ERROR ANALYSIS OF NEWMARK'S METHOD FOR THE SECOND ORDER EQUATION WITH INHOMOGENEOUS TERM

F. CHIBA and T. KAKO

Department of Computer Science,
The University of Electro-Communications,
1-5-1 Chofugaoka, Chofu, Tokyo 182-8585, Japan.
email: chiba@im.uec.ac.jp, kako@im.uec.ac.jp

Abstract

For the second order time evolution equation with a general dissipation term, we introduce a recurrence relation of Newmark's method. Deriving an energy inequality from this relation, we consider the stability and the convergence criteria of Newmark's method. We treat a dissipation term under the assumption that the coefficient damping matrix is constant in time and nonnegative. We can relax however the assumptions for the dissipation and the rigidity matrices to be arbitrary symmetric matrices.

Keywords: dissipation term, energy inequality, Newmark's method, recurrence relation, stability

1 Introduction of Newmark's method

We introduce the basic ideas of Newmark's method[9] for the second order time evolution equation in the d-dimensional Euclidean space $\mathbf{R}^d$. Let M, C and K be $d \times d$ matrices on $\mathbf{R}^d$ which are constant in time, and let $f(t)$ be a given $\mathbf{R}^d$-valued function on $[0, \infty)$. Throughout this paper, we assume that M is symmetric and positive definite:

$$M \geq mI, \; m > 0. \tag{1}$$

Here I is the identity matrix on $\mathbf{R}^d$. We consider the approximation method for the following initial value problem of the second order time evolution equation for an $\mathbf{R}^d$-valued function $u(t)$:

$$M\frac{d^2}{dt^2}u(t) + C\frac{d}{dt}u(t) + Ku(t) = f(t), \quad u(0) = u_0, \quad \frac{d}{dt}u(0) = v_0. \tag{2}$$

Let a_n, v_n and u_n be approximations of $(d^2/dt^2)u(t)$, $(d/dt)u(t)$ and $u(t)$ respectively at $t = \tau n$ with a positive time step τ and an integer n, and let $f_n = f(\tau n)$. Then Newmark's method for (2) is defined through the following relations:

$$\begin{cases} Ma_n + Cv_n + Ku_n = f_n, \\ u_{n+1} = u_n + \tau v_n + \frac{1}{2}\tau^2 a_n + \beta\tau^2(a_{n+1} - a_n), \\ v_{n+1} = v_n + \tau a_n + \gamma\tau(a_{n+1} - a_n). \end{cases} \tag{3}$$

The first relation corresponds to the equation (2), and the second and the third relations correspond to the Taylor expansions of $u(t+\tau)$ and $v(t+\tau)$ at $t = \tau n$ respectively. Here, β and γ are positive tuning parameters of the method.

An interpretation of the parameter β related to the acceleration $(d^2/dt^2)u(t)$ can be seen in [9]: The case $\beta = 1/6$ corresponds to the approximation that the acceleration is a linear function of t in each time interval; the case $\beta = 1/4$ corresponds to the approximation of the acceleration to be a constant function during the time interval which is equal to the mean of the initial and final values of acceleration; the case $\beta = 1/8$ corresponds to the case that the acceleration is a step function with the initial value for the first half of each time interval and the final value for the second half of the interval. It is often the case that γ is equal to $1/2$.

2 Iteration scheme of Newmark's method

Based on the formulas in (3), Newmark's method generates the approximation sequence $u_n, n = 0, 1, 2, \ldots, N$ by the following iteration scheme:

Step 1. For $n = 0$, compute a_0 from the initial data u_0 and v_0:

$$a_0 = M^{-1}(f_0 - Cv_0 - Ku_0).$$

Step 2. Compute a_{n+1} from f_{n+1}, u_n, v_n and a_n solving a linear equation:

$$\begin{aligned} a_{n+1} &= (M + \gamma\tau C + \beta\tau^2 K)^{-1} \\ &\quad \times[-Ku_n - (C + \tau K)v_n + \{(\gamma - 1)\tau C + (\beta - \tfrac{1}{2})\tau^2 K\}a_n + f_{n+1}]. \end{aligned}$$

Step 3. Compute u_{n+1} from u_n, v_n, a_n and a_{n+1}:

$$u_{n+1} = u_n + \tau v_n + (\tfrac{1}{2} - \beta)\tau^2 a_n + \beta\tau^2 a_{n+1}.$$

Step 4. Compute v_{n+1} from v_n, a_n and a_{n+1}:

$$v_{n+1} = v_n + (1 - \gamma)\tau a_n + \gamma\tau a_{n+1}.$$

Step 5. Replace n by $n + 1$, and return to Step 2.

Here, Step 1 is nothing but the first relation of (3) with $n = 0$. The expression of a_{n+1} in Step 2 is obtained by eliminating u_{n+1} and v_{n+1} from the second and the third equalities in (3) together with the first equality in (3) with n replaced by $n + 1$:

$$Ma_{n+1} + Cv_{n+1} + Ku_{n+1} = f_{n+1}.$$

Step 3 and Step 4 are from the second and the third relations of (3).

3 Recurrence relation of Newmark's method

Newmark's method (3) for the second order equation (2) is reformulated as follows in the recurrence relation([1], [2], [3] and [13]). Eliminating v_n and a_n from (3) by a series of tedious manual computations, we have

$$\begin{aligned} &(M + \beta\tau^2 K)D_{\tau\bar{\tau}}u_n \\ &\quad +\gamma C D_\tau u_n + \{(1-\gamma)C + \tau(\gamma - \tfrac{1}{2})K\}D_{\bar{\tau}}u_n + Ku_n = \{I + \tau(\gamma - \tfrac{1}{2})D_{\bar{\tau}} + \beta\tau^2 D_{\tau\bar{\tau}}\}f_n, \end{aligned} \tag{4}$$

where

$$\begin{cases} D_\tau u_n &= (u_{n+1} - u_n)/\tau, \\ D_{\bar\tau} u_n &= (u_n - u_{n-1})/\tau, \\ D_{\tau\bar\tau} u_n &= (D_\tau u_n - D_{\bar\tau} u_n)/\tau. \end{cases}$$

We confirmed this result by a formula manipulation software NCAlgebra[6] on Mathematica[12]. Especially, in the case $\gamma = 1/2$, we have:

$$(M + \beta\tau^2 K) D_{\tau\bar\tau} u_n + \frac{1}{2} C (D_\tau + D_{\bar\tau}) u_n + K u_n = (I + \beta\tau^2 D_{\tau\bar\tau}) f_n. \tag{5}$$

These recurrence relations are useful for the stability and error analyses of Newmark's method. See [8] and [10] for the case with $C \equiv 0$.

4 Derivation of an energy inequality

Taking a scalar product of (4) and $(D_\tau + D_{\bar\tau})u_n$, we can derive an energy inequality for Newmark's method. From this inequality, we obtain the stability conditions for Newmark's method. In the following, we use the usual Euclidean scalar product $(\cdot,\cdot)$ and the corresponding norm $\|\cdot\|$ in $\mathbf{R}^d$.

From now on, let T be a fixed positive number and N be a positive integer, and define $\tau := T/N$ and $n = 0, 1, 2, \ldots, N-1$. Let C be nonnegative: $C \geq 0$, let K be positive definite: $K \geq kI$, $k > 0$ and $m > 0$ is the smallest eigenvalue of M. We also assume that

$$\gamma \geq \frac{1}{2}.$$

The main result of this section is the following theorem for the energy inequality.

THEOREM 4.1 *Let $\{u_n\}_{n=0}^N$ be a sequence generated by the scheme in Section 2. Then for a positive constant τ_0 determined as below, there exists a positive constant C_0 such that the inequality:*

$$\|M^{1/2} D_\tau u_n\|^2 + \tau^2\{\beta - \tfrac{1}{2}(\gamma - \tfrac{1}{2}) - \tfrac{1}{2}\alpha\}\|K^{1/2} D_\tau u_n\|^2 + (1 - \tfrac{1}{2\alpha})\|K^{1/2} u_n\|^2 \leq C_0 \tag{6}$$

holds for all $\tau \leq \tau_0$ and an arbitrary positive constant α, where τ_0 is determined either as

$$\tau_0 > 0 \text{ when } \beta \geq \frac{\gamma}{2} \tag{7}$$

or as

$$0 < \tau_0 < \sqrt{\frac{m}{(\frac{1}{2}\gamma - \beta)\|K^{1/2}\|^2}} \text{ when } 0 \leq \beta < \frac{\gamma}{2}. \tag{8}$$

REMARK 4.1 *The constant C_0 in the theorem depends primarily on the coefficient matrices M, C and K, and secondarily on the tuning parameters β and γ and it also depends on the initial values u_0 and v_0 and the inhomogeneous term f, and finally on τ_0 in the way described above.*

PROOF. Using (4), we derive an energy inequality as follows (see [1] and [2] for the case $\gamma = 1/2$). Rearranging (4), we have

$$\begin{aligned} &(M + \beta\tau^2 K) D_{\tau\bar\tau} u_n + \frac{1}{2} C (D_\tau + D_{\bar\tau}) u_n \\ &\quad + (\gamma - \tfrac{1}{2}) C (D_\tau - D_{\bar\tau}) u_n + \tau(\gamma - \tfrac{1}{2}) K D_{\bar\tau} u_n + K u_n = g_n, \end{aligned} \tag{9}$$

where

$$g_n := \{I + \tau(\gamma - \frac{1}{2})D_{\bar{\tau}} + \beta\tau^2 D_{\tau\bar{\tau}}\} f_n. \tag{10}$$

We take a scalar product of (9) and $(D_\tau + D_{\bar{\tau}})u_n$. Since C is nonnegative, we have

$$\begin{aligned} &((M + \beta\tau^2 K)D_{\tau\bar{\tau}}u_n, (D_\tau + D_{\bar{\tau}})u_n) \\ &+(\gamma - \tfrac{1}{2})(C(D_\tau - D_{\bar{\tau}})u_n, (D_\tau + D_{\bar{\tau}})u_n) \\ &+\tau(\gamma - \tfrac{1}{2})(KD_{\bar{\tau}}u_n, (D_\tau + D_{\bar{\tau}})u_n) \\ &+(Ku_n, (D_\tau + D_{\bar{\tau}})u_n) - (g_n, (D_\tau + D_{\bar{\tau}})u_n) \\ &= -(\tfrac{1}{2}C(D_\tau + D_{\bar{\tau}})u_n, (D_\tau + D_{\bar{\tau}})u_n) \le 0. \end{aligned} \tag{11}$$

Using the assumption for M, C and K, we obtain the following lemma:

LEMMA 4.2 *When we put*

$$\begin{aligned} w_n := &((M + \beta\tau^2 K)D_\tau u_n, D_\tau u_n) + (Ku_{n+1}, u_n) \\ &+\tau(\gamma - \tfrac{1}{2})(CD_\tau u_n, D_\tau u_n) - \tfrac{1}{2}\tau^2(\gamma - \tfrac{1}{2})(KD_\tau u_n, D_\tau u_n), \end{aligned}$$

we have

$$w_n \le w_{n-1} + \tau(g_n, D_\tau u_n + D_\tau u_{n-1}). \tag{12}$$

PROOF. We omit the proof. □

Using the equality:

$$(Ku_{n+1}, u_n) = \tau(KD_\tau u_n, u_n) + (Ku_n, u_n),$$

we can modify the expression of w_n as

$$\begin{aligned} w_n = &\|M^{1/2}D_\tau u_n\|^2 + \tau^2\{\beta - \frac{1}{2}(\gamma - \frac{1}{2})\}\|K^{1/2}D_\tau u_n\|^2 \\ &+\tau(KD_\tau u_n, u_n) + \|K^{1/2}u_n\|^2 + \tau(\gamma - \tfrac{1}{2})\|C^{1/2}D_\tau u_n\|^2, \end{aligned} \tag{13}$$

Using (12) repeatedly, we have

$$\begin{array}{lll} w_n & \le & w_{n-1} + \tau(g_n, D_\tau u_n + D_\tau u_{n-1}) \\ w_{n-1} & \le & w_{n-2} + \tau(g_{n-1}, D_\tau u_{n-1} + D_\tau u_{n-2}) \\ \cdots\cdots & & \\ w_{n-m} & \le & w_{n-m-1} + \tau(g_{n-m}, D_\tau u_{n-m} + D_\tau u_{n-m-1}) \\ \cdots\cdots & & \\ w_1 & \le & w_0 + \tau(g_1, D_\tau u_1 + D_\tau u_0). \end{array}$$

Summing up these inequalities, we obtain

$$w_n \le w_0 + \sum_{i=1}^{n} \tau(g_i, D_\tau u_i + D_\tau u_{i-1}). \tag{14}$$

To modify (14), we show the next lemma.

LEMMA 4.3 *Under either the condition*

$$(7) \textit{ and } 0 < \delta \le m, \tag{15}$$

or

$$(8)\ and\ \tau \le \tau_0\ and\ 0 < \delta \le m - \tau_0^2(\frac{1}{2}\gamma - \beta)||K^{1/2}||^2, \tag{16}$$

we have

$$\delta||D_\tau u_i||^2 \le w_i. \tag{17}$$

PROOF. Let δ be a positive number. Using the assumption that M and K are positive definite, C is nonnegative and $\gamma \ge 1/2$, we have

$$\begin{aligned} w_i &- \delta||D_\tau u_i||^2 \\ &\ge ||M^{1/2}D_\tau u_i||^2 + \tau^2(\beta - \tfrac{1}{2}\gamma)||K^{1/2}D_\tau u_i||^2 + \tfrac{1}{4}||K^{1/2}(\tau D_\tau u_i + 2u_i)||^2 - \delta||D_\tau u_i||^2 \\ &\ge (m-\delta)||D_\tau u_i||^2 + \tau^2(\beta - \tfrac{1}{2}\gamma)||K^{1/2}D_\tau u_i||^2 + \tfrac{1}{4}||K^{1/2}(\tau D_\tau u_i + 2u_i)||^2. \end{aligned}$$

If β and γ satisfies the condition (7) and $\delta \le m$, then $w_i - \delta||D_\tau u_i||^2$ becomes nonnegative for any i.

On the other hand, if $\beta - \gamma/2 < 0$, then we have

$$\begin{aligned} w_i &- \delta||D_\tau u_i||^2 \\ &\ge m||D_\tau u_i||^2 - \tau^2(\tfrac{1}{2}\gamma - \beta)||K^{1/2}||^2||D_\tau u_i||^2 - \delta||D_\tau u_i||^2 \\ &= \{m - \tau^2(\tfrac{1}{2}\gamma - \beta)||K^{1/2}||^2 - \delta\}||D_\tau u_i||^2. \end{aligned}$$

If τ_0 satisfies the condition (8) and $\tau \le \tau_0$ then

$$m - \tau^2(\frac{1}{2}\gamma - \beta)||K^{1/2}||^2 - \delta > m - \tau_0^2(\frac{1}{2}\gamma - \beta)||K^{1/2}||^2 - \delta.$$

Hence, if $\delta \le m - \tau_0^2(\frac{1}{2}\gamma - \beta)||K^{1/2}||^2$, then $w_i - \delta||D_\tau u_i||^2 \ge 0$ for any i. Thus we obtain (17). □

Using Lemma 4.3, we have the next lemma.

LEMMA 4.4 *Under either the condition*

$$(7)\ and\ 0 < \delta \le \min\{m, 2/\tau_0\}, \tag{18}$$

or

$$(8)\ and\ \tau \le \tau_0\ and\ 0 < \delta \le \min\{m - \tau_0^2(\tfrac{1}{2}\gamma - \beta)||K^{1/2}||^2, 2/\tau_0\}, \tag{19}$$

we have, for $\tau \le \tau_0$,

$$w_n \le C_0, \quad n = 0, 1, 2, \ldots, N-1. \tag{20}$$

PROOF. Using (17), we modify (14) as follows:

$$\begin{aligned} w_n &\le w_0 + \tau\sum_{i=1}^{n}(g_i, D_\tau u_i + D_\tau u_{i-1}) \\ &\le w_0 + \tau\sum_{i=1}^{n}||g_i||(||D_\tau u_i|| + ||D_\tau u_{i-1}||) \\ &\le w_0 + \sum_{i=1}^{n}\{\frac{\tau}{\delta^2}||g_i||^2 + (\frac{\tau\delta^2}{2}||D_\tau u_i||^2 + \frac{\tau\delta^2}{2}||D_\tau u_{i-1}||^2)\} \\ &\le w_0 + \frac{\delta\tau}{2}\sum_{i=1}^{n}(w_i + w_{i-1}) + \frac{\tau}{\delta^2}\sum_{i=1}^{n}||g_i||^2 \\ &\le w_0 + \frac{\delta\tau}{2}w_n + \delta\tau\sum_{i=0}^{n-1}w_i + \frac{\tau}{\delta^2}\sum_{i=1}^{N-1}||g_i||^2. \end{aligned}$$

If $f(t)$ is bounded on $[0,T]$, then, for $i \geq 1$, we have

$$\begin{aligned} \|g_i\| &= \|\{I + \tau(\gamma - \tfrac{1}{2})D_{\bar\tau} + \beta\tau^2 D_{\tau\bar\tau}\}f_i\| \\ &= \|f_i + (\gamma - \tfrac{1}{2})(f_i - f_{i-1}) + \beta(f_{i+1} - 2f_i + f_{i-1})\| \\ &\leq (4\beta + 2\gamma) \sup_{0\leq t\leq T} \|f(t)\|. \end{aligned}$$

Hence we have

$$\begin{aligned} w_n &\leq w_0 + \frac{\delta\tau}{2} w_n + \delta\tau \sum_{i=0}^{n-1} w_i + \frac{\tau}{\delta^2} \sum_{i=1}^{N-1} \{(4\beta + 2\gamma) \sup_{0\leq t\leq T} \|f(t)\|\}^2 \\ &\leq w_0 + \frac{\delta\tau_0}{2} w_n + \delta\tau \sum_{i=0}^{n-1} w_i + \frac{T}{\delta^2}\{(4\beta + 2\gamma) \sup_{0\leq t\leq T} \|f(t)\|\}^2. \end{aligned}$$

Since $0 < 1 - \delta\tau_0/2$ from (18) or (19), we have

$$w_n \leq (1 - \frac{1}{2}\delta\tau_0)^{-1}[w_0 + \delta\tau \sum_{i=0}^{n-1} w_i + \frac{T}{\delta^2}\{(4\beta + 2\gamma) \sup_{0\leq t\leq T} \|f(t)\|\}^2].$$

Using the Gronwall inequality, we then obtain

$$w_n \leq (1 - \frac{1}{2}\delta\tau_0)^{-1}[w_0 + \frac{T}{\delta^2}\{(4\beta + 2\gamma) \sup_{0\leq t\leq T} \|f(t)\|\}^2] \exp(\delta(1 - \frac{1}{2}\delta\tau_0)^{-1}T),$$

where we use $N\tau = T$. Thus, we can define C_0 as follows

$$C_0 := (1 - \frac{1}{2}\delta\tau_0)^{-1}[w_0 + \frac{T}{\delta^2}\{(4\beta + 2\gamma) \sup_{0\leq t\leq T} \|f(t)\|\}^2] \exp(\delta(1 - \frac{1}{2}\delta\tau_0)^{-1}T), \tag{21}$$

where

$$\begin{aligned} w_0 = \|M^{1/2}D_\tau u_0\|^2 + \tau^2\{\beta - \frac{1}{2}(\gamma - \frac{1}{2})\}\|K^{1/2}D_\tau u_0\|^2 \\ + \tau(KD_\tau u_0, u_0) + \|K^{1/2}u_0\|^2 + \tau(\gamma - \frac{1}{2})\|C^{1/2}D_\tau u_0\|^2. \end{aligned} \tag{22}$$

□

Lastly, under either the condition (7) or (8), using the following inequality with an arbitrary α:

$$\begin{aligned} &\tau(K^{1/2}D_\tau u_n, K^{1/2}u_n) \\ &\quad \geq -\tau\, \|K^{1/2}D_\tau u_n\| \times \sqrt{\alpha} \times \tfrac{1}{\sqrt{\alpha}} \times \|K^{1/2}u_n\| \\ &\quad \geq -\tfrac{1}{2}\{\alpha\tau^2\|K^{1/2}D_\tau u_n\|^2 + \tfrac{1}{\alpha}\|K^{1/2}u_n\|^2\}, \end{aligned}$$

we obtain the energy inequality (6) from (20) in Lemma 4.4. From (14) we have

$$\text{If } f_n = g_n = 0, \text{ then } C_0 = w_0.$$

□

5 Stability conditions for Newmark's method

In this section, using the energy inequality (6) we derive stability conditions for Newmark's method. With respect to a parameter β, we divide the stability condition into two cases and lead to the next theorem.

THEOREM 5.1 *Let M and K be positive definite matrices and C be a nonnegative matrix. Let m and k be the smallest eigenvalues of M and K. Assume $\gamma \geq 1/2$. Then, Newmark's method for (2) in a time interval $[0,T]$ is stable in the following two cases with respect to β:*

Case 1: *If*

$$\frac{1}{2}\gamma < \beta,$$

then with $\tau_0 > 0$ and δ given in (18) we have, for $\tau < \tau_0$ and $N = T/\tau$,

$$\|u_n\| \leq \sqrt{\frac{(4\beta - 2\gamma + 1)}{(4\beta - 2\gamma)k} C_0}, \quad n = 0, 1, 2, \ldots, N. \tag{23}$$

Case 2: *If*

$$0 \leq \beta \leq \frac{1}{2}\gamma,$$

then for τ_0 satisfying

$$0 < \tau_0 < \sqrt{\frac{m}{(\frac{1}{2}\gamma - \beta)\|K^{1/2}\|^2}}, \tag{24}$$

we have for $\tau \leq \tau_0$ and $N = T/\tau$

$$\|u_n\| \leq \|u_0\| + \sqrt{\frac{C_0}{m - \tau_0^2(\frac{1}{2}\gamma - \beta)\|K^{1/2}\|^2}} \tau n, \quad n = 0, 1, 2, \ldots, N. \tag{25}$$

Here, in both cases, C_0 is given by (21) together with (22).

PROOF. We omit the proof. □

REMARK 5.1 *H. Fujii investigated in [4] and [5] the stability condition for the Rayleigh damping case with $C = aK + bM$, $a, b \in \mathbf{R}$.*

6 Relaxation of restrictions on C and K

To extend the applications of the stability theorem in Section 5 and also the convergence theorem in Section 7, we consider the case with weaker assumptions for C and K as in the next lemma. (See pp. 30–33 in [11].)

LEMMA 6.1 *For the second order equation:*

$$M\frac{d^2}{dt^2}u(t) + C\frac{d}{dt}u(t) + Ku(t) = f(t), \tag{26}$$

we assume that M, C and K are symmetric and satisfy the following conditions:

$$\begin{aligned}(Mw, w) &\geq m\|w\|^2, \ (Cw, w) \geq -c\|w\|^2, \\ (Kw, w) &\geq -c\|w\|^2 \ \text{for } w \in \mathbf{R}^d,\end{aligned} \tag{27}$$

where m and c are positive constants. Then, we can transform (26) into the equation:

$$M\frac{d^2}{dt^2}v(t) + (2\lambda M + C)\frac{d}{dt}v(t) + (\lambda^2 M + \lambda C + K)v(t) = e^{-\lambda t}f(t), \tag{28}$$

where

$$v(t) = e^{-\lambda t}u(t), (2\lambda M + C) \geq 0, (\lambda^2 M + \lambda C + K) \geq kI, \tag{29}$$

with positive constants λ and k.

PROOF. We omit the proof. □

So we can obtain the energy inequality based on (28).

7 Convergence of Newmark's method

Using the recurrence relation (4) and the stability theorem, we show in this section the convergence of Newmark's method together with its convergence order.

Let T be a fixed positive number and N be a positive integer, and define $\tau := T/N$. Let $u(t)$ be the solution of (2) and u_n $(n = 0, 1, 2, \dots, N)$ be the solution of Newmark's method for (2). We assume that M and K are positive definite and C is nonnegative.

The discretization error e_n is defined as $e_n := u(\tau n) - u_n$. Then we have the following theorem.

THEOREM 7.1 *We assume that $f \in C^2([0,T])$, then we have the estimate $\|e_n\| = O(\tau^l)$, where*

$$\begin{cases} l = 2 & \text{for } \gamma = \frac{1}{2}, \\ l = 1 & \text{for } \gamma > \frac{1}{2}. \end{cases}$$

PROOF. To prove this theorem, we first show the following two lemmas.

LEMMA 7.2 *At the mesh points $t = \tau n$ with $n = 0, 1$, we have*

$$e_0 = 0,\ e_1 = O(\tau^3).$$

PROOF. Since $u_0 = u(0)$, we have $e_0 = u(0) - u_0 = 0$. Next we estimate e_1. From Step 3 in the iteration scheme in Section 2, we have

$$u_1 = u_0 + \tau v_0 + (\tfrac{1}{2} - \beta)\tau^2 a_0 + \beta\tau^2 a_1,$$

where

$$v_0 \ = \ \tfrac{d}{dt}u(0),\ a_0 = \tfrac{d^2}{dt^2}u(0)$$

and

$$\begin{aligned} a_1 \ = \ & (M + \gamma\tau C + \beta\tau^2 K)^{-1} \\ & \times[f_1 - (C + \tau K)v_0 - Ku_0 + \{(\gamma - 1)\tau C + (\beta - \tfrac{1}{2})\tau^2 K\}a_0]. \end{aligned}$$

On the other hand, by Taylor's theorem, we have

$$\begin{aligned} u(\tau) &= u_0 + \tau v_0 + \tfrac{1}{2}\tau^2 a_0 + O(\tau^3) \\ &= u_0 + \tau v_0 + (\tfrac{1}{2} - \beta)\tau^2 a_0 + \beta\tau^2 a_0 + O(\tau^3). \end{aligned}$$

So, we obtain

$$\begin{aligned} e_1 \ = \ u(\tau) - u_1 &= \beta\tau^2 a_0 + O(\tau^3) - \beta\tau^2 a_1 \\ &= \beta\tau^2(a_0 - a_1) + O(\tau^3), \end{aligned}$$

and from the relation

$$a_0 = M^{-1}(f_0 - Cv_0 - Ku_0)$$

and the above expression of a_1, we have

$$\begin{aligned} a_1 &= (M^{-1}+O(\tau)) \times [(f_0+O(\tau)) - (Cv_0+O(\tau)) - Ku_0 + O(\tau)] \\ &= M^{-1}(f_0 - Cv_0 - Ku_0) + O(\tau). \end{aligned}$$

Thus we obtain $e_1 = O(\tau^3)$. □

At the mesh points $t = \tau n$, $n = 0, 1, 2, \ldots, N$, we have $u_n = u(\tau n) - e_n$ and $f_n = f(\tau n)$. Using the recurrence relation (4), we can prove the next technical lemma.

LEMMA 7.3 *Define p_n as*

$$\begin{aligned} p_n &:= (M+\beta\tau^2 K)D_{\tau\bar\tau}e_n \\ &\quad +[\{\gamma C D_\tau + (1-\gamma)CD_{\bar\tau}\} + \tau(\gamma-\tfrac{1}{2})KD_{\bar\tau}]e_n + Ke_n. \end{aligned} \tag{30}$$

Then it is expressed as follows:

$$p_n = -\tau(\gamma - \tfrac{1}{2})M\tfrac{d^3}{dt^3}u(\tau n) + O(\tau^2). \tag{31}$$

PROOF. Substituting $u(\tau n) - e_n$ for u_n in the recurrence relation (4), we have

$$\begin{aligned} p_n &= (M+\beta\tau^2 K)D_{\tau\bar\tau}e_n \\ &\quad +[\{\gamma C D_\tau + (1-\gamma)CD_{\bar\tau}\} + \tau(\gamma-\tfrac{1}{2})KD_{\bar\tau}]e_n + Ke_n \\ &= (M+\beta\tau^2 K)D_{\tau\bar\tau}u(\tau n) \\ &\quad +[\{\gamma C D_\tau + (1-\gamma)CD_{\bar\tau}\} + \tau(\gamma-\tfrac{1}{2})KD_{\bar\tau}]u(\tau n) + Ku(\tau n) \\ &\quad -\{I + \tau(\gamma-\tfrac{1}{2})D_{\bar\tau} + \beta\tau^2 D_{\tau\bar\tau}\}f(\tau n). \end{aligned}$$

Using the expressions

$$\begin{aligned} D_\tau u(\tau n) &= \{u(\tau(n+1)) - u(\tau n)\}/\tau, \\ D_{\bar\tau} f(\tau n) &= \{f(\tau n) - f(\tau(n-1))\}/\tau, \\ D_{\tau\bar\tau} u(\tau n) &= \{u(\tau(n+1)) - 2u(\tau n) + u(\tau(n-1))\}/\tau^2, \text{ etc.,} \end{aligned}$$

we rewrite the right hand side of the above formula. Applying Taylor's theorem to $u(\tau(n+1))$, $f(\tau(n-1))$, etc. at $t = \tau n$, we have

$$\begin{aligned} p_n &= M\tfrac{d^2}{dt^2}u(\tau n) + C\tfrac{d}{dt}u(\tau n) + Ku(\tau n) - f(\tau n) \\ &\quad +\tau(\gamma-\tfrac{1}{2})(C\tfrac{d^2}{dt^2}u(\tau n) + K\tfrac{d}{dt}u(\tau n) - \tfrac{d}{dt}f(\tau n)) + O(\tau^2). \end{aligned}$$

Using the equalities:

$$M\tfrac{d^2}{dt^2}u(\tau n) + C\tfrac{d}{dt}u(\tau n) + Ku(\tau n) - f(\tau n) = 0$$

and

$$C\tfrac{d^2}{dt^2}u(\tau n) + K\tfrac{d}{dt}u(\tau n) - \tfrac{d}{dt}f(\tau n) = -M\tfrac{d^3}{dt^3}u(\tau n),$$

we obtain (31). □

Proof of Theorem 7.1 (continued). Using Lemma 7.2, 7.3 and the stability theorem we can obtain the estimate of $||e_n||$. To apply Theorem 5.1 to (30) we consider a modification of (21). If we look the proof of Theorem 4.1 again, we can replace $(4\beta+2\gamma)\sup_{0\le t\le T}||f(t)||^2$ with $\sup_{1\le n\le N-1}||p_n||^2$. Then we have in this case

$$C_0 = (1-\tfrac{1}{2}\delta\tau_0)^{-1}\{w_0 + \tfrac{T}{\delta^2}\sup_{1\le n\le N-1}||p_n||^2\}\exp(\delta(1-\tfrac{1}{2}\delta\tau_0)^{-1}T).$$

Hence we have from (23) or (25)

$$\|e_n\| \leq C_1\sqrt{C_0} + \|e_0\| = C_1\sqrt{C_0},$$

where C_1 is independent of u_0, v_0 and p_n. By Lemma 7.2 and (22), where u_0 is replaced by e_0, we have

$$\begin{aligned} w_0 &= \|M^{1/2}D_\tau e_0\|^2 + \tau^2\{\beta - \tfrac{1}{2}(\gamma - \tfrac{1}{2})\}\|K^{1/2}D_\tau e_0\|^2 \\ &\quad +\tau(KD_\tau e_0, e_0) + \|K^{1/2}e_0\|^2 + \tau(\gamma - \tfrac{1}{2})\|C^{1/2}D_\tau e_0\|^2 \\ &= \|M^{1/2}\tfrac{1}{\tau}e_1\|^2 + \tau^2\{\beta - \tfrac{1}{2}(\gamma - \tfrac{1}{2})\}\|K^{1/2}\tfrac{1}{\tau}e_1\|^2 \\ &\quad +\tau(\gamma - \tfrac{1}{2})\|C^{1/2}\tfrac{1}{\tau}e_1\|^2 \\ &= O(\tau^4). \end{aligned}$$

From this we obtain with another constant C_2

$$C_0 \leq C_2\{O(\tau^4) + \sup_{1\leq n\leq N-1}\|p_n\|^2\}.$$

On the other hand, from Lemma 7.3, we have

$$\begin{cases} \sup_{1\leq n\leq N-1}\|p_n\|^2 = O(\tau^4) & \text{for } \gamma = \tfrac{1}{2}, \\ \sup_{1\leq n\leq N-1}\|p_n\|^2 = O(\tau^2) & \text{for } \gamma > \tfrac{1}{2}. \end{cases}$$

Thus we obtain the results. □

REFERENCES

[1] F. Chiba, and T. Kako., *On the stability of Newmark's β method*, RIMS Kôkyûroku 1040, Research Institute of Mathematical Sciences, Kyoto University, 1998, pp. 39–44.

[2] F. Chiba, and T. Kako., *On the stability and convergence of Newmark's β method*, The Institute of Statistical Mathematics Cooperative Report 110 Proceedings of 1997-Workshop on MHD Computations —Numerical methods and optimization techniques in controlled thermonuclear fusion research—, 1998, pp. 196–202.

[3] F. Chiba, and T. Kako., *Stability and error analyses by energy estimate for Newmark's method*, NIFS-PROC-40 Proceeding of 1998-Workshop on MHD Computations "Study on numerical methods related to plasma confinement", National Institute for Fusion Science, 1999, pp. 82–91.

[4] H. Fujii., *Finite-Element Galerkin Method for Mixed Initial-Boundary Value Problems in Elasticity Theory*, Center for Numerical Analysis, The University of Texas at Austin, October 1971.

[5] H. Fujii., *A Note on Finite Element Approximation of Evolution Equations*, RIMS Kôkyûroku 202, Research Institute of Mathematical Sciences, Kyoto University, 1974, pp. 96–117.

[6] J. W. Helton, and R. L. Miller., *NCAlgebra*, Department of Mathematics, University of California San Diego, 1994.

[7] T. Kako., *Spectral and numerical analysis of resistive linearized magnetohydrodynamic operators*, Series on Applied Mathematics Vol.5, Proceedings of the First China-Japan Seminar on Numerical Mathematics, Z.-C. Shi and T. Ushijima eds., Kinokuniya, 1993, pp. 60–66.

[8] M. Matsuki, and T. Ushijima., *Error estimation of Newmark method for conservative second order linear evolution*, Proc. Japan Acad., Vol. 69, Ser A (1993), pp. 219–223.

[9] N. M. Newmark., *A method of computation for structural dynamics*, Proceedings of the American Society of Civil Engineers, Journal of the Engineering Mechanics Division, Vol.85, No. EM 3 (1959), pp. 67–94.

[10] P. A. Raviart, and J. M. Thomas., *Introduction à l' Analyse Numérique des Equations aux Dérivées Partielles*, Masson, Paris, 1983.

[11] R. E. Showalter., *Mathematical Surveys and Monographs Vol. 49 Monotone Operators in Banach Space and Nonlinear Partial Differential Equations*, AMS, 1997

[12] S. Wolfram., *The Mathematica Book, Third Edition*, Addison Wesley, 1998.

[13] W. L. Wood., *A further look at Newmark, Houbolt, etc., time-stepping formulate*, International Journal for Numerical Methods in Engineering, Vol. 20 (1984), pp. 1009–1017.

Numerical Conformal Mapping by the Charge Simulation Method *

Kaname AMANO, Dai OKANO and Hidenori OGATA
Department of Computer Science, Faculty of Engineering, Ehime University

Abstract
We present a method of numerical conformal mapping of a bounded Jordan domain onto the unit disk, which is a basic problem of conformal mapping. We reduce the mapping problem to a Dirichlet problem of the Laplace equation with a pair of conjugate harmonic functions and employ the charge simulation method, where the conjugate harmonic functions are approximated by a linear combination of complex logarithmic potentials. We give some schemes of approximating mapping function which is continuous and analytic in the problem domain using the principal value of logarithmic function in computation. Numerical examples show that high accuracy is obtained if the problem domain has no reentrant corners.

Keywords: numerical conformal mapping, Riemann's mapping theorem, charge simulation method, continuity, analyticity

1 Introduction

Conformal mappings are familiar in science and engineering. However exact mapping functions are not known except for some special domains. The numerical conformal mapping has been an attractive subject in scientific computation.

Symm [29, 30, 31] proposed an integral equation method of numerical conformal mappings of interior, exterior and doubly-connected Jordan domains onto the unit disk, its exterior and a circular annulus, respectively. He expressed a pair of conjugate harmonic functions by a complex single-layer logarithmic potential and reduced the mapping problem to the singular Fredholm integral equation of the first kind. The integral equation method was improved by Hayes, Kahaner and Kellner [16] and Hough and Papamichael [18, 19]. Amano [1, 2, 3, 6], with these points as background, proposed a charge simulation method of numerical conformal mappings of the interior, exterior and doubly-connected domains. He approximated the conjugate harmonic functions by a linear combination of complex logarithmic potentials and reduced the mapping problem directly to a system of simultaneous linear equations.

See Gaier [15], Henrici [17], Trefethen [32] and Kythe [25] for surveys of numerical conformal mappings.

We here present some schemes of approximating mapping function of the numerical conformal mapping of a bounded Jordan domain onto the unit disk. The approximate mapping function is continuous and analytic in the problem domain using the principal value of logarithmic function.

2 Charge Simulation Method

The charge simulation method is a solver for potential problems. We have two schemes, i.e., the conventional scheme [26, 28] and the invariant scheme [27].

Consider the two-dimensional Dirichlet problem of the Laplace equation

$$\Delta g(z) = 0 \quad \text{in } D, \tag{1}$$
$$g(z) = b(z) \quad \text{on } C, \tag{2}$$

where D is the problem domain with the boundary C and $b(z)$ is the boundary data. We abbreviate (x, y) as (z) for $z = x + iy$.

In the conventional scheme, the solution is approximated by a linear combination of logarithmic potentials

$$G(z) = \sum_{i=1}^{N} Q_i \log|z - \zeta_i|, \tag{3}$$

where the points $\zeta_1, \zeta_2, \ldots, \zeta_N$, called *charge points*, are placed outside D. The unknown constants $Q_1, Q_2, \ldots, Q_N$, called *charges*, are determined to satisfy the boundary condition (2) at the same number of points $z_1, z_2, \ldots, z_N$, called *collocation points*, placed on C. That is to say, they are solutions of the simultaneous linear equations

$$\sum_{i=1}^{N} Q_i \log|z_j - \zeta_i| = b(z_j) \quad (j = 1, 2, \ldots, N), \tag{4}$$

*Supported by the Grant-in-Aid for Scientific Research of the Ministry of Education, Science, Sports and Culture in Japan (09440081).

which are called *collocation condition*. This simple approximation can be highly accurate, but does not remain invariant with respect to trivial affine transformations [27].

In the invariant scheme, the solution is approximated by

$$G(z) \simeq Q_0 + \sum_{i=1}^{N} Q_i \log|z - \zeta_i|, \tag{5}$$

and the constant term Q_0 and the charges $Q_1, Q_2, \ldots, Q_N$ are determined under the constraint $\sum_{i=1}^{N} Q_i = 0$. That is to say, they are solutions of the simultaneous linear equations

$$\sum_{i=1}^{N} Q_i \log|z_j - \zeta_i| = b(z_j) \quad (j = 1, 2, \ldots, N), \tag{6}$$

$$\sum_{i=1}^{N} Q_i = 0. \tag{7}$$

It is called the *invariant scheme* and satisfies mathematically nice and physically natural properties.

The approximation $G(z)$ exactly satisfies the Laplace equation. If D is bounded, the maximum principle for harmonic functions tells us that the error takes its maximum value somewhere on C and is estimated as

$$\begin{aligned} |G(z) - g(z)| &\leq \max_{z \in C} |G(z) - b(z)| \\ &\simeq \max_{1 \leq j \leq N} |G(z_{j+1/2}) - b(z_{j+1/2})|, \end{aligned} \tag{8}$$

where $z_{j+1/2}$ is an intermediate point on C between the collocation points z_j and z_{j+1}. It is known that the error decays exponentially with respect to N if C is smooth and $b(z)$ is analytic [20, 21, 22, 23, 24].

We extend the charge simulation method to the complex function and, in application to the numerical conformal mapping, approximate a pair of conjugate harmonic functions by a linear combination of complex logarithmic potentials.

3 Problem and Theorem

We are concerned here with the basic problem of conformal mapping, i.e., the mapping of a domain D bounded by a closed Jordan curve C given in the z-plane onto the unit disk $|w| < 1$ in the w-plane.

Theorem 1 (Riemann) *The mapping function $w = f(z; z_0)$ is uniquely determined by the normalization conditions $f(z_0; z_0) = 0$ and $f'(z_0; z_0) > 0$, where z_0 is an arbitrary point in D and is called the normalization point.*

We take $z_0 = 0$ and abbreviate $f(z; 0)$ as $f(z)$, which does not lose generality. Then the normalization conditions are $f(0) = 0$ and $f'(0) > 0$.

Figure 1 shows the situation of the problem.

4 Numerical Method

4.1 Conventional Scheme

The mapping function is expressed as

$$f(z) = z \exp(g(z) + \mathrm{i}h(z)), \tag{9}$$

where $g(z)$ and $h(z)$ are conjugate harmonic functions in D. They should satisfy the boundary condition $|f(z)| = 1$ $(z \in C)$ and the normalization condition $f'(0) > 0$, i.e.,

$$g(z) = -\log|z| \quad (z \in C) \tag{10}$$

and

$$h(0) = 0, \tag{11}$$

respectively. From (9), the normalization condition $f(0) = 0$ is satisfied. Conversely, if (10) and (11) are satisfied, (9) is the mapping function of the problem. The uniqueness of the solution tells us that the problem is reduced to finding the conjugate harmonic functions $g(z)$ and $h(z)$ satisfying (10) and (11).

Amano [1, 6] applied the conventional scheme of the charge simulation method (3) to $g(z) + \mathrm{i}h(z)$ and obtained the following scheme of the numerical conformal mapping.

Scheme 1 *The approximate mapping function is expressed as*

$$F(z) = z \exp(G(z) + \mathrm{i}H(z)), \tag{12}$$

$$\begin{aligned} G(z) + \mathrm{i}H(z) &= \sum_{i=1}^{N} Q_i \log(z - \zeta_i) + \mathrm{i}\Theta_0 \\ &= \sum_{i=1}^{N} Q_i \left\{ \log|z - \zeta_i| + \mathrm{i} \arg\left(1 - \frac{z}{\zeta_i}\right) \right\}, \end{aligned} \tag{13}$$

where Θ_0 is the constant of rotation determined by the normalization condition (11). The charges $Q_1, Q_2, \ldots, Q_N$ are the solutions of the N simultaneous linear equations

$$\sum_{i=1}^{N} Q_i \log|z_j - \zeta_i| = -\log|z_j| \quad (j = 1, 2, \ldots, N). \tag{14}$$

4.2 New Continuous Schemes

The same mapping function is also expressed as

$$f(z) = \frac{z}{r_D} \exp(g(z) + \mathrm{i}h(z)), \tag{15}$$

where r_D is a constant to be $f'(0) = 1/r_D > 0$, i.e., the mapping radius of D at $z = 0$ (strictly speaking, at the normalization point $z = z_0$). The boundary condition $|f(z)| = 1$ $(z \in C)$ and the normalization condition $f'(0) = 1/r_D$ are

$$g(z) - \log r_D = -\log|z| \quad (z \in C) \tag{16}$$

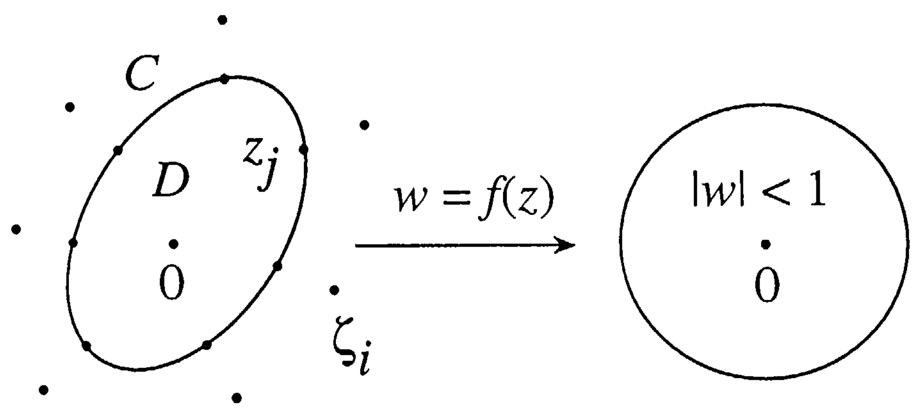

Figure 1: Conformal mapping by the charge simulation method. The normalization conditions are $f(0) = 0$ and $f'(0) > 0$, and ζ_i and z_j are the charge points and the collocation points, respectively.

and

$$g(0) + \mathrm{i}h(0) = 0, \tag{17}$$

respectively. The problem is reduced to finding $g(z)$ and $h(z)$ satisfying (16) and (17).

We apply the invariant scheme of the charge simulation method (5) to $g(z) + \mathrm{i}h(z)$ under the constraint

$$\sum_{i=1}^{N} Q_i = -1 \tag{18}$$

instead of (7) as proposed by Amano and Inoue [11], and have the approximate mapping function

$$F(z) = \frac{z}{R_D}\exp(G(z) + \mathrm{i}H(z)), \tag{19}$$

$$G(z) + \mathrm{i}H(z) = Q_0 + \sum_{i=1}^{N} Q_i \log(z - \zeta_i), \tag{20}$$

where Q_0 is a complex constant.

4.2.1 Starlike Case

Assume that C is starlike with respect to the origin. We rewrite (20) to

$$G(z) + \mathrm{i}H(z) = Q_0 + \sum_{i=1}^{N} Q_i \left\{\log\left(1 - \frac{z}{\zeta_i}\right) + \log(-\zeta_i)\right\} \tag{21}$$

for $H(z)$ to be continuous in D using the principal value of logaritmic function. Note that discontinuity of $\mathrm{Arg}(1 - z/\zeta_i)$ appears on the radial line behind ζ_i. The normalization condition (17) requires

$$G(0) + \mathrm{i}H(0) = Q_0 + \sum_{i=1}^{N} Q_i \log(-\zeta_i) = 0. \tag{22}$$

We eliminate Q_0 from (21) and (22), and obtain the following scheme of the numerical conformal mapping.

Scheme 2 *If C is starlike with respect to the origin, the approximate mapping function is expressed*

$$F(z) = \frac{z}{R_D}\exp(G(z) + \mathrm{i}H(z)), \tag{23}$$

$$G(z) + \mathrm{i}H(z) = \sum_{i=1}^{N} Q_i \log\left(1 - \frac{z}{\zeta_i}\right), \tag{24}$$

where the charges $Q_1, Q_2, \ldots, Q_N$ and the mapping radius r_D are the solutions of the $N + 1$ simultaneous linear equations

$$\sum_{i=1}^{N} Q_i \log\left|1 - \frac{z_j}{\zeta_i}\right| - \log R_D = -\log|z_j| \tag{25}$$

$$(j = 1, 2, \ldots, N),$$

$$\sum_{i=1}^{N} Q_i = -1. \tag{26}$$

If C is starlike with respect to a point z_s in D, which may be different from the the normalization point z_0, we obtain the following scheme of the numerical conformal mapping in a similar way.

Scheme 3 *If C is starlike with respect to z_s in D, the approximate mapping function is expressed as*

$$F(z) = \frac{z}{R_D}\exp(G(z) + \mathrm{i}H(z)), \tag{27}$$

$$G(z) + \mathrm{i}H(z) = \sum_{i=1}^{N} Q_i \left\{\log\left(1 - \frac{z - z_s}{\zeta_i - z_s}\right) - \log\left(1 - \frac{-z_s}{\zeta_i - z_s}\right)\right\}, \tag{28}$$

where the charges $Q_1, Q_2, \ldots, Q_N$ and the mapping radius r_D are the solutions of the $N+1$ simultaneous linear equations

$$\sum_{i=1}^{N} Q_i \left\{ \log\left|1 - \frac{z_j - z_s}{\zeta_i - z_s}\right| - \log\left|1 - \frac{-z_s}{\zeta_i - z_s}\right| \right\} - \log R_D = -\log|z_j| \quad (j = 1, 2, \ldots, N), \tag{29}$$

$$\sum_{i=1}^{N} Q_i = -1. \tag{30}$$

4.2.2 General Case

If C is not starlike, using the constraint (18), we rewrite (20) to

$$\begin{aligned} G(z) + \mathrm{i}H(z) &= Q_0 + Q_1 \log(z - \zeta_1) \\ &\quad + \sum_{i=2}^{N} \left(\sum_{k=1}^{i} Q_k - \sum_{k=1}^{i-1} Q_k \right) \log(z - \zeta_i) \\ &= Q_0 + \sum_{i=1}^{N-1} \left(\sum_{k=1}^{i} Q_k \right) (\log(z - \zeta_i) - \log(z - \zeta_{i+1})) \\ &\quad + \left(\sum_{k=1}^{N} Q_k \right) \log(z - \zeta_N) \\ &= Q_0 + \sum_{i=1}^{N-1} \left(\sum_{k=1}^{i} Q_k \right) \log\left(\frac{z - \zeta_i}{z - \zeta_{i+1}} \right) \\ &\quad - \log(z - \zeta_N) \end{aligned} \tag{31}$$

for $H(z)$ to be continuous in D using the principal value of logarithmic function. The discontinuity of $\mathrm{Arg}((z - \zeta_i)/(z - \zeta_{i+1}))$ appears on the straight line connenting ζ_i and ζ_{i+1}. The discontinuity of $\log(z - \zeta_N)$ term is mentioned later. The normalization condition (17) requires

$$\begin{aligned} &G(0) + \mathrm{i}H(0) \\ &= Q_0 + \sum_{i=1}^{N-1} \left(\sum_{k=1}^{i} Q_k \right) \log\left(\frac{\zeta_i}{\zeta_{i+1}} \right) - \log(-\zeta_N) \\ &= 0. \end{aligned} \tag{32}$$

We eliminate Q_0 from (31) and (32), and obtain the following scheme of the numerical conformal mapping.

Scheme 4 *Whether C is starlike or not, the approximate mapping function is expressed as*

$$F(z) = \frac{z}{R_D} \exp(G(z) + \mathrm{i}H(z)), \tag{33}$$

$$\begin{aligned} &G(z) + \mathrm{i}H(z) = \\ &\sum_{i=1}^{N-1} Q^i \left\{ \log\left(\frac{z - \zeta_i}{z - \zeta_{i+1}} \right) - \log\left(\frac{\zeta_i}{\zeta_{i+1}} \right) \right\} \\ &- \log\left(1 - \frac{z}{\zeta_N}\right), \end{aligned} \tag{34}$$

where the unknown constants, the partial sums of charges,

$$Q^i = \sum_{k=1}^{i} Q_k \quad (i = 1, 2, \ldots, N-1) \tag{35}$$

and the mapping radius r_D are the solutions of the N simultaneous linear equations

$$\begin{aligned} &\sum_{i=1}^{N-1} Q^i \left(\log\left|\frac{z_j - \zeta_i}{z_j - \zeta_{i+1}}\right| - \log\left|\frac{\zeta_i}{\zeta_{i+1}}\right| \right) - \log R_D \\ &= \log\left|1 - \frac{z_j}{\zeta_N}\right| - \log|z_j| \\ &\qquad (j = 1, 2, \ldots, N). \end{aligned} \tag{36}$$

The charge point ζ_N should be placed for discontinuity of $\mathrm{Arg}(1 - z/\zeta_N)$ not to intersect D also in this scheme.

5 Some Remarks

1. We can obtain the mapping radius with Scheme 1 by

$$R_D = \exp\left(-\sum_{i=1}^{N} Q_i \log|-\zeta_i| \right). \tag{37}$$

   However, it is advisable to use schemes with a constant term [11].

2. The maximum modulus theorem for analytic functions tells us that the error takes its maximum value somewhere on C and is estimated as

$$\begin{aligned} E_F(z) &= |F(z) - f(z)| \\ &\le \max_{z \in C} |F(z) - f(z)| = E_F. \end{aligned} \tag{38}$$

   The collocation condition means $|F(z_j)| = 1$ $(j = 1, 2, \ldots, N)$, so that

$$\begin{aligned} E_M &= \max_{z \in C} ||F(z)| - 1| \\ &\simeq \max_{1 \le j \le N} ||F(z_{j+1/2})| - 1|. \end{aligned} \tag{39}$$

   Many examples imply

$$E_F \simeq E_M, \tag{40}$$

   which is useful to estimate errors when analytical solutions are unknown.

3. A simple method of charge placement is

$$\begin{aligned} \zeta_j &= z_j + \frac{q}{2}|z_{j+1} - z_{j-1}| \\ &\quad \cdot \exp\left\{ \mathrm{i}\left(\arg(z_{j+1} - z_{j-1}) - \frac{\pi}{2} \right) \right\}, \end{aligned} \tag{41}$$

   where $q > 0$ is a parameter called *assignment factor*. It gives numerical results of high accuracy in many problems [5].

6 Numerical Examples

We show numerical examples on typical problems.

1. an eccentric circle: $|z - x_0| < 1$ ($x_0 = 0.75$),

$$\begin{aligned}(a)\ z_j &= x_0 + e^{i\theta_j},\\ (b)\ z_j &= r_j e^{i\theta_j},\\ r_j &= x_0 \cos\theta_j + \sqrt{1 - x_0^2 \sin^2\theta_j},\\ \theta_j &= 2\pi(j-1)/N.\end{aligned}$$

2. an ellipse: $x^2/a^2 + y^2 < 1$ ($a = 4$),

$$\begin{aligned}(a)\ z_j &= a\cos\theta_j + i\sin\theta_j,\\ (b)\ z_j &= r_j e^{i\theta_j},\\ r_j &= a/\sqrt{\cos^2\theta_j + a^2\sin^2\theta_j},\\ \theta_j &= 2\pi(j-1)/N.\end{aligned}$$

3. a Cassini's oval: $|z^2 - 1| < a^2$ ($a = 1.0219$),

$$\begin{aligned}z_j &= r_j e^{i\theta_j},\\ r_j &= \sqrt{\cos 2\theta_j + \sqrt{\cos^2 2\theta_j + a^4 - 1}},\\ \theta_j &= 2\pi(j-1)/N,\end{aligned}$$

4. a square: $\{(|x| < 1) \cap (|y| < 1)\}$,

$$|z_{j+1} - z_j| = 8/N, \quad z_1 = 1.$$

5. an L-shaped polygon: $\{(|x-0.5| < 1) \cap (|y-0.5| < 1)\} - \{(x \geq 0.5) \cap (y \geq 0.5)\}$,

$$|z_{j+1} - z_j| = 8/N, \quad z_1 = 0.5 + 0.5i.$$

We use (41) for charge placement, and compute

$$\begin{aligned}E_M &= \max_{1\leq j\leq N}(|F(z_{j+1/2})| - 1),\\ E_F &= \max_{1\leq j\leq N}\{|F(z_j) - f(z_j)|,\\ &\qquad |F(z_{j+1/2}) - f(z_{j+1/2})|\}\end{aligned}$$

for error estimation.

Table 1 shows the numerical results by Scheme 1. Errors are estimated by E_F for the eccentric circle and the Cassini's oval since thier analytical solutions are known, and by E_M for the others. The relation (40) holds in the former cases. They show that high accuracy is obtained if the problem domain has no reentrant corners. The error decays exponetially with respect to N in the problems 1-4. As q increases, i.e., the charges move away from the boundary, the error first increases and then decreases till the coefficient matrix of the linear equations becomes numerically singular.

We can obtain an approximate inverse mapping function $z = F^{-1}(w)$ by the same algorithm using the boundary correspondence between z_j and $F(z_j)$ established in advance [4]. Figure 2 is an example of the bidirectional numerical conformal mapping by the charge simulation method.

7 Concluding Remarks

The charge simulation method has first been applied to the numerical conformal mappings of interior, exterior and doubly-connected domains onto the unit disk, its exterior and a circular annulus, respectively [1, 2, 3, 6]; and then mappings of unbounded multiply-connected domains onto the parallel, circular and radial slit domains [7, 8, 9, 10, 12]. The latter three problems are important in the two-dimensional potential flow analysis [13].

The method presented here for the mapping of a bounded Jordan domain onto the unit disk has the following advantages.

- High accuracy by simple computation for domains with curved boundaries.
- Explicit forms of approximate mapping function continuous and analytic in the problem domain using the principal value of logarithmic function in computation.

These characteristics are important from applicational viewpoints. See Amano, Okano and Ogata [14] for details of the continuity problem of the approximate mapping function.

Numerical experiments by Schemes 2-4 should be made in future studies.

References

[1] Amano, K.: Numerical conformal mapping based on the charge simulation method (in Japanese), Trans. Inform. Process. Soc. Japan, **28** (7), 697-704 (1987).

[2] Amano, K.: Numerical conformal mapping of exterior domains based on the charge simulation method (in Japanese), Trans. Inform. Process. Soc. Japan, **29** (1), 62-72 (1988).

[3] Amano, K.: Numerical conformal mapping of doubly-connected domains by the charge simulation method (in Japanese), Trans. Inform. Process. Soc. Japan, **29** (10), 914-924 (1988).

[4] Amano, K.: A bidirectional method for numerical conformal mapping based on the charge simulation method (in Japanese), J. Inform. Process. Soc. Japan, **31** (5), 623-632 (1990).

[5] Amano, K.: Error properties of numerical conformal mapping based on the charge simulation method (in Japanese), Trans. Inform. Process. Soc. Japan, **32** (1), 1-10 (1991).

[6] Amano, K.: A charge simulation method for the numerical conformal mapping of interior, exterior and doubly-connected domains, J. Comput. Appl. Math., **53** (3), 353-370 (1994).

Table 1: Numerical results with $N = 64$ simulation charges.

1. an eccentric circle E_F	(a)	2.7E-05	2.6E-06	6.2E-06	7.5E-06
	(b)	4.4E-07	1.0E-10	1.3E-13	8.7E-12
q		2	4	6	8
2. an ellipse E_M	(a)	2.7E-05	1.2E-05	1.5E-05	1.5E-05
	(b)	4.5E-05	1.8E-06	1.6E-08	1.8E-09
q		2	4	6	8
3. a Cassini's oval E_F		1.3E-04	1.0E-05	1.0E-06	9.3E-05
q		1	2	3	4
4. a square E_M		1.6E-05	2.1E-07	2.4E-09	1.5E-10
q		3	6	9	12
5. an L-shaped polygon E_M		9.5E-02			
q		0.5			

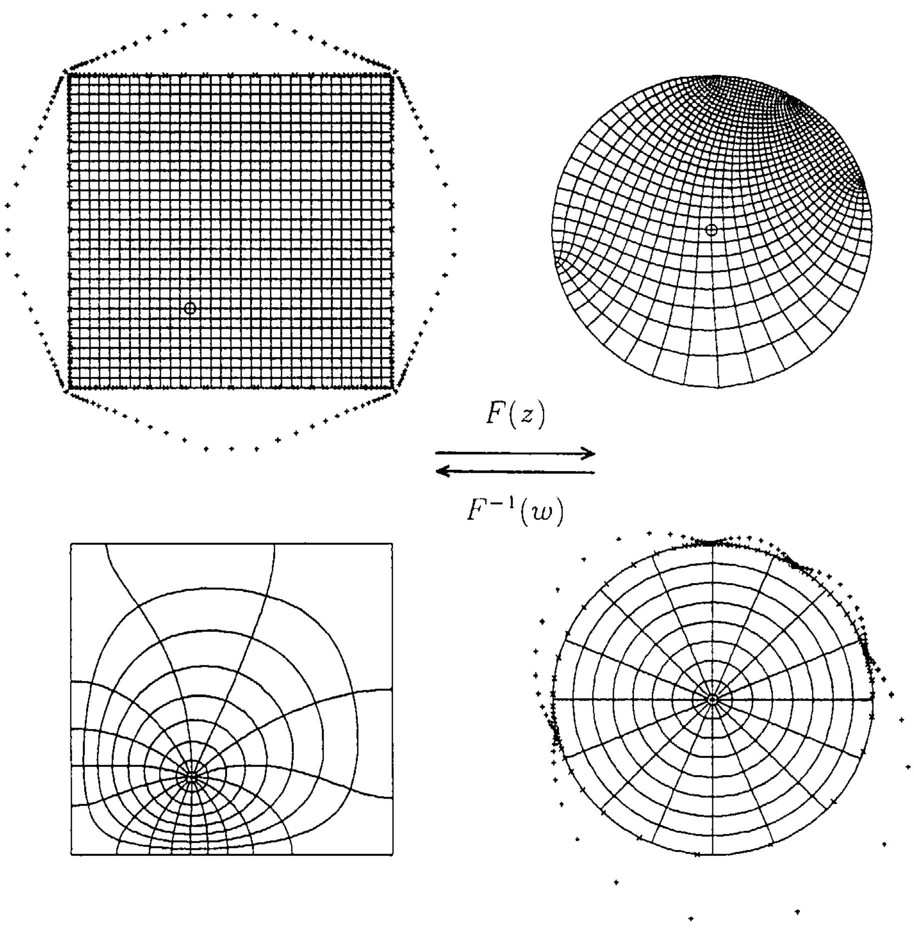

Figure 2: Bidirectional numerical conformal mapping between a square domain and the unit disk.

[7] Amano, K.: Numerical conformal mapping onto the circular slit domains (in Japanese), Trans. Inform. Process. Soc. Japan, **36** (2), 219–225 (1995).

[8] Amano, K.: Numerical conformal mapping onto the radial slit domains by the charge simulation method (in Japanese), Trans. Japan SIAM, **5** (3), 267–280 (1995).

[9] Amano, K., Shibuya, Y., Tsuchie, M. and Sugihara, M.: Numerical conformal mapping onto the parallel slit domains by the charge simulation method (in Japanese), Trans. Japan SIAM, **6** (4), 353–371 (1996).

[10] Amano, K.: A charge simulation method for numerical conformal mapping onto circular and radial slit domains, SIAM J. Sci. Comput., **19** (4), 1169–1187 (1998).

[11] Amano, K. and Inoue, T.: Dilatation invariance of the numerical conformal mapping by the charge simulation method (in Japanese), Trans. Japan SIAM, **8** (1), 1–17 (1998).

[12] Amano, K., Okano, D., Shimohira, H. and Sugihara, M.: Numerical conformal mapping onto parallel, circular and radial slit domains, Advances in Numerical Mathematics (Proceedings of The Fourth Japan-China Joint Seminar on Numerical Mathematics, Edited by Kawarada, H., Nakamura M. and Shi, Z.), Mathematical Sciences and Applications, **12**, 1–10 (1999).

[13] Amano, K., Okano, D., Shimohira, H., Okamoto, T. and Igaue Y.: Potential flow analysis by the numerical conformal mapping, Information, **3** (1), 73–88 (2000, to appear).

[14] Amano, K., Okano, D. and Ogata, H.: Numerical conformal mapping onto the unit disk with concentric circular slits by the charge simulation method (in Japanese), Trans. Inform. Process. Soc. Japan, **41** (4), (2000, to appear).

[15] Gaier, D.: Konstruktive Methoden der konformen Abbildung, Springer-Verlag, Berlin (1964).

[16] Hayes, J.K., Kahaner, D.K. and Kellner, R.G.: An improved method for numerical conformal mapping, Math. Comput, **26** (118), 327–334 (1972).

[17] Henrici, P.: Applied and Computational Complex Analysis **3**, John Wiley & Sons, New York (1986).

[18] Hough, D.M. and Papamichael, N.: The use of splines and singular functions in an integral equation method for conformal mapping, Numer. Math., **37**, 133–147 (1981).

[19] Hough, D.M. and Papamichael, N.: An integral equation method for the numerical conformal mapping of interior, exterior and doubly-connected domains, Numer. Math, **41**, 287–307 (1983).

[20] Katsurada, M.: A mathematical study of the charge simulation method II, J. Fac. Sci. Univ. Tokyo Sect. IA Math., **36** (1), 135–162 (1989).

[21] Katsurada, M.: Asymptotic error analysis of the charge simulation method in a Jordan region with an analytic boundary, J. Fac. Sci. Univ. Tokyo Sect. IA Math., **37** (3), 635–657 (1990).

[22] Katsurada, M.: Charge simulation method using exterior mapping functions, Japan J. Indust. Appl. Math., **11** (1), 47–61 (1994).

[23] Katsurada, M. and Okamoto, H.: A mathematical study of the charge simulation method I, J. Fac. Sci. Univ. Tokyo Sect. IA Math., **35** (3), 507–518 (1988).

[24] Katsurada, M. and Okamoto, H.: The collocation points of the fundamental solution method for the potential problem, Computers Math. Appl., **31** (1), 123–137 (1996).

[25] Kythe, P.K.: Computational Conformal Mapping, Birkhaüser, Boston (1998).

[26] Murashima, S.: Charge Simulation Method and Its Applications (in Japanese), Morikita, Tokyo (1983).

[27] Murota, K.: On "invariance" of schemes in the fundamental solution method (in Japanese), Trans. Inform. Process. Soc. Japan, **34** (3), 533–535 (1993).

[28] Okamoto, H. and Katsurada, M.: A rapid solver for the potential problems (in Japanese), Bull. Japan SIAM, **2** (3), 2–20 (1992).

[29] Symm, G.T.: An integral equation method in conformal mapping, Numer. Math., **9**, 250–258 (1966).

[30] Symm, G.T.: Numerical mapping of exterior domains, Numer. Math., **10**, 437–445 (1967).

[31] Symm, G.T.: Conformal mapping of doubly-connected domains, Numer. Math., **13**, 448–457 (1969).

[32] Trefethen, L.N. (ed.): Numerical Conformal Mapping, North-Holland, Amsterdam (1986).