$$\begin{aligned}
{\Lambda_0}^0 = \frac{1}{Q}(q^{-1}aa^\star + q(aa^\star)^{-1} + q^{-1}nn^\star) \quad &, \quad {\Lambda_3}^3 = \frac{1}{Q}(qaa^\star + q^{-1}(aa^\star)^{-1} - qnn^\star), \\
{\Lambda_3}^0 = \frac{1}{Q}(aa^\star - (aa^\star)^{-1} - nn^\star) \quad &, \quad {\Lambda_0}^3 = \frac{1}{Q}(aa^\star - (aa^\star)^{-1} + q^2 nn^\star) \\
{\Lambda_z}^0 = na^\star, \quad {\Lambda_z}^3 = qna^\star \quad &, \quad {\Lambda_0}^z = \frac{2q}{Q}n(a^\star)^{-1}, \quad {\Lambda_3}^z = \frac{-2}{Q}n(a^\star)^{-1} \\
{\Lambda_z}^{\bar{z}} = 2a^{-1}a^\star \quad &, \quad {\Lambda_z}^z = 0.
\end{aligned} \tag{15}$$

The remaining generators are obtained by complex conjugation. Note that the commutation relations (13-14) permit us to take a real, $a^\star = a$, then ${\Lambda_z}^{\bar{z}} = {\Lambda_{\bar{z}}}^z = 2$.

From the commutation relations (13-14), we see that $[a^{-1}a^\star, n] = [a^{-1}a^\star, a] = 0$ which implies that ${\Lambda_z}^{\bar{z}}$ is central, i.e. $[{\Lambda_z}^{\bar{z}}, {\Lambda_N}^M] = 0$. From $[a, nn^\star] = [a^\star, nn^\star] = 0$ we deduce that $[{\Lambda_3}^3, {\Lambda_0}^3] = [{\Lambda_3}^3, {\Lambda_3}^0] = [{\Lambda_0}^3, {\Lambda_3}^0] = 0$. A straighforward computation shows that ${\Lambda_0}^0$ is central, $[{\Lambda_0}^0, {\Lambda_N}^M] = 0$, and

$$\begin{aligned}
{\Lambda_3}^0{\Lambda_z}^0 - q^2{\Lambda_z}^0{\Lambda_3}^0 &= (q - q^{-1}){\Lambda_0}^0{\Lambda_z}^0, && (16)\\
{\Lambda_z}^0{\Lambda_{\bar{z}}}^0 - {\Lambda_{\bar{z}}}^0{\Lambda_z}^0 &= (q - q^{-1})Q{\Lambda_3}^0({\Lambda_3}^0 + q^{-1}{\Lambda_0}^0), && (17)\\
{\Lambda_3}^0{\Lambda_0}^z - q^2{\Lambda_0}^z{\Lambda_3}^0 &= (q - q^{-1}){\Lambda_0}^0{\Lambda_0}^z, && (18)\\
{\Lambda_0}^z{\Lambda_{\bar{z}}}^0 - q^2{\Lambda_{\bar{z}}}^0{\Lambda_0}^z &= q^2(q - q^{-1}){\Lambda_3}^0{\Lambda_{\bar{z}}}^z, && (19)\\
{\Lambda_0}^3{\Lambda_0}^z - q^{-2}{\Lambda_0}^z{\Lambda_0}^3 &= (q - q^{-1}){\Lambda_0}^0{\Lambda_0}^z, && (20)\\
\frac{Q}{4}({\Lambda_0}^{\bar{z}}{\Lambda_0}^z - {\Lambda_0}^z{\Lambda_0}^{\bar{z}}) &= (q - q^{-1}){\Lambda_0}^3({\Lambda_0}^3 - q{\Lambda_0}^0), && (21)\\
{\Lambda_0}^z{\Lambda_z}^0 - q^{-2}{\Lambda_z}^0{\Lambda_0}^z &= 0 && (22)\\
{\Lambda_3}^3{\Lambda_z}^0 - q^2{\Lambda_z}^0{\Lambda_3}^3 &= q^{-1}(q - q^{-1}){\Lambda_z}^0({\Lambda_0}^3 + \frac{Q}{4}{\Lambda_z}^{\bar{z}}), && (23)\\
{\Lambda_3}^3{\Lambda_0}^z - q^{-2}{\Lambda_0}^z{\Lambda_3}^3 &= -q(q - q^{-1}){\Lambda_0}^z({\Lambda_3}^0 - \frac{1}{Q}{\Lambda_{\bar{z}}}^z). && (24)
\end{aligned}$$

The remaining commutation relations are obtained by substituting into (16-24) the following relations

$$ {\Lambda_z}^0 = q^{-1}{\Lambda_z}^3, \quad {\Lambda_0}^z = -q{\Lambda_3}^z, \quad {\Lambda_3}^3 + q^{-1}{\Lambda_0}^3 = {\Lambda_0}^0 + q{\Lambda_3}^0 \tag{25}$$

deduced from the form of the boost generators (15). From these relations and the commutation rules (16-24) we can also show that the orthogonality condition $G^{NM}{\Lambda_N}^L{\Lambda_M}^K = G^{LK}$ and $G_{LK}{\Lambda_N}^L{\Lambda_M}^K = G_{NM}$ are satisfied if

$$ {\Lambda_z}^0 = \frac{Q}{4}{\Lambda_z}^{\bar{z}}{\Lambda_0}^z(q^{-1}{\Lambda_0}^0 + {\Lambda_3}^0), \quad {\Lambda_{\bar{z}}}^0 = q^{-2}\frac{Q}{4}{\Lambda_{\bar{z}}}^z{\Lambda_0}^{\bar{z}}(q^{-1}{\Lambda_0}^0 + {\Lambda_3}^0), \tag{26}$$

$$ {\Lambda_0}^z = \frac{1}{Q}{\Lambda_{\bar{z}}}^z{\Lambda_z}^0(q{\Lambda_0}^0 - {\Lambda_0}^3), \quad {\Lambda_0}^{\bar{z}} = q^2\frac{1}{Q}{\Lambda_z}^{\bar{z}}{\Lambda_{\bar{z}}}^0(q{\Lambda_0}^0 - {\Lambda_0}^3). \tag{27}$$

In fact by substituting (15) and (25) into $G^{NM}\Lambda_N^{\ z}\Lambda_M^{\ 0} = 0 = -q^{-\frac{3}{2}}\Lambda_0^{\ z}\Lambda_0^{\ 0} + q^{\frac{1}{2}}(\frac{q\Lambda_z^{\ z}\Lambda_{\bar{z}}^{\ 0} + q^{-1}\Lambda_{\bar{z}}^{\ z}\Lambda_z^{\ 0}}{Q} + \Lambda_3^{\ z}\Lambda_3^{\ 0})$, we get

$$\frac{q^{-1}}{Q}\Lambda_{\bar{z}}^{\ z}\Lambda_z^{\ 0} = q^{-2}\Lambda_0^{\ z}\Lambda_0^{\ 0} + q^{-1}\Lambda_0^{\ z}\Lambda_3^{\ 0} = q^{-1}\Lambda_0^{\ z}(q^{-1}\Lambda_0^{\ 0} + \Lambda_3^{\ 0})$$

leading to the left relation of (26). The same procedure gives from $G^{NM}\Lambda_N^{\ \bar{z}}\Lambda_M^{\ 0} = 0$ the right relation of (26) and from $G_{NM}\Lambda_z^{\ N}\Lambda_0^{\ M} = 0$ and $G_{NM}\Lambda_{\bar{z}}^{\ N}\Lambda_0^{\ M} = 0$ the relations (27).
By substituting (27) into (26) and by using $\Lambda_z^{\ \bar{z}}\Lambda_{\bar{z}}^{\ z} = 4$ obtained from (15), we get $(q\Lambda_0^{\ 0} - \Lambda_0^{\ 3})(q^{-1}\Lambda_0^{\ 0} + \Lambda_3^{\ 0}) = 1$ leading to

$$\Lambda_3^{\ 3} = q\Lambda_3^{\ 0} + (\Lambda_0^{\ 0} + q\Lambda_3^{\ 0})^{-1}, \quad \Lambda_0^{\ 3} = q\Lambda_0^{\ 0} - q(\Lambda_0^{\ 0} + q\Lambda_3^{\ 0})^{-1}. \tag{28}$$

Then all generators of the boost can be given in terms of $\Lambda_0^{\ 0}$, $\Lambda_z^{\ 0}$, $\Lambda_{\bar{z}}^{\ 0}$ and $\Lambda_3^{\ 0}$. As in [3], we set $\Lambda_0^{\ 0} = \gamma$ and $\Lambda_i^{\ 0} = \gamma V_i$ where γ is a real c-number given by

$$\gamma = (1 - |v|_q^2)^{-\frac{1}{2}}, \tag{29}$$

V_i are the components of the velocity operator and $|v|_q^2 = -\frac{G^{ij}}{G^{00}}V_iV_j$ is its length which is also a c-number.

3 The addition rules of velocities in the noncommutative special relativity

In [3] the evolution of free particles in the coordinate system X_N, $(N = 0, 1, 2, 3)$ of the Minkowski space-time are described in terms of states $|\mathcal{P}\rangle = |t, x_3, \tau^2\rangle$ belonging to the Hilbert space $\mathcal{H}_{\mathcal{M}}$. Here t, x_3 and τ^2 are the time, the coordinate x_3 and the proper-time respectively. They are eigenvalues of the set of commuting operators which are the time X_0, the component X_3 of the space coordinates and the length of the four-vector X_N, $G^{NM}X_NX_M = -\tau^2 = -q^{-\frac{3}{2}}X_0^2 + \frac{q^{\frac{3}{2}}}{Q}Z\overline{Z} + \frac{q^{-\frac{1}{2}}}{Q}\overline{Z}Z + q^{\frac{1}{2}}X_3^2$, where $Z = X_1 + iX_2$ and $\overline{Z} = X_1 - iX_2$. τ^2 is real, bi-invariant and central. Then $|\mathcal{P}\rangle$ is a common eigenstate of X_0, X_3 and τ^2

$$X_0|\mathcal{P}\rangle = t|\mathcal{P}\rangle, \quad X_3|\mathcal{P}\rangle = x_3|\mathcal{P}\rangle, \quad and \quad \tau^2|\mathcal{P}\rangle = \tau^2|\mathcal{P}\rangle. \tag{30}$$

Since the coordinate system transforms under quantum Lorentz group with tensorial product as $X'_N = \Lambda_N^{\ M} \otimes X_M$ we have assumed in [3] that the Hilbert state $|\mathcal{P}\rangle$ transforms into $|\mathcal{P}'\rangle$ as

$$|\mathcal{P}'\rangle = |sym_q\rangle \otimes |\mathcal{P}\rangle \tag{31}$$

where $|\mathcal{P}'\rangle$ describes the evolution of the particle in the coordinate system X'_N. Note that the coordinates X'_N fulfil the same commutation relations as those of X_N, then $|\mathcal{P}'\rangle = |t', x'_3, \tau^2\rangle$ satisfy also

$$X'_0|\mathcal{P}'\rangle = t'|\mathcal{P}'\rangle, \quad X'_3|\mathcal{P}'\rangle = x'_3|\mathcal{P}'\rangle, \quad and \quad \tau^2|\mathcal{P}'\rangle = \tau^2|\mathcal{P}'\rangle. \tag{32}$$

The state $|sym_q\rangle$ belongs to the Hilbert state $\mathcal{H}_{\mathcal{L}}$ where the quantum Lorentz generators act. It is a common eigenstate of a set of commuting operators of (15). Since all the generators of the boost can be written in terms of $\Lambda_N^{\ 0} = \gamma V_N$, the set of commuting operators of the boost are ${\Lambda_0}^0 = \gamma$ and ${\Lambda_3}^0 = \gamma V_3$ and therefore, $|sym_q\rangle = |v_3, \gamma\rangle$ is a common eigenstate of γ or the length of the velocity $-\frac{G^{ij}}{G^{00}} V_i V_j$ and V_3

$$\gamma|v_3,\gamma\rangle = \gamma|v_3,\gamma\rangle \quad and \quad V_3|v_3,\gamma\rangle = v_3|v_3,\gamma\rangle. \tag{33}$$

The coordinates X'_N act on the transformed state $|\mathcal{P}'\rangle$ as

$$X'_0|\mathcal{P}'\rangle = ({\Lambda_0}^0 \otimes X_0)|\mathcal{P}'\rangle + ({\Lambda_0}^k \otimes X_k)|\mathcal{P}'\rangle = {\Lambda_0}^0|v_3\gamma\rangle \otimes X_0|\mathcal{P}\rangle + {\Lambda_0}^k|v_3\gamma\rangle \otimes X_k|\mathcal{P}\rangle, \tag{34}$$

$$X'_i|\mathcal{P}'\rangle = ({\Lambda_i}^0 \otimes X_0)|\mathcal{P}'\rangle + ({\Lambda_i}^k \otimes X_k)|\mathcal{P}'\rangle = {\Lambda_i}^0|v_3\gamma\rangle \otimes X_0|\mathcal{P}\rangle + {\Lambda_i}^k|v_3\gamma\rangle \otimes X_k|\mathcal{P}\rangle. \tag{35}$$

In the case where we boost a particle at rest described by the state $|\mathcal{P}_0\rangle = |t, 0, \tau^2\rangle$ satisfying $X_0|\mathcal{P}_0\rangle = t|\mathcal{P}_0\rangle$, $X_i|\mathcal{P}_0\rangle = 0|\mathcal{P}_0\rangle$ and $\tau^2|\mathcal{P}_0\rangle = \tau^2|\mathcal{P}_0\rangle$ it is shown in [3] that this state is unique and transforms under the quantum Lorentz group as:

$$|\mathcal{P}\rangle = |t, x_3, \tau^2\rangle = |v_3, \gamma\rangle \otimes |\mathcal{P}_0\rangle. \tag{36}$$

x_3 and v_3 are quantized and read

$$x_3^{(l,m)} = q^{-1}(\frac{q^{2m}}{\gamma^{(l)}} - 1)t \ , \ v_3^{(l,m)} = q^{-1}(\frac{q^{2m}}{\gamma^{(l)}} - 1) \tag{37}$$

where

$$\gamma^{(l)} = \frac{(q^{(2l+1)} + q^{-(2l+1)})}{Q}, \tag{38}$$

$l = 0, \frac{1}{2}, 1, \ldots.\infty$ and m runs by integer steps over the range $-l \leq m \leq l$. In the following the states $|v_3^{(l,m)}, \gamma^{(l)}\rangle$ describing the boost will be noted $|l, m\rangle$. They form an orthonormal basis

$$\langle l, m|l', m'\rangle = \delta_{l',l}\delta_{m',m} \tag{39}$$

satisfying

$$\gamma|l,m\rangle = \gamma^{(l)}|l,m\rangle \quad and \quad V_3|l,m\rangle = v_3^{(l,m)}|l,m\rangle. \tag{40}$$

By setting ${\Lambda_0}^3 = \gamma V^3$, we obtain from (28) $V^3 = q - \frac{q}{\gamma^2(1+qV_3)}$ and ${\Lambda_3}^3 = q\gamma V_3 + \frac{1}{\gamma(1+qV_3)}$ from which we deduce

$$V^3|l,m\rangle = -q(\frac{q^{-2m}}{\gamma^{(l)}} - 1)|l,m\rangle = v^{3(l,m)}|l,m\rangle \quad and \quad {\Lambda_3}^3|l,m\rangle = (q^{2m} + q^{-2m} - \gamma^{(l)})|l,m\rangle. \tag{41}$$

From the orthogonality condition $G^{NM}\Lambda_N^{\ 0}\Lambda_M^{\ 0} = G^{00} = -q^{-\frac{3}{2}} = -q^{-\frac{3}{2}}\gamma^2 + q^{\frac{1}{2}}\gamma^2(\frac{qV_zV_{\bar{z}} + q^{-1}V_{\bar{z}}V_z}{Q} + V_3V_3)$ and the commutation relation $V_zV_{\bar{z}} - V_{\bar{z}}V_z = (q - q^{-1})QV_3(V_3 + q^{-1})$ obtained from (17), we get

$$V_{\bar{z}}V_z = \frac{1}{\gamma^2}(q^{-2}(1 + qV_3)(1 - q^3V_3) - q^{-2}), \tag{42}$$

$$V_zV_{\bar{z}} = \frac{1}{\gamma^2}(q^{-2}(1 + qV_3)(1 - q^{-1}V_3) - q^{-2}). \tag{43}$$

On the other hand, the commutation relations

$$V_3 V_z - q^2 V_z V_3 = (q - q^{-1}) V_z \ \ and \ \ V_3 V_{\bar{z}} - q^{-2} V_{\bar{z}} V_3 = -q^{-2}(q - q^{-1}) V_{\bar{z}} \tag{44}$$

obtained from (16), show that [3]

$$V_z|l,m\rangle = (\alpha^1_{(l,m)})^{\frac{1}{2}}|l,m+1\rangle \ \ and \ \ V_{\bar{z}}|l,m\rangle = (\alpha^2_{(l,m)})^{\frac{1}{2}}|l,m-1\rangle \tag{45}$$

where

$$\alpha^1_{(l,m)} = \langle l,m|V_{\bar{z}}V_z|l,m\rangle = \frac{1}{(\gamma^{(l)})^2}(q^{2m-1}Q\gamma^{(l)} - q^{4m} - q^{-2}), \tag{46}$$

$$\alpha^2_{(l,m)} = \langle l,m|V_z V_{\bar{z}}|l,m\rangle = \frac{1}{(\gamma^{(l)})^2}(q^{2m-3}Q\gamma^{(l)} - q^{4m-4} - q^{-2}). \tag{47}$$

From (27) we get

$$V^z = \frac{2}{Q\gamma}V_z(q^{-1} + V_3)^{-1} \ \ and \ \ V^{\bar{z}} = \frac{2q^2}{Q\gamma}V_{\bar{z}}(q^{-1} + V_3)^{-1} \tag{48}$$

implying

$$V^z|l,m\rangle = \frac{2}{Q}q^{-2m+1}V_z|l,m\rangle = \frac{2}{Q}q^{-2m+1}(\alpha^1_{(l,m)})^{\frac{1}{2}}|l,m+1\rangle = (\beta^1_{(l,m)})^{\frac{1}{2}}|l,m+1\rangle \tag{49}$$

where

$$\beta^1_{(l,m)} = \langle l,m|V^{\bar{z}}V^z|l,m\rangle = \frac{4}{(Q\gamma^{(l)})^2}(q^{-2m+1}Q\gamma^{(l)} - q^{-4m} - q^{+2}). \tag{50}$$

The same procedure applied to $V^{\bar{z}}$ gives

$$V^{\bar{z}}|l,m\rangle = (\beta^2_{(l,m)})^{\frac{1}{2}}|l,m-1\rangle \tag{51}$$

where

$$\beta^2_{(l,m)} = \langle l,m|V^z V^{\bar{z}}|l,m\rangle = \frac{4}{(Q\gamma^{(l)})^2}(q^{-2m+3}Q\gamma^{(l)} - q^{-4m-4} - q^{+2}). \tag{52}$$

Now we want to consider successive boost transformations given in terms of left-coaction on the coordinates as:

$$X''_N = (\Delta \otimes I)\Delta_L(X_N) = (i \otimes \Delta_L)\Delta_L(X_N) = \Lambda_N^{\ K} \otimes \Lambda_K^{\ M} \otimes X_N. \tag{53}$$

As noted above X''_N fulfil the same commutation relation as X_N and $\Lambda''^{\ M}_N = \Lambda_N^{\ K} \otimes \Lambda_K^{\ M}$ fulfil the same commutation relations as $\Lambda_N^{\ M}$. Then a state describing the evolution of a particle in the coordinate system X''_N reads $|\mathcal{P}''\rangle = |t'', x''_3, \tau^2\rangle$. It is a common eigenstate of X''_0, X''_3 and τ^2 with eigenvalues t'', x''_3 and τ^2 respectively. As assumed above, the transformed states may be written as

$$|\mathcal{P}''\rangle = |sym''_q\rangle \otimes |\mathcal{P}\rangle = |sym'_q\rangle \otimes |sym_q\rangle \otimes |\mathcal{P}\rangle = |sym'_q\rangle \otimes |\mathcal{P}'\rangle \tag{54}$$

where $|sym''_q\rangle = |v''_3, \gamma''\rangle$ is a common eigenstate of $\gamma'' = \Lambda''^0_0$ and V''_3 deduced from $\Lambda''^0_3 = V''_3\gamma''$. Since Λ''^M_N fulfil the same commutation relations as Λ_N^M, $|v''_3, \gamma''\rangle$ has the same form as (37) and may be written as $|l_3, m_3\rangle$ satifying

$$\gamma''|l_3, m_3\rangle = \gamma^{(l_3)}|l_3, m_3\rangle \quad and \quad V''_3|l_3, m_3\rangle = v_3^{(l_3,m_3)}|l_3, m_3\rangle \tag{55}$$

where

$$\gamma^{(l_3)} = \frac{q^{(2l_3+1)} + q^{-(2l_3+1)}}{Q} \quad , \quad v_3^{(l_3,m_3)} = q^{-1}(\frac{q^{2m_3}}{\gamma^{(l_3)}} - 1) \tag{56}$$

and $l_3 = 0, \frac{1}{2}, 1, ...\infty$ and m_3 runs by integer steps over the range $-l_3 \leq m_3 \leq l_3$.
We are now ready to state the addition rule of the velocity out of the coaction on the generators of the boost. In fact let $|sym'_q\rangle = |l_2, m_2\rangle$ and $|sym_q\rangle = |l_1, m_1\rangle$. For l_2 and l_1 fixed, the basis $|l_2, m_2\rangle \otimes |l_1, m_1\rangle = |l_2, l_1, m_2, m_1\rangle$ contains $(2l_2+1)(2l_1+1)$ linear independent states satisfying

$$\langle l_2, l_1, m_2, m_1|l_2, l_1, m'_2, m'_1\rangle = \delta_{m_2,m'_2}\delta_{m_1,m'_1} \tag{57}$$

$$\sum_{m_2,m_1} |l_2, l_1, m_2, m_1\rangle\langle l_2, l_1, m_2, m_1| = 1. \tag{58}$$

The m_2 sum is performed over all values $-l_2 \leq m_2 \leq l_2$ and analogueously in the m_1 case over the interval $-l_1 \leq m_1 \leq l_1$. Thus in the basis $|l_2, l_1, m_2, m_1\rangle$ the state $|l_3, m_3\rangle$ reads

$$|l_3, m_3\rangle = \sum_{m_2,m_1} |l_2, l_1, m_2, m_1\rangle\langle l_2, l_1, m_2, m_1|l_3, m_3\rangle \tag{59}$$

where the coefficients $\langle l_2, l_1, m_2, m_1|l_3, m_3\rangle$ are the quantum-analogue of the Clebsch-Gordon coefficients. Now we are ready to state that

$$m_3 = m_2 + m_1 \quad , \quad -l_3 \leq m_3 \leq l_3 \quad and \quad |l_2 - l_1| \leq l_3 \leq l_2 + l_1. \tag{60}$$

From the coaction on the boost generators we get

$$\Delta({\Lambda_0}^0) = \gamma'' = {\Lambda_0}^0 \otimes {\Lambda_0}^0 + \frac{1}{2}({\Lambda_0}^z \otimes {\Lambda_{\bar{z}}}^0 + {\Lambda_0}^{\bar{z}} \otimes {\Lambda_z}^0) + {\Lambda_0}^3 \otimes {\Lambda_3}^0 \tag{61}$$

$$\Delta({\Lambda_3}^0) = V''_3\gamma'' = {\Lambda_3}^0 \otimes {\Lambda_0}^0 + \frac{1}{2}({\Lambda_3}^z \otimes {\Lambda_{\bar{z}}}^0 + {\Lambda_3}^{\bar{z}} \otimes {\Lambda_z}^0) + {\Lambda_3}^3 \otimes {\Lambda_3}^0. \tag{62}$$

By replacing (25) into (62) we get

$$\begin{aligned} V''_3\gamma'' = {\Lambda_3}^0 \otimes {\Lambda_0}^0 - \frac{q^{-1}}{2}({\Lambda_0}^z \otimes {\Lambda_{\bar{z}}}^0 \quad &+ \quad {\Lambda_0}^{\bar{z}} \otimes {\Lambda_z}^0) + ({\Lambda_0}^0 + q{\Lambda_3}^0 - q^{-1}{\Lambda_0}^3) \otimes {\Lambda_3}^0 = \\ {\Lambda_3}^0 \otimes ({\Lambda_0}^0 + q{\Lambda_3}^0) \quad &+ \quad q^{-1}{\Lambda_0}^0 \otimes ({\Lambda_0}^0 + q{\Lambda_3}^0) \\ -q^{-1}({\Lambda_0}^0 \otimes {\Lambda_0}^0 + \frac{1}{2}({\Lambda_0}^z \otimes {\Lambda_{\bar{z}}}^0 \quad &+ \quad {\Lambda_0}^{\bar{z}} \otimes {\Lambda_z}^0) + {\Lambda_0}^3 \otimes {\Lambda_3}^0) = \\ {\Lambda_3}^0 \otimes ({\Lambda_0}^0 + q{\Lambda_3}^0) \quad &+ \quad q^{-1}{\Lambda_0}^0 \otimes ({\Lambda_0}^0 + q{\Lambda_3}^0) - q^{-1}\gamma''. \end{aligned}$$

By applying the latter relation to the state (59), we get

$$V''_3\gamma''|l_3, m_3\rangle \quad = \quad v_3^{(l_3,m_3)}\gamma^{(l_3)}|l_3, m_3\rangle = q^{-1}(q^{2m_3} - \gamma^{(l_3)})|l_3, m_3\rangle = \tag{63}$$

$$\sum_{m_2,m_1} (v_3^{(l_2,m_2)}\gamma^{(l_2)}(1 + qv_3^{(l_1,m_1)})\gamma^{(l_1)}|l_2,l_1,m_2,m_1\rangle\langle l_2,l_1,m_2,m_1|l_3,m_3\rangle +$$

$$(q^{-1}\gamma^{(l_2)}(1 + qv_3^{(l_1,m_1)})\gamma^{(l_1)} - q^{-1}\gamma^{(l_3)})|l_2,l_1,m_2,m_1\rangle\langle l_2,l_1,m_2,m_1|l_3,m_3\rangle = \quad (64)$$

$$\sum_{m_2,m_1} q^{-1}(q^{2(m_2+m_1)} - \gamma^{(l_3)})|l_2,l_1,m_2,m_1\rangle\langle l_2,l_1,m_2,m_1|l_3,m_3\rangle. \quad (65)$$

By identifying (63) with (65) and by applying $\langle l_2,l_1,m_2,m_1|$ from the left we get because of linear independence

$$(q^{2m_3} - q^{2(m_2+m_1)})\langle l_2,l_1,m_2,m_1|l_3,m_3\rangle = 0 \quad (66)$$

which implies $m_3 = m_2 + m_1$ and

$$\langle l_2,l_1,m_2,m_1|l_3,m_3\rangle = 0 \quad if \quad m_2 + m_1 \neq m_3. \quad (67)$$

By applying (61) to (59) and then $\langle l_2,l_1,m_2,m_1|$ from the left, we get

$$\gamma^{(l_3)}\langle l_2,l_1,m_2,m_1|l_3,m_3\rangle = \quad (68)$$

$$(\gamma^{(l_1)}q^{-2m_2} + \gamma^{(l_2)}q^{2m_1} - q^{-2(m_2-m_1)})\langle l_2,l_1,m_2,m_1|l_3,m_3\rangle$$

$$+ \frac{1}{2}(\beta^1_{(l_2,m_2-1)}\alpha^2_{(l_1,m_1+1)})^{\frac{1}{2}}\langle l_2,l_1,m_2-1,m_1+1|l_3,m_3\rangle$$

$$+ \frac{1}{2}(\beta^2_{(l_2,m_2+1)}\alpha^1_{(l_1,m_1-1)})^{\frac{1}{2}}\langle l_2,l_1,m_2+1,m_1-1|l_3,m_3\rangle. \quad (69)$$

For $m_2 = l_2$ and $m_1 = l_1$ or $m_2 = -l_2$ and $m_1 = -l_1$ the second and third terms of the right hand side of this relation vanish implying $l_3 = l_2 + l_1$. We note in these cases that the upper value $m_3^{max} = l_2 + l_1$ and the lower value $m_3^{min} = -(l_2 + l_1)$ appear once, thus the upper value of l_3 is $l_3^{max} = l_2 + l_1$ and

$$|l_3^{max}, l_3^{max}\rangle = |l_2,l_1,l_2,l_1\rangle, \quad (70)$$

$$|l_3^{max}, -l_3^{max}\rangle = |l_2,l_1,-l_2,-l_1\rangle. \quad (71)$$

Since they are $(2l_2+1)(2l_1+1)$ linear independent states $|l_2,l_1,m_2,m_1\rangle$ and $m_3^{max} = l_2 + l_1$ appears only once we may make a similar demonstration to one of the additions of two angular momentum, called the triangle rule in the text book of quantum mechanics [7], to show that all values of $l_3 = l_2 + l_1, l_2 + l_1 - 1, ...|l_2 - l_1|$ appear precisely once and the number of states $|l_3, m_3\rangle$ is equal the number of basis $|l_2,l_1,m_2,m_1\rangle$

$$\sum_{|l_2-l_1|}^{l_2+l_1} (2l_3+1) = (2l_2+1)(2l_1+1). \quad (72)$$

Therefore, the projection of the velocity on the quantization direction is given in terms of quantum number $m_3 = m_2 + m_1$ and $l_3 = l_2 + l_1, l_2 + l_1 - 1, ...|l_2 - l_1|$ with $-l_3 \leq m_3 \leq l_3$. Now we are ready to compute explicitly the q-deformed Clebsch-Gordon coefficients. We start from

$$\begin{aligned}\Delta({\Lambda_z}^0) = \gamma'' V_z'' &= {\Lambda_z}^0 \otimes {\Lambda_0}^0 + \frac{1}{2}({\Lambda_z}^z \otimes {\Lambda_{\bar z}}^0 + {\Lambda_z}^{\bar z} \otimes {\Lambda_z}^0) + {\Lambda_z}^3 \otimes {\Lambda_3}^0 = \\ &= {\Lambda_z}^0 \otimes {\Lambda_0}^0 + 1 \otimes {\Lambda_z}^0 + {\Lambda_z}^3 \otimes {\Lambda_3}^0 = \\ &= {\Lambda_z}^0 \otimes ({\Lambda_0}^0 + q{\Lambda_3}^0) + 1 \otimes {\Lambda_z}^0\end{aligned}$$

where we have used (25), ${\Lambda_z}^z = 0$ and ${\Lambda_z}^{\bar z} = 2$. By applying the latter relation to (59), we obtain

$$\gamma^{(l_3)}(\alpha^1_{(l_3,m_3)})^{\frac{1}{2}}|l_3,m_3\rangle = \sum_{m_2,m_1} \gamma^{(l_2)}(\alpha^1_{(l_2,m_2)})q^{2m_1}|l_2,l_1,m_2+1,m_1\rangle\langle l_2,l_1,m_2,m_1|l_3,m_3\rangle + \gamma^{(l_1)}(\alpha^1_{(l_1,m_1)})|l_2,l_1,m_2,m_1+1\rangle\langle l_2,l_1,m_2,m_1|l_3,m_3\rangle$$

which gives, because of linear independence, the condition

$$\begin{aligned}&\gamma^{(l_3)}(\alpha^1_{(l_3,m_3)})^{\frac{1}{2}}\langle l_2,l_1,m_2,m_1|l_3,m_3+1\rangle = \\ &\gamma^{(l_2)}(\alpha^1_{(l_2,m_2-1)})^{\frac{1}{2}}q^{2m_1}\langle l_2,l_1,m_2-1,m_1|l_3,m_3\rangle + \gamma^{(l_1)}(\alpha^1_{(l_1,m_1-1)})^{\frac{1}{2}}\langle l_2,l_1,m_2,m_1-1|l_3,m_3\rangle.\end{aligned}$$

The same procedure for $\Delta({\Lambda_{\bar z}}^0) = \gamma'' V_{\bar z}''$ gives the condition

$$\begin{aligned}&\gamma^{(l_3)}(\alpha^2_{(l_3,m_3)})^{\frac{1}{2}}\langle l_2,l_1,m_2,m_1|l_3,m_3-1\rangle = \\ &\gamma^{(l_2)}(\alpha^2_{(l_2,m_2+1)})^{\frac{1}{2}}q^{2m_1}\langle l_2,l_1,m_2+1,m_1|l_3,m_3\rangle + \gamma^{(l_1)}(\alpha^2_{(l_1,m_1+1)})^{\frac{1}{2}}\langle l_2,l_1,m_2,m_1+1|l_3,m_3\rangle\end{aligned}$$

These conditions give recursion relations for the calculation of Clebsch-Gordon coefficients. For example, in the case where $l_2 = \frac{1}{2}$ and $l_1 = \frac{1}{2}$ we obtain the following Clebsch-Gordon coefficients:

$$\langle\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}|1,0\rangle = (qQ)^{-\frac{1}{2}} \quad \textit{and} \quad \langle\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}|1,0\rangle = (q^{-1}Q)^{-\frac{1}{2}} \tag{73}$$

leading to

$$|1,1\rangle = |\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\rangle \;,\; |1,-1\rangle = |\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\rangle, \tag{74}$$

$$|1,0\rangle = (qQ)^{-\frac{1}{2}}|\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\rangle + (q^{-1}Q)^{-\frac{1}{2}}|\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\rangle, \tag{75}$$

$$|0,0\rangle = -(q^{-1}Q)^{-\frac{1}{2}}|\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\rangle + (qQ)^{-\frac{1}{2}}|\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\rangle. \tag{76}$$

For the case $l_2 = \frac{1}{2}$ and $l_1 = 1$ we obtain

$$|\frac{3}{2},\frac{3}{2}\rangle = |\frac{1}{2},1,\frac{1}{2},1\rangle \;,\; |\frac{3}{2},-\frac{3}{2}\rangle = |\frac{1}{2},1,-\frac{1}{2},-1\rangle, \tag{77}$$

$$|\frac{3}{2},\frac{1}{2}\rangle = (\frac{q^{-1}Q}{Q^2-1})^{\frac{1}{2}}|\frac{1}{2},1,\frac{1}{2},0\rangle + q(\frac{1}{Q^2-1})^{\frac{1}{2}}|\frac{1}{2},1,-\frac{1}{2},0\rangle, \tag{78}$$

$$|\frac{3}{2},-\frac{1}{2}\rangle = (\frac{qQ}{Q^2-1})^{\frac{1}{2}}|\frac{1}{2},1,-\frac{1}{2},0\rangle + q^{-1}(\frac{1}{Q^2-1})^{\frac{1}{2}}|\frac{1}{2},1,\frac{1}{2},-1\rangle \tag{79}$$

$$|\frac{1}{2},\frac{1}{2}\rangle = q(\frac{1}{Q^2-1})^{\frac{1}{2}}|\frac{1}{2},1,\frac{1}{2},0\rangle - (\frac{q^{-1}Q}{Q^2-1})^{\frac{1}{2}}|\frac{1}{2},1,-\frac{1}{2},0\rangle, \tag{80}$$

$$|\frac{1}{2},-\frac{1}{2}\rangle = -q^{-1}(\frac{1}{Q^2-1})^{\frac{1}{2}}|\frac{1}{2},1,-\frac{1}{2},0\rangle + (\frac{qQ}{Q^2-1})^{\frac{1}{2}}|\frac{1}{2},1,\frac{1}{2},-1\rangle. \tag{81}$$

Conclusion:

In this paper we have shown how the addition rule of the velocity in the noncommutative special relativity is derived from the left coaction (53) on a quantum coordinate system of the noncommutative Minkowski space-time. The states describing the evolution of a particle in different coordinate systems tied by quantum boost cotransformations (54) are computed explicitly and the recursion formulas giving the q-deformed Glebsch-Gordon coefficients of the transformed states are investigated.

Acknowledgments: I am particularly grateful to the Abdus Salam International Centre for Theoretical Physics, Trieste, Italy, for hospitality.

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