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FRACTAL DIFFUSION EQUATIONS: MICROSCOPIC MODELS WITH ANOMALOUS DIFFUSION AND ITS GENERALIZATIONS

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Abstract

To describe the "anomalous" diffusion the generalized diffusion equations of fractal order are deduced from microscopic models with anomalous diffusion as Comb model and Levy flights. It is shown that two types of equations are possible: with fractional temporal and fractional spatial derivatives. The solutions of these equations are obtained and the physical sense of these fractional equaitions is discussed. The relation between diffusion and conductivity is studied and the well-known Einstein relation is generalized for the anomalous diffusion case. It is shown that for Levy flight diffusion the Ohm's law is not applied and the current depends on electric field in a nonlinear way due to the anomalous character of Levy flights. The results of numerical simulations, which confirmed this conclusion, are also presented.

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1 Introduction.

Classical diffusion, in which diffusing particle hops only to nearest sites, has been thoroughly studied, and many methods, related to the research of this phenomenon, have been developed. In distinction of this the random walks with an anomalous power character are ,however, studied less. One of the well known examples are random walks on percolation clusters (random fractals), which have a sub-diffusion character [1],[2]:

$$< X^2(t) > \sim t^{\frac{2}{2+\theta}} \tag{1}$$

Here t is diffusion time, $< X^2(t) >$ is a random mean square (rms) displacement during the time, θ is a critical index of the anomalous diffusion. Let's note too that the critical exponent of anomalous diffusion θ depends on the space dimension: $\theta_2 \sim 0.8, \theta_3 \sim 1.3$. The change of diffusion character is caused by two reasons : strong tortuous (twistness) of percolation ways and presence of impasses - " dead ends on current ways at least. This problem was formulated many years ago in the [3],[4]- as a problem of "ant in labirint" and it is still not solved.

To take into account an influence of impasses for diffusion character the model of comb stucture was put forward [5] and [6]. This model consists of one-dimensional backbone with fingers of infinite lengths - see fig.1. Using the technique of the generating functions it was shown, that the root-mean-square displacement along an axis of structure depends on time in the anomalous way (1) with the exponent $\theta = 2$.

This model is one of the few exactly solvable models with unusual diffusion properties. So in this paper we consider this model in more detail. The generalized diffusion equation, describing random walks along an axis of structure, was deduced. It essentially differs from the usual diffusion equation, having the form of the continuity equation: instead of the first derivative on time the derivative of the fractional order 1/2 arises. The expression for a diffusion current remains the former - see also [7]. The generalization for a multidimensional case is performed. The relation of the diffusion on the comb model with a problem of continuous time random walks (CTRW) is established [8],[9]. A further development of the model is a study of random walks on the comb structure with random distribution of fingers over lengths [10] , [11]. In particular it was shown for a power law distribution $f(l) \sim l^{-\gamma}, 1 \leq \gamma \leq 2$ rms depends on time in the following power way :

$$< X^2(t) > \sim t^{\frac{\gamma}{2}} \tag{2}$$

It is connected with that at these values $1 \leq \gamma \leq 2$ all moments of the power distribution over lengths are diverged. Recently an analogous behavior was obtained in the continuum description of the anomalous diffusion on the comb structure [12],[13]. It is necessary to note that the diffusion problem on random comb structure is not yet solved , and the above called results were obtained in the effective medium approximation. We discuss this problem and show that results essentially depend on the way of averaging. The expressions for averaged probabilities

(Green functions) are found in two limiting ways. The form of operator evolution in the fractal temporal derivative is deduced in the effective medium approximation. The drift of particle and the influence of an electrical field for the statistical properties of random walks are also studied.

The second part of this paper is devoted to another anomalous random walks - super-diffusion via Levy flights [14],[15]. At Levy flights particles may hop for an arbitrary large distance with a power probability, so that rms displacement per unit time appears to be infinite. The study of Levy diffusion is of interest as a microscopical model with an unusual diffusion and in a connection with possible applications to hopping conductivity in disordered media and to other fields. The generalization of Levy diffusion for a finite length of hop is discussed. In this case Levy flights are alternated by the usual diffusion. The most interesting one is a research on the relation between diffusion and conductivity in the super-diffusion case. It is shown that due to a super-diffusion character of random walks the current and electric field are connected in a nonlinear way. The index of the nonlinearity is described by the exponent of the anomalous Levy diffusion.

The paper organized as follows. In section 2 the exact solution of the random walks on comb structure is obtained. Namely the generalized fractal diffusion equations for the anomalous case are deduced in two different ways. In section 3 the generalization for a multidimensional case is made. The connection between problems of diffusion on comb structure and continuous time random walks is considered in section 4. The transition to usual diffusion due to a finite length of finger is traced in section 5. Section 6 is devoted to diffusion on the random comb model. Some variants of effective medium approximation (EMA) are considered. In section 7 another variant of EMA, based on microscopical approach, is studied. The drift on the comb structure is considered in section 8. In the last sections 9-11 the diffusion via Levy flights and in an electric field are studied. It is shown that a relation between diffusion and conductivity is nonlinear. The results of numerical simulations are also discussed. Section 12 concludes the paper and the discussion of results is given.

2 The diffusion on comb structure.

A feature of the diffusion in the considered model consists of that the displacement in the X-direction is possible only along an axis of structure (at y = 0). This means that diffusion coefficient D_{xx} is different from zero only at y = 0:

$$D_{xx} = D_1 \delta(y) \tag{3}$$

i.e. *X- component of the diffusion current is equal to:*

$$J_x = -D_1 \delta(y) \frac{\partial \rho}{\partial x} \tag{4}$$

The diffusion along fingers is considered as usual: $D_{yy} = D_2$. Thus, the random walks on the comb structure is described by the tensor of diffusion:

$$\hat{D} = \begin{pmatrix} D_1\delta(y) & 0 \\ 0 & D_2 \end{pmatrix}$$

Accordingly, we obtain the following diffusion equation:

$$[\frac{\partial}{\partial t} - D_1\delta(y)\frac{\partial^2}{\partial x^2} - D_2\frac{\partial^2}{\partial y^2}]G(x,y,t) = \delta(x)\delta(y)\delta(t) \tag{5}$$

Here $G(x,y,t)$ is the Green function of the diffusing problem. To solve the equation we use the following reception. Let's rewrite equation (5) as the usual diffusion equation with a non-uniform right part:

$$[\frac{\partial}{\partial t} - D_2\frac{\partial^2}{\partial y^2}]\rho = D_1\delta(y)\frac{\partial^2\rho}{\partial x^2} \tag{6}$$

The solution of the homogeneous equation (6) is well known and has the Gaussian form:

$$G(y,t) = \frac{exp(-\frac{y^2}{4D_2t})}{\sqrt{\pi D_2 t}} \tag{7}$$

Thus we obtain the integral equation for the concentration of the diffusing particles:

$$\rho(x,y,t) = \int G(y-y',t-t')D_1\delta(y')\frac{\partial^2\rho(x,y',t')}{\partial x^2}dy'dt' \tag{8}$$

After integration over y' one obtains the closed equation for the concentration of particles on an axis of the structure (y=0):

$$\rho(x,o,t) = D_1\frac{\partial^2}{\partial x^2}\int_{-\infty}^{t}\frac{\rho(x,o,t')}{\sqrt{\pi D_2(t-t')}}dt' \tag{9}$$

It is easy to see, that the right-hand side of formula (9) is the integral of the fractional order 1/2 [16],[17]. Therefore using the operator of fractional differentiation of a degree 1/2 we obtain the required diffusion equation:

$$\frac{\partial^{\frac{1}{2}}\rho(x,t)}{\partial t^{\frac{1}{2}}} = D_1\frac{\partial^2\rho(x,t)}{\partial x^2} \tag{10}$$

The integro-differential form of the diffusion equation (10) is a consequence of random disappearance and subsequent birth of particles at the axis of structure at diffusion (leaving and returning to an axis of structure). Let's mark that this equation describes the diffusion problem with a non-conserving number of particles.

To find the solution at arbitrary values of the coordinate y we use another direct approach. Let's use a mixed (s, k,y)-representation :

$$[s + D_1k^2\delta(y) - D_2\frac{\partial^2}{\partial y^2}]\rho(s,k,y) = \delta(y) \tag{11}$$

Let's find the solution (11) in the form :

$$G(s,k,y) = g(s,k)exp(-\lambda|y|) \tag{12}$$

After necessary calculations one has :

$$G(k,y,s) = \frac{exp(-(\sqrt{\frac{s}{D_2}}|y|)}{2\sqrt{sD_2}+D_1k^2} \quad (13)$$

Using Fourier transformations , we obtain :

$$G(x,y,t) = \int_0^\infty \frac{exp(-\frac{x^2}{4D_1\tau} - \frac{-D_2(\tau+|y|)^2}{4t})}{\sqrt{2\pi t^3}} \quad (14)$$

To obtain this expression the following identity was used:

$$\int_0^\infty exp(-\alpha\tau)d\tau = \frac{1}{\alpha} \quad (15)$$

The distribution of particles on an axis of structure is described by the same expression at y=0.

Let's note that the complete number of particles on an axis of structure decreases or in other words this diffusion problem is the one with a non-conserving number of particles:

$$< G >= \int G(x,o,t)dx = \frac{1}{\sqrt{2D_2t}} \quad (16)$$

Taking into account last remark to calculate the displacement along an axis of structure:

$$< X^2(t) >= \frac{< X^2G >}{< G >} = D_1\sqrt{\frac{t}{D_2}} \quad (17)$$

Let's return to the equation for $G(x,0,t)$. As follows from (13) in (s, k) - representation it has the form:

$$[2\sqrt{sD_2} + D_1k^2]\rho(s,k) = 0 \quad (18)$$

It is easy to see that this equation consists of the Fourier representation of the fractional derivative on time [16],[17],[18]. So we recover the diffusion equation for a density of particles on an axis of the structure in the form (10).

So the consideration of random walks on comb structure shows that the problem with anomalous diffusion and with non-conserving number of particles should be described by the diffusion equation with temporal derivative of the fractional order.

3 Multidimensional case.

Let's generalize these results for a multidimensional case. First let's begin with a three-dimensional comb structure. Such a structure is formed by attaching the additional fingers to the existing two-dimensional comb structure that points in the direction parallel to the Z axis. Hence in the three-dimensional case displacements in the X-direction are possible only along the intersections of the planes $y = 0$ and $z = 0$. In other words the diffusion coefficient is not zero , i.e. $D_{xx} = D_1\delta(y)\delta(z)$. Accordingly , a displacement in the y-direction is possible only if $z = 0$, and a displacement along z axis is ordinary. Thus, we have the following diffusion tensor:

$$\hat{D} = \begin{pmatrix} D_1\delta(y)\delta(z) & 0 & 0 \\ 0 & D_2\delta(y) & 0 \\ 0 & 0 & D_3 \end{pmatrix}$$

So the corresponding diffusion equation in the mixed (s, k, y, z) - representation is :

$$[s + D_1k^2\delta(y)\delta(z) - D_2\delta(z)\frac{\partial^2}{\partial x^2} - D_3\frac{\partial^2}{\partial y^2}]\rho(s, k, y, z) = 0 \tag{19}$$

Let's find a solution for (19) in the form:

$$\rho(s, k, y, z) = g(s, k)exp(-\lambda_2|y| - \lambda_3|z|) \tag{20}$$

Substituting (20) into Eq. (19) yields the following formulas for the parameters λ_2 and λ_3 and the function $g(s, k)$:

$$\lambda_3^2 = s/D_3,\ \lambda_2^2 = \frac{2\lambda_3 D_3}{D_2}$$
$$g(s, k) = \frac{1}{2\lambda_2 D_2 + D_1k^2} \tag{21}$$

Consequently for the mean-square displacement along the x and y axes we then have :

$$< X^2(t) >\sim t^{1/4},\ < Y^2(t) >\sim t^{1/2} \tag{22}$$

Hence in the N-dimensional case the diffusion tensor is described by the matrix :

$$\hat{D} = \begin{pmatrix} D_1\delta(x_2)\ldots\delta(x_N) & 0 & \ldots \\ 0 & D_2\delta(x_3)\ldots\delta(x_N) & \ldots \\ \vdots & \vdots & \vdots \\ \ldots & D_{N-1}\delta(x_N) & 0 \\ 0 & \ldots & D_N \end{pmatrix}$$

Accordingly we find a solution for the N-dimensional diffusion problem in the form

$$\rho(s, k, x_2, x_3\ldots, x_N) = g(s, k)exp(-\lambda_2|x_2| - \lambda_3|x_3| - \ldots - \lambda_n|x_N|) \tag{23}$$

Here the parameters λ_N are linked through the formulas:

$$2\lambda_N = s/D_N,\ \lambda_{N-1}^2 = \frac{2\lambda_N D_N}{D_{N-1}}, \ldots,\ \lambda_2^2 = \frac{2\lambda_3 D_3}{D_2} \tag{24}$$

and the function $g(s, k)$ is defined in the expression (21). The formulae (23) and (24) give the complete solution of the multidimensional problem. For instance it is easy to calculate the mean-square displacement along the main axis of the structure:

$$< X_N^2(t) >\sim t^{1/2(N-1)} \tag{25}$$

For the next lateral finger the mean-square displacement is

$$< X_{N-1}^2(t) >\sim t^{1/2(N-2)} \tag{26}$$

...And for the axis, from which only fingers of infinite length emerge, we have

$$< X_2^2(t) >\sim t^{1/2} \tag{27}$$

Thus random walks on a multidimensional comb structure is of a hierarchical nature and there are many variants of behavior of the mean-square displacements along the axes of the structure.

4 Continuous-time random walks.

The above problem of a random walk on an N-dimensional comb structure is connected to the problem of diffusion in a medium with traps (continuous -time random walk). The difference between the two problems consists in that in diffusion in a medium with traps the particles do not disappear, but only delay at each site with a certain probability. The total number of diffusing particles is conserved [19], [20]. For a comb structure the transition to the problem with a continuous distribution over delay time occurs if we study the following quantity:

$$\tilde{G}(x,t) = \int G(x,y,t)dy \tag{28}$$

According to (13) the function $\tilde{G}(x,t)$ is described by the equation :

$$[s + \frac{D_1 k^2 s^{1/2}}{D_2}]\tilde{G} = 1 \tag{29}$$

Hence in the case of a medium with traps the diffusion equation has the form of the continuity equation for a medium with temporal dispersion:

$$\frac{\partial \rho(x,t)}{\partial t} - \frac{\partial J}{\partial x} = 0 \tag{30}$$

where

$$J = -\frac{D_1}{2D_2}\frac{\partial}{\partial x}\int \frac{\partial \rho(x,\tau)}{\partial \tau}\frac{\partial \tau}{|t-\tau|^{1/2}} \tag{31}$$

Diffusion is still anomalous with the exponent $\theta = 2$. Let's consider the three-dimensional case and examine the Green function averaged over the y and z axes , i.e. the function $\tilde{G}(s,k) = \int\int G(s,k,y,z)dydz$. According to (23), for this function, we have the equation:

$$[s + D_1 k^2 (\frac{4sD_3}{D_2})^{3/4}]\tilde{G} = 1 \tag{32}$$

Hence the diffusion equation has the form of the continuity equation with a diffusion current:

$$J \sim -\frac{\partial}{\partial x}\int \frac{\partial \rho(x,\tau)}{\partial \tau}\frac{\partial \tau}{|t-\tau|^{3/4}} \tag{33}$$

Further we study the Green function averaged over one coordinate z :

$$\tilde{G}(s,k,y,t) = \frac{exp(-\lambda_2|y|)}{\lambda_3(2\lambda_2 D_2 + D_1 k^2)} \tag{34}$$

Accordingly, the motion along the axis $y = 0$ is described by the equation:

$$[s^{3/4} + ADk^2(s)^{1/2}]\tilde{G} = 0 \tag{35}$$

where $A = const.$

The number of particles on the $y = 0$ axis is not conserved because particles are also in the dead ends . As result of this the diffusion current contains a fractional temporal derivative of order 1/2. So in the N-dimensional case the equation for the function $\tilde{G}_m$, averaged over the m coordinates has the form:

$$[s^\beta + s^\nu k^2]\tilde{G}_m(s,k) = 0 \tag{36}$$

where $\beta = (N - m + 1)/4$ and $\nu = (N - m - 1)/4$

5 Transition to usual diffusion at finite lengths of fingers.

Up to this point we have studied comb structures with infinitely long fingers. Transition to usual diffusion due to the finite length of fingers is studied below. The reflecting boundary conditions are used as boundary conditions :

$$J(y = \pm L) = 0 \tag{37}$$

Thus random walks on the comb structure with finite length of fingers is described by equation (6) and boundary conditions (37). The solution of the homogeneous equation (6) with boundary conditions (37) is well known:

$$G_L(y,t) = \sum_{m=0}^{\infty} exp(-\frac{D_2 t(m\pi)^2}{L^2})cos(\frac{m\pi y}{L}) \tag{38}$$

Thus, we receive the integral equation for concentration:

$$\rho(x,y,t) = \int G_L(y - y', t - t')D_1\delta(y')\frac{\partial^2\rho(x,y',t')}{\partial x^2}dy'dt' \tag{39}$$

It has the most simple form in (s,k,y) - representation:

$$\rho(k,y,s) = -\frac{D_1 k^2}{L}\sum_0^{\infty}\frac{\rho(k,o,s)}{s + (\frac{m\pi}{L})^2}cos(m\pi y/L) \tag{40}$$

At y = 0 one obtains the closed equation for $\rho(k,0,s)$:

$$\hat{K}(s,L)\rho(s,k) = -D_1 k^2\rho(s,k) \tag{41}$$

Here the operator $\hat{K}$ is equal:

$$\hat{K}^{-1}(s,L) = \frac{1}{sL} + \frac{cth[L\sqrt{sD_2}]}{2\sqrt{sD_2}} \tag{42}$$

Here $cotan(x)$ is the hyperbolic function. At infinite length of fingers one obtains the result (18), obtained above:

$$\hat{K}(\infty,L) = 2\sqrt{sD_2} \tag{43}$$

On asymptotic large times we obtain the usual diffusion equation with a diffusion coefficient, depending on the length of the fingers :

$$[s + const\frac{D_1 k^2}{L}]\rho(s,k) = 0 \tag{44}$$

The structure, which we studied before , had a finger of equal length L. Now we consider a case when N fingers have various lengths $L_1, L_2, ...L_N$ and that this pattern repeats periodically. The distance between the sites on the structure's axis is a. To understand how a random walk on a such structure may be described we analyze the case of two lengths L_1 and L_2. We write the second derivative with respect to the coordinate x in the finite-difference form and introduce the notations $K_1 = K(s, L = L_1), K_2 = K(s, L = L_2)$ and also denote the particle concentration by F_1 at the point on the axis to which a finger of length L_1 is attached. (F_2 is introduced in a similar way.) Then the following system of equations , describing the behavior of the particles on the axis may be written:

$$\begin{aligned} K_1F_1(x) &= \frac{D_1}{a^2}(F_2(x+a) + F_2(x-a) - 2F_1(x)) \\ K_2F_2(x) &= \frac{D_1}{a^2}(F_1(x+a) + F_1(x-a) - 2F_2(x)) \end{aligned} \tag{45}$$

or in the k-representation:

$$(K_1 - 2D_1/a^2)F_1(k) + 2D_1/a^2 cos(ka)F_2(k) = 0 \tag{46}$$

$$(K_2 - 2D_1/a^2)F_2(k) + 2D_1/a^2 cos(ka)F_1(k) = 0 \tag{47}$$

Setting the determinant of this equation to zero we can find the dispersion relationship between the parameters s and k or in other words, the analog of the diffusion equation in the (s,k)-representation:

$$T_1(s)T_2(s) - C^2 cos^2(ka) = 0 \tag{48}$$

where $T(s) = K(s) - C$, and $C = 2D_1/a^2$. From Eq.(48) with equal finger lengths and as $a \to 0$ we obtain Eq.(10) as expected.

Thus to describe random walks on N fingers of differing length we must set up a system of N equations. Such a system emerges because diffusion strongly depends on what finger (and what length) are involved in the random walk of a particle. The above analysis suggests that in the case of a comb structure with N fingers, the determinant takes the form:

$$\begin{vmatrix} T_1 & exp(ika) & \dots & \dots & exp(-ika) \\ exp(-ika) & T_2 & exp(ika) & \dots & \dots \\ 0 & exp(-ika) & T_3 & exp(ika) & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ exp(ika) & \dots & \dots & exp(-ika) & T_N \end{vmatrix} .$$

Thus instead of the ordinary diffusion equation we have the N-channel diffusion equation or instead of a simple dispersion law $s = Dk^2$, valid for ordinary diffusion, we have the equation of N-th order. Moreover according to (42) the form of operator depends on the relation between the parameter s and the diffusion time $t_i = D/L_i^2$ along the finger.

Let's analyze the solutions of N-channel equations by qualitative reasonings. Suppose that finger lengths differ substantially and consequently the hierarchy of times related to the diffusion along these fingers appears. Over short times the diffusion is anomalous and as time increases it is replaced by ordinary diffusion with a diffusion coefficient, depending on the length of the particular finger:

$$< X^2(t) >\sim t^{1/2}, \quad t << t_1 << \ldots << t_n \tag{49}$$

$$< X^2(t) >\sim D_1 t/L_m, \qquad t_m << t << t_{m+1} \tag{50}$$

6 Diffusion on random comb structure.

In this section the diffusion on the comb structure with a random distribution of fingers over lengths is studied. It is believed that this model better describes the random walks on percolation clusters. In accordance with the above results the concentration of diffusing particles is satisfied to equation (39) . So one represents the procedure of the averaging over a random distribution of fingers in the following form:

$$< \rho(x,y,t) >= \int G_L(y-y', t-t') D_1 \delta(y') \frac{\partial^2 \rho(x,y',t')}{\partial x^2} dy' dy dt' f(L) dL \tag{51}$$

Here $< \rho(x,t) >$ is the averaged concentration , $f(L)$ is the function of the distribution of fingers over the lengths. But to solve the considered problem and to find the solution of (51) it is necessary to make additional assumptions. The simplest of them are considered below.

6.1 "Consequent" way of averaging

Let's insert in (51) the average value of $< G_L >$ instead of the unknown exact Green function, which is the solution of (51). This approach also supposes also that a diffusing particle passes through fingers in consequent way and that it diffuses along i finger before goes to next $i+1$ finger. It is equivalent to the consequent connection of resistors in the conductivity problem, so let's call it the "consequent" averaging on configurations of finger lengths. Let's calculate the averaged Green function for some distributions at the "consequent" averaging over configurations.

A. Let's consider the Gauss distribution:

$$f(l) = \frac{1}{l_0} exp(-\frac{l^2}{l_0^2})$$

In this case the following expression for the average Green function, which is used in formula (51), is obtained:

$$< G_L(\tau) >= \frac{1}{L} l \sum_{m=o}^{\infty} \int exp(-D_2 \tau m^2 \pi^2 / L^2 - L^2/L_0^2) dL/L_0$$

$$\sim \frac{1}{L_0} \sum_{m=0}^{\infty} exp(-2\sqrt{D_2\tau}\pi m/L_0) \sim [(1 - exp(-2\pi\sqrt{D_2\tau}/L_0)L_0]^{-1} \tag{52}$$

Here the integral is calculated by the method of the fastest descent . From (52) it is easy to find the asymptotic behavior of the averaged Green function at different values of the parameter $\tau = t - t'$:

$$G(\tau) \sim \begin{cases} (D_2\tau)^{-1/2} & \text{if } \tau << L_0^2/D_2 \\ 1/L_0) & \text{if } \tau >> L_0^2/D_2 \end{cases}$$

Correspondingly, in the s -representation (Laplace -representation) one has:

$$< G(s) >= \int G(\tau)exp(s\tau)d\tau \sim \begin{cases} (sD_2)^{-1/2} & \text{if } s >> D_2/L_0^2 \\ (sL_0)^{-1} & \text{if } s << D_2/L_0^2 \end{cases}$$

From this formula and (51) it is easy to find asymptotic dependences of rms on times: anomalous on the small times and usual D_1t/L_0 at big times.

B. Let's consider a power distribution of fingers over lengths:

$$f(l) = Al^{-\gamma}, 1 < \gamma < 2$$

In accordance with formula (51) we have:

$$< G_L(\tau) >= A \sum_{m=o}^{\infty} \int exp(-D_2\tau(m\pi/L)^2 - (1+\gamma)lnL)dL$$

$$\sim (\sum_{m=1}^{\infty} (m^2\tau D_2)/\pi^2)^{-\gamma/2} + 1) \tag{53}$$

In the Laplace s-representation one has

$$< G(s) >\sim (s^{-1} + Cs^{(\gamma-2)/2}) \tag{54}$$

Here C is a constant . So at small values of the parameter s (corresponding to big times) and at values $\gamma < 2$ one has the following asymptotic diffusion equation:

$$[Cs^{(2-\gamma)/2} + D_1k^2]\rho(s,k) = 0 \tag{55}$$

Consequently rms is equal to

$$< X^2(t) >\sim t^{(2-\gamma)/2} \tag{56}$$

Let's note that the value $\gamma = 2$ is critical for the change of the diffusion character. The transition to a new anomalous diffusion dependence is caused by the infinity value of the average finger length at the values $\gamma < 2$:

$$< L >= \int lf(l)dl = \infty$$

At the values $\gamma > 2$ the average length is finite, so the diffusion has a usual character. Formally this transition occurred due to the member $m = 0$ in the sum (53).

As noted above this approach of the averaging corresponds to the consequent connection of resistors, so from the Einstein relation

$$\sigma = q^2 nD/kT$$

one obtains the expression for the effective length :

$$L_e =< l > \tag{57}$$

Here L_e is determined by the formula : $D_e = D_1/L_e$. For a case of power distribution over lengths $L_e = \infty$ and as a result of this a new power anomalous diffusion behavior appears.

6.2 "Parallel" way of averaging.

Further consider the assembly of comb structures,which have fixed finger length, and suppose that the probability of the structure with fixed length l is described by the same distribution function over finger lengths. This approach corresponds to the parallel connection of resistors. In this case according to (41) one has the Green function for the problem with fixed length of finger:

$$G_l(s,k) = \int_0^\infty exp(-D_1 k^2 \tau - \hat{K}(s,l)\tau)d\tau \tag{58}$$

The averaging over finger lengths is very simple in this case:

$$< G(x,t) >= \int G_l(x,t) f(l) dl =$$
$$\int\int exp[-\frac{x^2}{4D_1\tau} - \frac{\sqrt{sD_2}\tau + st}{D_2/2sl + cth(\sqrt{s/D_2}l}]d\tau ds f(l) dl \tag{59}$$

A. One can easily find arbitrary moments of the average Green function with Gauss distribution function. After necessary calculations one has that the rms depends in an anomalous way at small times and in usual way at big times:

$$< X^2(t) >\sim \begin{cases} (D_1(t/D_2)^{1/2} & \text{if } t << L_0^2/D_2 \\ (D1t/L_0)^{-1} & \text{if } t >> L_0^2/D_2 \end{cases}$$

B. Let's calculate the average Green function for a case of the power distribution:

$$< G(x,t) >= \int G_l(x,t) f(l) dl =$$
$$\int\int exp[-\frac{x^2}{4D_1\tau} - \frac{\sqrt{sD_2}\tau + st}{D_2/2sl + cth(\sqrt{s/D_2}l}]f(l) dl d\tau ds/\sqrt{\pi D_1 \tau} \tag{60}$$

Note that the Green function $< G >$ is not normalized for a unit, namely:

$$< G(x,t) >\sim t^{-\gamma/2} \tag{61}$$

From (60) and (61) ones calculates rms along structure axis:

$$< x^2(t) >=< x^2 G(x,t) > / < G(x,t) >\sim D_1 \frac{t}{D_2} \tag{62}$$

Note that at this averaging the diffusion on random comb structure has the same index $\theta = 2$, as in the case of comb structure with infinite length of fingers. It is connected of that the effective length at "parallel" averaging is determined by the relation:

$$L_e =< \frac{1}{l} >^{-1}$$

and has a finite value for power distribution also.

In the general case effective length must has value, which is limited by these values:

$$< \frac{1}{l} >^{-1} \leq L_e \leq < l >$$

7 Microscopic description of diffusion on the random comb model: effective medium approximation.

Below we use the microscopic approach to study random walks on comb structure, developed in [12]. They are characterized by two types of functions: first $g(x_i, x', t)$ is the probability density to reach the point X on the structure axis after time t , if it was in the point X' at the moment t = 0. The set of probability density $\phi_i(y, t)$ to find a particle at the i fingers at the moment t belong to the second type [12],[13]. For convenience let's use the Laplace images of the functions. The function g is described by the equation for $x = ai, i = 0, \pm1, \pm2...$, a is a distance between fingers:

$$sg = D\frac{\partial^2 g}{\partial x^2} + \delta(x - x') \tag{63}$$

and the function φ is described by the equation:

$$s\varphi = D\frac{\partial^2 \varphi}{\partial y^2} \tag{64}$$

with reflecting boundary conditions :

$$\frac{\partial \varphi}{\partial y} = 0 \tag{65}$$

In the points of the backbone, in which fingers connect to it, the continuity conditions have a form:

$$\varphi(s, o) = g(s, x_i, x'), \tag{66}$$

$$J_i|_{y=0} = j|_{x=x_i-0} - j|_{x-x_i+0} \tag{67}$$

where

$$J_i = -D\frac{\partial \varphi_i}{\partial y}, j_i = -D\frac{\partial g}{\partial x}$$

The solution of (64) with the reflecting conditions (65) has the form:

$$\varphi(s, o) = g(s, x_i, x')[ch(y\sqrt{\frac{s}{D}}) - sh(y\sqrt{\frac{s}{D}})th(l_i\sqrt{\frac{s}{D}})] \tag{68}$$

So one can write the boundary conditions as follows :

$$\varphi(s,o) = g(s,x_i,x'), \tag{69}$$

$$j|_{x=x_i-0} - j|_{x=x_i+0} = -D\frac{\partial \varphi_i}{\partial y} = \sqrt{sD}th(l_i\sqrt{\frac{s}{D}})g(s,x_i,x) \tag{70}$$

This relation shows that the equation for the function $g(s,x_i,x)$ may be continued over an axis of the comb structure, if one uses the δ - like functions with the power $\sqrt(sD)th(l_i\sqrt{\frac{s}{D}})$, introduced at the points x_i. Consequently, one can write the diffusion equation in the form:

$$\hat{T}g = D\frac{\partial^2 g}{\partial x^2} + \delta(x-x') \tag{71}$$

where the evolution operator $\hat{T}$ is equal to :

$$\hat{T} = s + \sqrt{sD}\sum_{i=-N}^{N} th(l_i\sqrt{\frac{s}{D}})\delta(x-x_i) \tag{72}$$

Further simplification consists of the substitution of the evolution operator for an average meaning $< T >$:

$$< \hat{T} >= s + \sqrt{sD}\lim_{N\to\infty}\frac{\sum_{i=-N}^{N} th(l_i\sqrt{\frac{s}{D}})}{2aN} \tag{73}$$

It corresponds to another form of the effective medium approximation (EMA), different from that described above. At the fixed length of the fingers it is easy to obtain :

$$< \hat{T} >= s + \sqrt{sD}\frac{th(L\sqrt{\frac{s}{D}}}{a} \tag{74}$$

One can see that this expression essentially differs from exact solution of this problem - see (42). Nevertheless it correctly describes an asymptotic diffusion behavior . So we have an anomalous diffusion at small times $\frac{D}{L^2} \gg s \gg \frac{D}{a^2}$ and usual at big times :

$$< X^2(t) > \sim Dt\frac{a}{L} \tag{75}$$

For a case of the power distribution of fingers one obtains:

$$< T >= s + \sqrt{sD}\int_a^\infty th(l\sqrt{\frac{s}{D}})f(l)\partial l \sim s + const(\frac{s}{D})^{\gamma/2} \tag{76}$$

Correspondingly we obtain the new anomalous diffusion behavior :

$$< X^2(t) > \sim (t)^{\frac{\gamma}{2}} \tag{77}$$

8 Drift on the comb structure model.

The appearance of the electrical field leads to an anisotropy of random walks. In weak fields the anisotropy parameter $\alpha(E) << 1$ is small and is proportional to a field. Accordingly the field current equals: $J = n\mu E$. In the comb structure the mobility tensor is analogous to the diffusion coefficient. The equation for the diffusion on comb structure and in an electrical field has the following form:

$$[\frac{\partial}{\partial t} - \delta(y)(D_1\frac{\partial^2 \rho}{\partial x^2} + \mu_1\frac{\partial^2}{\partial y^2}) - (D_2\frac{\partial^2}{\partial y^2}\mu_2\frac{\partial^2}{\partial y^2})]\rho(x,y,t;E) = 0 \tag{78}$$

Let's assume that the field is directed only along an axis of structure $\vec{E} = E(1,0,0)$. Accordingly,the Green function in mixed (s,k,y) -representation is equal to:

$$G(s,k,y;E) = \frac{exp(-\sqrt{s/D_2}|y|)}{2\sqrt{sD_2} + D_1k^2 + ik\mu_1 E} \tag{79}$$

After Fourier transformations , we obtain :

$$G(x,y,t) = \int_0^\infty \frac{exp(-\frac{(x-\mu_1 E\tau)^2}{4D_1\tau} - \frac{-D_2(\tau+|y|)^2}{4t})}{\sqrt{2\pi t^3}} \tag{80}$$

Let's find the first moment of the Green function in a field:

$$< X(t) >= \mu_1 E\sqrt{\pi t/2} \tag{81}$$

Let's emphasize that the response to a constant electrical field appears as a time-dependent one. Namely, the velocity decreases with time according to the power way:

$$< V >= \mu_1 E\sqrt{\pi/2t} \tag{82}$$

This result means that in the anomalous diffusion problem with drift it is impossible to find such an inertial system of the coordinates, which is moved with constant speed and in which the diffusion remained only as in the usual diffusion case.

Let's consider also the influence of an electrical field on a returning probability. In the usual diffusion case the drift leads to the exponential reduction of it:

$$G(o,t;E) = \frac{exp(-(\mu E)^2 t)}{\sqrt{\pi t}} \tag{83}$$

In our case it is easy to see that for large time values there is only power reduction of the probability:

$$G(o,t;E) \sim ((\mu E)^2 t)^{3/4} \tag{84}$$

This result can be easily understood. The electrical field acts on particles only when they are on a structure axis. But most of the time a particle remains on the fingers, outside the axis, so a more slightly power dependence is obtained.

9 Levy flight diffusion.

As it was discussed above another microscopical model with anomalous diffusion is a model with Levy flight diffusion. A feature of the Levy flight diffusion is that in each step a particle may move for an arbitrarily large distance, so that the root-mean-square displacement per unit time appears to be infinite [14]. Numerical simulation of diffusion via Levy hops shows that the points visited by a diffusing particle form spatially well-separated clusters. From more in-depth consideration one can see that each cluster consists of a set of clusters, so that a structure of self-similar clusters appears [15]. So one can say that Levy diffusion is a random walk among self-similar clusters.

The probability distribution function in the Fourier representation has the form:

$$P(k,t) \propto e^{-A|k|^{\mu}t} \tag{85}$$

where A and μ are positive magnitudes, $1 < \mu < 2$. Such stable distributions are called Levy distributions. A more detailed discussion of Levy hops is given in [21].

The study of Levy diffusion is of interest as a microscopical model with unusual diffusion, but also in connection with some possible applications, for example , to the hopping conductivity problem in inhomogeneous medium [22].

9.1 Discrete distribution of Levy random walks.

Let us consider a one-dimensional discrete analog of a Levy flight [14]. Let the probability, that a particle occupies the l-th site after n steps, be $P_n(l)$ and let $f(l)$ be the probability distributions of hops over lengths. So the master equation for complex diffusion has the form:

$$P_{n+1}(l) = \sum_{m=-\infty}^{\infty} f(l-m)P_n(m) \tag{86}$$

To simulate a Levy flight the following function is used for $f(l)$:

$$f(l) = \sum_{n=0}^{\infty} a^{-n}(\delta_{l,-b^n} + \delta_{l,b^n}) \tag{87}$$

where $\delta_{n,m}$ is the Kronecker delta and a and b are the parameters of the Levy flight. Then after Fourier transformation the structure function for such a random walk is equal to:

$$\lambda = \int f(l)exp(ikl)dl = \sum_{n=0}^{\infty} a^{-n}\cos(kb^n) \tag{88}$$

Note that the structure function $\lambda(k)$ satisfies the functional equation:

$$\lambda(k) = a\lambda(kb) + cos(k) \tag{89}$$

Hence at $k \to 0$ the structure function is a power law function with exponent $\mu = ln(a)/ln(b)$. One can establish the non-analytic power-law behavior at $k \to 0$ by means of a Mellin transformation, or with the help of Poisson formulae for set summation . For details see [14].

9.2 Transition from ordinary diffusion to Levy diffusion.

In this section , in addition to Levy hops we allow for ordinary diffusion. The simplest way to do this is to introduce a finite hop length ξ at each step. So we obtain a random walk in which ordinary diffusion alternates with Levy hops. However, due to the super-linear time dependence of the rms displacement for Levy diffusion, on small scales (times) the main contribution to the random walk is provided by ordinary diffusion, while at long times Levy hops contribute most to the random walks. Accordingly, the hop-length distribution function has the form:

$$f(l) = \sum_{n=0}^{\infty} a^{-n}(\delta_{l,-(b^n+\xi)} + \delta_{l,(b^n+\xi)} \tag{90}$$

Hence the structure function is :

$$\lambda = \sum_{n=0}^{\infty} a^{-n} \cos(kb^n + k\xi) \tag{91}$$

In the limit of small length ($b \to 0$) this formula turns into the expression corresponding to ordinary diffusion:

$$\lim_{b\to 0} \lambda(k,\zeta) = \frac{a-1}{a} cos(k\xi) \tag{92}$$

10 Nonlinear relation between diffusion and conductivity.

10.1 Einstein relation and its generalization.

Below the particle drift or the relation between diffusion and conductivity is studied when there is Levy diffusion in the system. For the case of usual classical diffusion and linear response (Ohm's law) this problem was considered by A. Einstein and the well-known Einstein relation was obtained. However in the case of Levy hops a question about the existence of an Einstein relation arises. The problem is that the diffusion coefficient, defined in the usual way as $D = \lim\limits_{t\to\infty} \frac{x^2(t)}{t}$, diverges in a Levy flight diffusion case.

Consequently, there are two possibilities: Either the particle mobility tends to infinity, which is nonsense from a physical point of view, or the Einstein relation is broken. Below it will be shown that instead of the Einstein relation a new nonlinear relation between mobility and diffusion coefficient appears.

Let us recall the well-known Einstein arguments. Let there be in the system the diffusion $J_d = -D\nabla n$ and the field $J_f = \mu E n$ currents. In the equilibrium the diffusion current J_d is compensated by the field current J_f, and the distribution function must have Boltzmann's form:

$$J_d + J_f = 0,\ N_{eq} \propto e^{-U/kT} \tag{93}$$

where U is the potential energy, T is temperature, and k is Boltzmann's constant.

Before applying analogous arguments to Levy flights consider the assumptions used in deriving the Einstein relation. There are the three following assumptions:

i) the Boltzmann's statistics

ii) the expression for the diffusion current in the usual classical form

iii) the linear Ohm's law

Let us try to understand which of these assumptions need to be modified. Firstly, the assumption about Boltzmann's statistics is not essential, since its type is determined by the statistical properties of the system, and we will retain it. Secondly, the diffusion current has a different form and we write it in a general operator form:

$$J_d = -\hat{K}n = -iA\vec{k}|k|^{\mu-2}n \tag{94}$$

And finally we write the field current as $J_f = nV$, where V is the drift velocity.

By taking a definition for the derivative of the fractional order in the form of the set [23], one can get a general formula for the drift velocity:

$$\vec{V} = e^{U/kT} \lim_{\epsilon \to 0} (\triangle^2 + \epsilon)^{(\mu-2)/4} \nabla \exp(-\frac{U}{kT}) \tag{95}$$

where $\triangle$ is the Laplace operator.

In a homogeneous electrical field $U = -qEr$ we recover that the drift velocity depends on the electric field in a nonlinear way:

$$V = Aq\vec{E}\frac{|q\vec{E}|^{\mu-2}}{(kT)^{\mu-1}} \tag{96}$$

It should be emphasized that this nonlinearity occurs in arbitrarily weak fields and is a consequence of the unusual character of diffusion. The power of nonlinearity is described by the critical index of the Levy hop diffusion.

This is a preliminary result, which we obtain below in an exact way.

11 Random walks of Levy and particle drift in the electric field.

Let us now introduce an anisotropy into the random walk on self-similar clusters. By virtue of the specific nature of Levy hops a particle can move in one hop over an arbitrary distance b^n. For this reason a small anisotropy $(1+\alpha)$, with $\alpha = qEs/kT$, when particles move on a small distance s, becomes exponentially large on large distances b^n. Since at each step a diffusing particle leaves a site, the sum of probabilities W_+ and W_- of motions parallel and anti-parallel, respectively, to the field must be equal to 1:

$$W_+ + W_- = 1$$

Hence we get the expressions for probabilities of motion parallel and anti-parallel to the field:

$$W_\pm = \frac{(1 \pm \alpha)^{b^n}}{(1+\alpha)^{b^n} + (1-\alpha)^{b^n}} \tag{97}$$

Therefore, the structure function $\lambda(k;E)$ in the case of diffusion via Levy hops in the electrical field equals:

$$2\lambda(k;E) = \sum_{n=0}^{\infty} a^{-n}[\cos(kb^n) + i\sin(kb^n)(W_+ - W_-)] \tag{98}$$

As for usual diffusion the second term contains the drift velocity for small $k \to 0$:

$$V = i\frac{\partial\lambda(k;E)}{\partial k}|_{k\to 0} = \sum_{n=0}^{\infty}(\frac{b}{a})^n * (W_+ + W_-) \approx \sum_{n=0}^{\infty}(\frac{b}{a})^n \tanh(\alpha b^n) \tag{99}$$

where $\tanh(x)$ is the hyperbolic tangent.

Using the Poisson formula we obtain after some calculations the formula for the velocity:

$$V(E) = \alpha/2 + \alpha^{\mu-1}\left[\sum_{m=-\infty}^{\infty}\int_1^{\infty}\tanh(z)z^{-\gamma_m}dz + \int_o^{\alpha}\tanh(z)z^{-\gamma_m}dz\right] \tag{100}$$

where the exponent is equal to:

$$\gamma_m = \mu + 2\pi i m/\ln b.$$

It is easy to see that for weak fields the second term in brackets is less than the first term. Thus, in arbitrarily weak electric fields one can get the nonlinear field dependence of velocity (96).

11.1 Transitions from ordinary diffusion to Levy diffusion and from Ohm's law to nonlinear response.

Anisotropy is introduced into these random walks using the method described above: we replace the hop length with the quantity $b^n + \xi$. Thus the structure function in an electric field and for finite hop length is:

$$\lambda(k,\xi,\alpha) = \sum_{n=0}^{\infty} a^{-n}[\cos(kb^n + k\xi) + i\sin(kb^n + \xi)(W_+ - W_-)] \tag{101}$$

And after calculations by Poisson's method we obtain the following results: in arbitrarily weak fields the velocity is nonlinear in the field, eq. (96), and crosses over to linear behavior in strong fields:

$$V \simeq E\xi^{2-\mu}, \qquad qE\xi/kT \gg 1 \tag{102}$$

Thus the particle velocity in an electric field has two asymptotic limits in accordance with two diffusing regimes: Levy hops and ordinary diffusion.

12 Numerical simulations.

Below the results of numerical simulations of diffusion via Levy hops are reported. Let's briefly explain the algorithm of simulations. Probabilities of left and right walks are determined as probabilities to have a random value from [0;0.5] and [0.5;1] correspondingly. The anisotropy of random walks is simulated by decreasing the length of [0;0.5] for the quantity W_- in anti-parallel field and increasing [0.5;1] for W_+ in a parallel field. The simulations are made at different values of the parameters a and b . As the probability decreases rather rapidly we can confine ourselves to finite terms in the sum (96). For example, at $a = 50, b = 10, n = 6$ and $a = 6, b = 3, n = 12$. But we proceed so that at every hop the sum of all probabilities with finite numbers of hops equals to 1, that is particles do not stay in the site.

The results of random walks , fig.3, are in accordance with known results [15]. The step-like dependence of rms as a function of time is easy to understand as follows. The particle diffuses at nearest sites mainly, making the cluster from visited sites, and hops with small probability at big distance (at next step) and again diffuses at nearest sites and so on. The electric field leads to the particle drift. The dependence of the average displacement as a function of the time is presented in fig.4 at different values of anisotropy. As expected it has a linear dependence and from this linear dependence it is easy to find the particle velocity by the standard way : $V =< X > /N$. The value of the nonlinear dependence index is determined from numerical simulation data as

$$\mu_{exp} = 1 + \frac{ln(V/V_0)}{ln(\alpha/aplha_0)} \tag{103}$$

The results are presented in fig.5. The main distortion in the simulations is due to the random character of walks and is founded in the calculations from values of average displacements at zero fields.

13 Discussion.

We have studied random walks on the comb model and found that the existence of fingers on comb model - analog of "dead ends" in the current-carrying paths of percolation systems leads to the anomalous nature of the random walk. We have established that for diffusion problems , in which the number of particles is not conserved, the generalized diffusion equations must be the fractal temporal derivative equations : instead of a first temporal derivative , the equation must contain the fractional-order derivative. Fractional temporal derivatives emerge due to the random disappearance and reappearance of diffusing particles (the departure of particles from axis and their return). Let's stress that in our consideration the fractal temporal diffusion equations are deduced in an obvious way. The physical sense of fractional temporal derivative is clear. Usually the fractal diffusion equations are postulated [25], [26] and [18], and questions about the possibility of its application are arised.

When we examine random walks in a medium with traps, the same problems appear. As noted earlier, the problem of diffusion in a medium with traps differs from the problem of diffusion along the axis of a comb structure. The difference lies in the fact that the particles do not disappear, but delay at each site with a certain probability. The total number of diffusing particles is conserved. In other words we have the law of mass conservation, expressed by a continuity equation. However, the anomalous nature of diffusion, due to the capture of particles by the traps, leads to an unusual expression for the diffusion current with fractional temporal derivative. Note that mathematically the generalized diffusion equations in both problems are different and describe different physical situations. First, in a diffusion along the axis of a comb structure the number of particles is not conserved. Second, the diffusion currents are different.

The problem of diffusion on the comb model with random distribution of fingers over lengths is considered in the effective medium approximation. It is widely used in the conductivity problem of the random inhomogeneous medium and is considered as a general approach. So a diffusion on the random comb model is studied by EMA. As it was shown the average length of finger $L =< \frac{1}{l} >^{-1}$ has a finite value in the case of power distribution over lengths and at "parallel" way of averaging. As a result of this the time dependence of rms remains the same -the square ones. But in the case of "consequent" way of averaging the average length of finger is diverged $L =< l >= \infty$ and it leads to the change of a diffusion character:

$$< X^2(t) >\sim t^{1-\gamma/2} \tag{104}$$

Let's note that due to a randomness a diffusing particle moves more rapidly: $1 - \gamma/2 < 1/2$. It is connected with the existence of short fingers and the diffusing particle rapidly passes these short fingers and then returns to the axis of the structure to continue the random walks. In the case of the comb structure with an infinite length of fingers a particle returns to the axis with a small probability $P(t) \sim t^{-1/2}$. If the time evolution operator $\hat{T}$ is substituted by the averaged meaning ,that one obtains :

$$< X^2(t) >\sim t^{\gamma/2} \tag{105}$$

but it has a rather paradox physical sense: existence of many short fingers leads to "slow" diffusion. So a further careful study of diffusion on random comb structure is necessary.

The generalized relation between diffusion and conductivity is obtained for a sub-diffusion case. It has the form of the well- known Einstein relation for the diffusion coefficient and the particle mobility, depending on the time.

In the second part of the paper the Levy flight diffusion is considered. The main result consists of the nonlinear dependence of the particle mobility in weak electric fields. Usually theoreticians expand the current in powers of the electric field of the electric field:

$$J = \sigma E + \chi |E|^2 E + \dots \tag{106}$$

Our result essentially differs from those , obtained by such a method. In the microscopical model of Levy hops we show that current depends on an electric field in a nonlinear way due to unusual regime of diffusion in space, i.e. there is no linear term, corresponding to Ohm's law, in the field expansion of the current (106). In other words if there is an usual diffusion in the system, so the Ohm's law exists , in the case of anomalous diffusion as Levy hops the response of system has a nonlinear character.

We consider the transition from ordinary diffusion to Levy flight by introducing a finite displacement length ξ at each step. The new parameter $qE\xi/kT$, which determines whether the particle mobility behaves linearly or nonlinearly, appears in the problem. In other words a new physical length L_E governed by the electric field emerges in such diffusion problems:

$$L_E = \frac{kT}{qE} \tag{107}$$

To appreciate the significance of this quantity we consider an ordinary random walk in an external electric field. Let's imagine that the medium is partitioned into the blocks of size L_E. Then we study the particle behavior within a single block. With a probability of order unity the particles leave the block when it moves along the field and does not leave the block when it moves against the field. Briefly speaking within a block, whose linear size is of order L_E, ordered motion prevails over diffusion. This makes it possible to estimate the particle velocity to be:

$$V = \frac{L_E}{t_E} \tag{108}$$

where t_E is the diffusion time for the distance L_E. For ordinary diffusion $t_E = L_E^2/D$ and we have the well-known Einstein relation:

$$V = q^2 DE/kT \tag{109}$$

For a Levy flight diffusion the same estimates give the nonlinear dependence of velocity. And for the case of two diffusion limits we have two different: linear and nonlinear expressions for mobility. Recently the deduced nonlinear behavior of the velocity due to the unusual nature of diffusion was confirmed by the independent numerical simulations of particle drift in the presence of Levy diffusion [24].

Strictly speaking, the diffusion on clusters has an sub-diffusion character, but the nature is still not clear. As it was shown above we can distinguish between two limiting sub- and super-diffusion cases if we know the response of the system of an electric field. An attempt to detect the mobility nonlinearity by computing modeling was not successful [7], since over a desired range of fields the electric field in inhomogeneous media induces traps. Such traps are sections of the current paths , directed against the electric field.

As for experimental results many researches have observed the nonlinear power dependence of the current in inhomogeneous media with exponents close to anomalous diffusion index and have different explanations of this phenomenon - see [27], [28]. In our opinion, the nonlinear behavior may be explained in an universal way as a result of anomalous nature of random walks in inhomogeneous media. But comparisons of theoretical and experimental results require further study.

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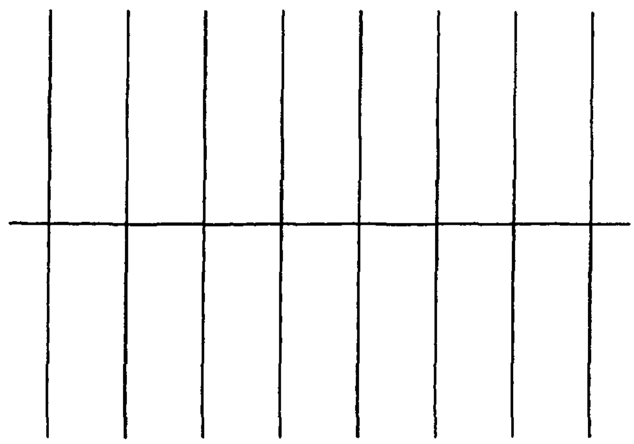

Figure 1. Comb structure:
the conducting axis (y=0) has fingers going to infinity.

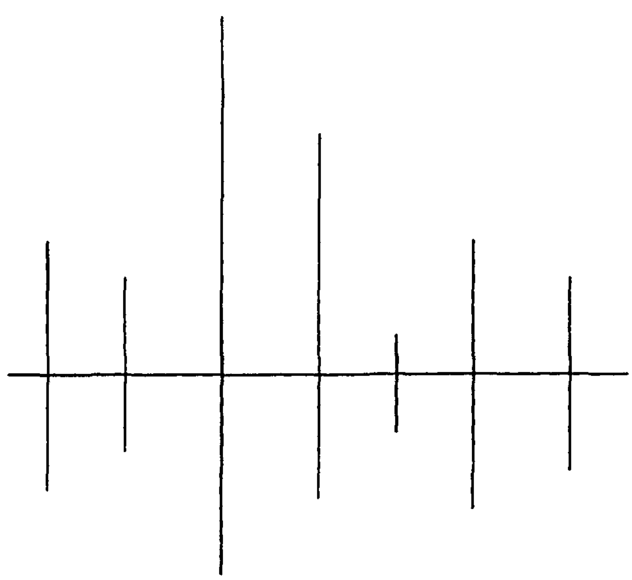

Figure 2. Random comb structure.

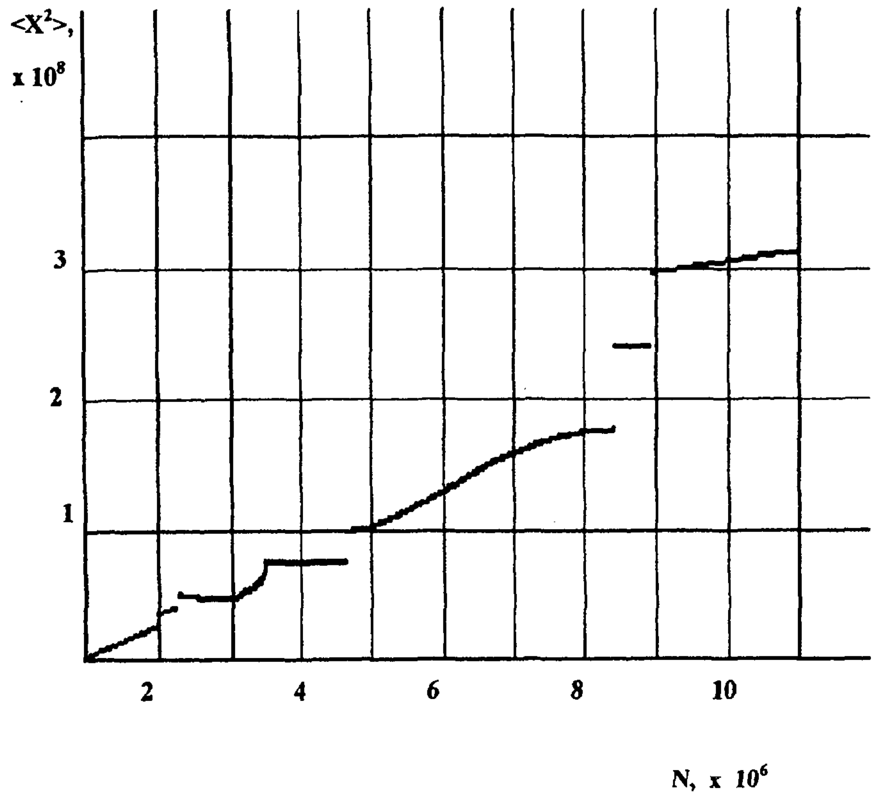

Figure 3. Typical dependence of r.m.s. displacement X^2 from number of hops N.

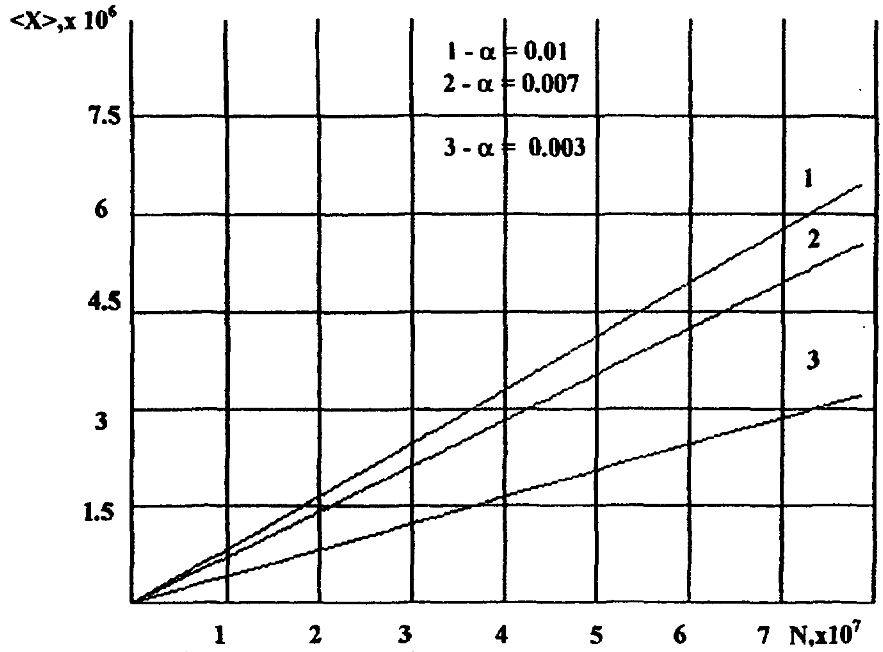

Figure 4. The dependence of the average displacement <X> from number of hops N at different values of anisotropy α.

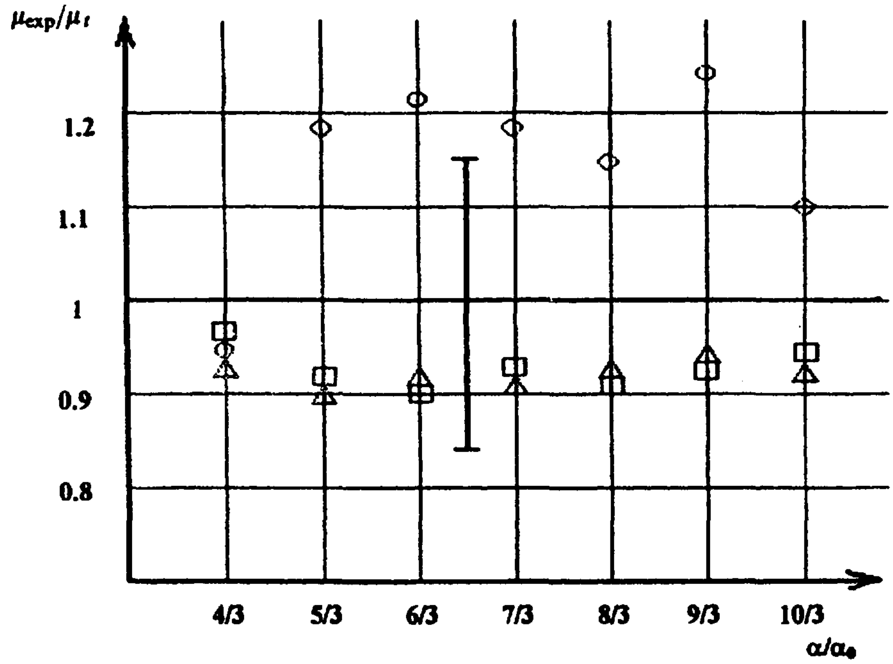

Figure 5. The dependence of relation $\mu_{\exp} / \mu_t$ at different values of anisotropy.