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QUANTIZATION AND HALL EFFECT: NECESSITIES AND DIFFICULTIES

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ABSTRACT

The quantization procedure is a necessary tool for a proper understanding of many interesting quantum phenomena in modern physics. In this note, we focus on geometrical framework for such procedures, particularly the group-theoretic approach and their difficulties. Finally we look through the example of Hall effect as a quantized macroscopic phenomenon with group-theoretic quantisation approach.
Keywords: quantization, group theory, quantum Hall effect.

Introduction:

Quantization is the problem of constructing the quantum formulation of a system from its classical description, or in other words it is the method by which one makes the transition from classical mechanics to quantum mechanics, satisfying the classical limit as $h \to 0$ where h is the Planck's constant. In spite of being an old subject, it still has a pivotal role in modern physics either as a tool for understanding physical phenomena or as a field of research as there is no one comprehensive quantizing solution for all situations. Many quantization schemes have been developed over the years such as the first one canonical quantization,[1] geometric quantization,[2] deformation quantization,[3] path integral quantization[4] and the group theoretical quantization.[5,6] Despite their differences all these schemes are based more on less on the same axioms of quantum theory. For completeness, we list the axioms as:

A1) To each physical system corresponds a Hilbert space H, a physical state is represented by a vector $\psi \in H$.

A2) Each physical observable f is represented by a self-adjoint operator $\hat{f}$ on H, the spectrum of $\hat{f}$ is the set of possible values of f.

A3) Quantum mechanics does not predict in general the value of the observable f for the state ψ ; it gives only the expectation value:

$$\langle f \rangle_\psi = \frac{\langle \psi, f\psi \rangle}{\langle \psi, \psi \rangle} \qquad (1)$$

A4) Time evolution of the state vector from t_1 to t_2 define by a unitary operator U in the absence of an external influence as:

$$\psi(t_2) = U(t_2 - t_1)\psi(t_1)\,; \qquad (2)$$

where the unitary operator U(t) is generated by a self-adjoint generator $\hat{H}$ called the Hamiltonian as follows:

$$U(t) = \exp\left(-it\hat{H}\right); \qquad (3)$$

Given these axioms there are usually two points of departure in any quantization program [7]:

i) Start with the system localized on the configuration manifold M and makes the replacement of the classical Poisson bracket by the quantum commutator bracket as in the case of the canonical quantization
ii) Start with the system localized on the phase space Γ, or more generally, on an arbitrary symplectic manifold and then make the transition to quantum theory. This is the case of the group theoretic quantisation which we shall briefly describe in the second section. The third section is devoted to quantum Hall effect, starting with its brief review and followed by the quantization of this phenomena within the group theoretic quantization scheme.

Group Theoretic Quantization Program:

Group theoretic quantization (GTQ) arises as an answer of the following question.[8] Given a classical dynamical system with a phase space Γ (which may be a general symplectic manifold, not merely a vector space) what is the equivalent of the canonical commutation relations? GTQ is based on the basic idea that the quantum state space of such system should be an irreducible representation of the canonical group of symplectomorphisms of the phase space. Working on symplectic manifolds, group-theoretic quantization[5,8,9] shares the same tool as geometric quantization[10] in that it uses symplectic geometry to describe the classical phase space. Given a configuration space Q of a particle, its cotangent bundle T^*Q forms the momenta phase space Γ of the particle. To convert the phase space into a symplectic manifold (Γ, ω) a symplectic structure needs to be identified. This structure is simply a non-degenerate closed two-form on the manifold. It differs however from geometric quantization by not assuming the necessity of a quantization bundle and the necessity of the state space as functions over the configuration (or half of the phase space). The following steps summarizes the program based on the work of Isham.[5] It is worth noting that one can reorder the steps below without changing the essential behaviour of the program; we can start with step three rather than step one which is more convenient when there is an external magnetic field put on the system.[11]

Step 1:

It was argued[5] that the first step in constructing the correct canonical algebra lies in finding the Lie group G that acts on Γ transitively and symplectically. With the Lie algebra $L(G)$ of the group one can identify the one-parameter subgroup of the symplectic transformations. This group will generate a family of vector fields ξ^A on Γ with $A \in L(G)$, and they satisfy the vector field commutation relations $[\xi^A, \xi^B] = \xi_{[AB]}$ where [AB] denotes the Lie bracket of A and B in $L(G)$.

Step 2:

Vector fields of this type are locally Hamiltonian that is to each ξ^A there will be an exact 1-form (observable) $f^A \in C^\infty(\Gamma, \Re)$ such that:

$$\omega^A = \xi^A \lrcorner \omega = df^A \qquad (4)$$

with ω as the given symplectic structure on Γ. The Poisson bracket algebra of these observables should then be isomorphic to $L(G)$.

Step 3:

The set of observables $\{f^A, A \in L(G)\}$ should satisfy the requirement that the set has a closed Poisson bracket algebra and large enough to generate any other observables in $C^\infty(\Gamma, \Re)$. An important obstacle called cocycle obstruction may arise here in the closure condition of Poisson algebra. One should circumvent this difficulty by going to a (central) extension of $L(G)$ for otherwise the quantization process breaks down. This is usually the case for systems with external fields with the 'quantization bundle' appearing as a possible solution.

Step 4:
Once the canonical group is identified with all the requirements mentioned before satisfied, the pivotal stage in the quantization (kinematical quantization) consists in finding and classifying self-adjoint representations of $L(G)$ or the unitary irreducible representations of G (or their universal covers).

Hall Effect Within GTQ Scheme:

In order to make this note self-contained, we will give a brief review of Hall effect. Hall effect is observed in a system of charged particle moving under the influence of a constant magnetic field perpendicular on the plane of motion more than a century ago in the context of classical electrodynamics. A transversal conductance has been observed to appear due to the effect of the magnetic field. This conductance has been called Hall conductance and the whole phenomena is called Hall effect. Mathematically the Hall (transversal) conductance can be given by

$$\sigma_{xy} = ne/B \qquad (5)$$

A century after that i.e. in 1980, Klitzing[12] the Hall effect into the quantum domain where he observed that under high magnetic fields (>15T) and low temperatures (~0^0K) the Hall conductance is quantized as:

$$\sigma_{xy} = \nu \frac{e^2}{h} \qquad (6)$$

where ν being an integer known as the integral filling factor.

Two years later, Tsui and Stomer[13] observed the same phenomena but now with a fractional filling factor (ν); this case is called fractional quantum Hall effect (FQHE). In fact this phenomena received much attention from physicists and mathematicians due to the variety of ideas and mathematical structures that entered into the physics of two-dimensional condensed matter systems. In order to give a theoretical explanation of QHE, one can go back to the Landau problem within Schrodinger picture and work with the system of an impurity in the Hall sample i.e. a potential in the Hamiltonian of the system as was done in [14]. In another perspective, the high accuracy and the universality of Hall conductance led to more sophisticated mathematical tools like topology and geometry in explaining some fundamental features as was done in [15].

The application of the group theoretic quantization programme here focusses only on the kinematical quantisation which means that we only classify the Hall states. This classification will involve parameters that reflect the nature of the states and hence they will be related physically to the filling factors ν. The external magnetic field introduced in the problem should be homogeneous in order to keep the proposed original symmetry of the system. If the symmetry is broken (or the magnetic field is inhomogeneous) Isham's programme will have to be further modified and this may be considered as deficiencies of this scheme. Another visible deficiency appears in the study of quantum mechanics on the sphere without and with magnetic field; we found that both of the systems have the same canonical Lie algebra and further work has to be done to really obtain the global features that differentiate these systems.

In a previous work[16] the quantization scheme has been applied to Hall systems with torus and sphere as the underlying configuration space. Here we focus on the case where the configuration space is the sphere as well as the punctured sphere for an attempt to explain the fractional case. To begin let's consider a charged particle on a sphere interacting with a monopole field. So, the system is defined by (S,ω) as defined in the previous section where ω is the two form defined in the streographic coordinates as[17]:

$$\omega = dz \wedge dp_{\bar{z}} + d\bar{z} \wedge dp_z + F\frac{dz \wedge d\bar{z}}{\left(1+|z|^2\right)}, \qquad (7)$$

It is shown[18] that the canonical group for this system is SO(3) where there is no magnetic field ($F = 0$), but this is obstructed when there exists a magnetic field. To lift this obstruction we pass to the covering group which is SU(2). Classifying the SU(2) representations would eventually classify the possible quantization of the Hall system on the sphere. It is clear that we can only classify the integer Hall states so long as we are working within the framework of single particle. In order to classify the fractional Hall states we have to consider a many-body problem, we project this topologically in our framework by the removing n-points from the configuration space. Thus in this case, the system becomes a charged particle moving in a multiply connected space (n-punctured sphere). From the physical viewpoint, it is the system of (n+1)–particles moving on the sphere; each one see the others as punctures incorporating the fact that it is impossible for two particles to share the same position at the same time. Within this configuration space the new symplectic form defined as[17]:

$$w_F = g'(w)g'(\bar{w})\left[d\bar{w} \wedge dp_w + dw \wedge dp_{\bar{w}} + \frac{F dw \wedge d\bar{w}}{\left(1+\prod|w-w_i|^2\right)}\right] + \qquad (8)$$
$$\left[p_{\bar{w}} g'(w) g''(\bar{w}) - p_w g'' g(w) g'(\bar{w})\right] dw \wedge d\bar{w}.$$

where the relation between the old and new coordinates is given by

$$z = \prod_{i=1}^{n} \frac{1}{(w-w_i)} \equiv \frac{1}{g(w)}, \qquad (9)$$

From the above symplectic form it is clear that another symmetry has entered the problem namely the symmetry of exchange of punctures. This symmetry could be reflected somehow in the wave function of the system and as a consequence leads to the fractional statistics relying on the fact that the exchange symmetry group in 2d-space is the braid group. It is found[17] that the canonical group for the parent sphere is SU(2) but, it is shown later this is only defined locally as there is a transitivity problem for the action of SU(2) on the punctured sphere. To overcome this problem we find the canonical group of the system by studying the detailed geometry of the configuration space realized as a Riemann surface.

It is well known that any Riemann surface[19] could be written as H/Γ due to the uniformization theorem, where H is the upper half plane and Γ is a discrete subgroup of the isometry group of the H, isomorphic to the fundamental group of H/Γ. Based on this and taking in mind that $H \equiv SL(2,R)/SO(2)$ one can define the canonical group following Isham [5] to be the group $V * SL(2,R)/SO(2)$ where V is the smallest vector space for which the action of $SL(2,R)/SO(2)$ on it defines H/Γ orbit. Here we focus on the quotient group

$$G_c = SL(2,R)/SO(2). \qquad (10)$$

The state vectors of the system are sections in an appropriate bundle or as functions on H but not Γ-automorphic ones as in the case without magnetic field [20]. The states vector transform under the action of the fundamental group as:

$$\psi(\gamma z) = \left(\frac{cz+d}{|cz+d|}\right)^{2eB} \psi(z)$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$.

With this in mind and taking into consideration that the punctured sphere S_p^2 pictorially is identified with the boundary of the fundamental region under the action of Γ, this leads to something equivalent to the boundary conditions in the usual quantum mechanics and from this we get that the magnetic field B is quantized as[20] :

$$B = n/2e, \qquad n \in Z \qquad (11)$$

This result also could be gotten within the vector bundle framework . So, we can classify the Hall states using this parameter but the problem is still not totally solved.

However many difficulties appear in this process of quantization and here we list some of them as questions to further researched upon. First how can we present the symmetry of the exchange of the punctures which reflect mathematically, the topology of the configuration space and physically, the many body effect. It is known that the braid group is the right group defines such symmetry in two dimensional space and its representation leads to the understanding of the fractional statistics but the problem how it will reappear in our case as part of the canonical group exactly is a difficulty. Is it really that the right group is the braid group and why not the mapping class group which is a subgroup of the former. Now whatever the group is, the other question can be asked about the global structure of the canonical group is whether it is the direct or semi-direct product of the former group with that defined by (10)?And what are the physical observables that correspond to the new generators? If we argue that the canonical group is defined only by (10), can we guess that the effect of the exchange symmetry will reappear in the twisted representation of that group? All these problems will effect the classification of the group representation and how they relate with the filling factor.

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