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Available at: http://www.ictp.trieste.it/~pub_off IC/2001/32

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# THE CARDY-VERLINDE FORMULA AND TOPOLOGICAL AdS-SCHWARZSCHILD BLACK HOLES

Donam Youm[1]
*The Abdus Salam International Centre for Theoretical Physics, Trieste, Italy.*

**Abstract**

We consider the brane universe in the background of the topological AdS-Schwarzschild black holes. The induced geometry of the brane is that of a flat or an open radiation dominated FRW-universe. Just like the case of a closed radiation dominated FRW-universe, the temperature and entropy are simply expressed in terms of the Hubble parameter and its time derivative when the brane crosses the black hole horizon. We propose the modified Cardy-Verlinde formula which is valid for any values of the curvature parameter $k$ in the Friedmann equations.

MIRAMARE – TRIESTE
May 2001

[1]E-mail: youmd@ictp.trieste.it

Verlinde made an interesting proposal [1, 2] that the Cardy formula [3] for a two-dimensional conformal field theory (CFT) can be generalized to an arbitrary spacetime dimensions and such generalized formula, called the Cardy-Verlinde formula, is closely related to the Friedmann equation at the moment when the cosmological bounds on the thermodynamic quantities of the holographic dual theory are saturated. This result is later shown to hold for the holographic duals to various bulk backgrounds [4, 5, 6, 7, 8, 9, 10, 11]. The quantum effects to the Cardy-Verlinde formula were studied in Refs. [12, 13, 14]. In this note, we consider the brane universe in the bulk background of the AdS-Schwarzschild black holes with the event horizon having zero and negative curvatures, the so-called topological black holes. (Cf. Some aspects of the brane cosmology in such bulk background were explored also in Refs. [15, 16].) The brane universes under consideration are therefore flat and open universes. We propose the modified Cardy-Verlinde formula which holds for any values of the curvature parameter $k$ in the Friedmann equations. We show that even for flat and open brane universes the proposed cosmological bounds on the temperature and entropy of the holographic dual theory are saturated and thereby simply expressed in terms of the Hubble parameter and its time derivative when the brane crosses the horizon of the AdS-Schwarzschild black hole.

It is believed that black holes in asymptotically flat spacetime should have spherical horizon [17, 18]. However, when a spacetime has negative cosmological constant, a black hole can have non-spherical horizon. Four-dimensional black hole solutions whose horizon is an arbitrary genus Riemann surface were studied in Refs. [19, 20, 21, 22, 23, 24]. In Ref. [25], such black hole solutions were generalized to arbitrary spacetime dimensions. The solution has the following form:

$$\begin{aligned} ds^2_{n+2} &= -h(a)dt^2 + \frac{1}{h(a)}da^2 + a^2\gamma_{ij}(x)dx^i dx^j, \\ h(a) &= k - \frac{w_{n+1}M}{a^{n-1}} + \frac{a^2}{L^2}, \qquad \omega_{n+1} = \frac{16\pi G_{n+2}}{n\mathrm{Vol}(M^n)}, \end{aligned} \tag{1}$$

where $\gamma_{ij}$ is the horizon metric for a constant curvature manifold $M^n$ with the volume $\mathrm{Vol}(M^n) = \int d^n x\sqrt{\gamma}$, $G_{n+2}$ is the $(n+2)$-dimensional Newton's constant, $M$ is the ADM mass of the black hole and $L$ is the curvature radius of the background AdS spacetime. The horizon geometry of the black hole is elliptic, flat and hyperbolic for $k = 1, 0, -1$, respectively. The Bekenstein-Hawking entropy and the Hawking temperature of the black hole are

$$S = \frac{a^n_H \mathrm{Vol}(M^n)}{4G_{n+2}}, \qquad \mathcal{T} = \frac{h'(a_H)}{4\pi} = \frac{(n+1)a_H^2 + (n-1)kL^2}{4\pi L^2 a_H}, \tag{2}$$

where $a_H$ is the horizon, defined as the largest zero of $h(a)$, and the prime denotes the derivative w.r.t. $a$. As for the $k = -1$ case, the requirement of positivity of temperature enforces an inequality on the value of $a_H$, namely that $a_H > L\sqrt{(n-1)/(n+1)}$.

We consider an $n$-brane moving in the background of the above AdS-Schwarzschild black hole. The metric on the brane is given by the induced metric

$$ds_{n+1}^2 = -\left[h(a) - \frac{1}{h(a)}\left(\frac{da}{dt}\right)^2\right]dt^2 + a^2\gamma_{ij}dx^i dx^j. \tag{3}$$

In terms of a new time coordinate $\tau$, called the cosmic time, satisfying

$$\frac{1}{h(a)}\left(\frac{da}{d\tau}\right)^2 - h(a)\left(\frac{dt}{d\tau}\right)^2 = -1, \tag{4}$$

the brane metric (3) takes the standard Robertson-Walker form

$$ds_{n+1}^2 = -d\tau^2 + a^2(\tau)\gamma_{ij}dx^i dx^j, \tag{5}$$

with the cosmic scale factor $a$. The equation of motion for the brane action can be translated into [2]

$$\frac{dt}{d\tau} = \frac{\sigma a}{h(a)}, \tag{6}$$

where the parameter $\sigma$ is related to the brane tension. From Eqs. (4,6), we obtain the following Friedmann equation for the radiation dominated brane universe:

$$H^2 = \frac{\omega_{n+1}M}{a^{n+1}} - \frac{k}{a^2}, \tag{7}$$

by fine-tuning the brane tension to $\sigma = 1/L$ so that the cosmological constant term in the Friedmann equation vanishes. Here, $H \equiv \dot{a}/a$ is the Hubble parameter, where the overdot denotes the derivative w.r.t. $\tau$. Taking the $\tau$-derivative of Eq. (7), we obtain the second Friedmann equation

$$\dot{H} = -\frac{n+1}{2}\frac{\omega_{n+1}M}{a^{n+1}} + \frac{k}{a^2}. \tag{8}$$

According to the AdS/CFT correspondence, thermodynamic quantities of the CFT at high temperature can be identified with the corresponding thermodynamic quantities of the bulk AdS black hole [26]. Since the standard GKPW prescription [27, 28] does not fix the overall scale of the boundary metric, we are free to re-scale the boundary metric to be of the following form:

$$ds_{CFT}^2 = \lim_{a\to\infty}\left[\frac{L^2}{a^2}ds_{n+2}^2\right] = -dt^2 + L^2\gamma_{ij}dx^i dx^j. \tag{9}$$

Since the CFT time is rescaled by the factor $L/a$ w.r.t. the AdS time, the energy $E$ and the temperature $T$ of the CFT are rescaled by the same factor w.r.t. the corresponding thermodynamic quantities of the AdS black hole:

$$E = M\frac{L}{a}, \qquad T = \mathcal{T}\frac{L}{a} = \frac{1}{4\pi a}\left[(n+1)\frac{a_H}{L} + (n-1)\frac{kL}{a_H}\right], \tag{10}$$

whereas the entropy $S$ of the CFT is given by the Bekenstein-Hawking entropy (2) of the AdS black hole without re-scaling. Note, in terms of the energy density $\rho = E/V$ and the pressure $p = \rho/n$ of the CFT within the volume $V = a^n\mathrm{Vol}(M^n)$, the Friedmann equations (7,8) take the following standard forms:

$$H^2 = \frac{16\pi G}{n(n-1)}\rho - \frac{k}{a^2}, \tag{11}$$

$$\dot{H} = -\frac{8\pi G}{n-1}(\rho + p) + \frac{k}{a^2}, \tag{12}$$

where $G = (n-1)G_{n+2}/L$ is the Newton's constant on the brane. From these Friedmann equations, we obtain the energy conservation equation $\dot{\rho} = nH(\rho + p)$.

The Friedmann equations (11,12) can be respectively put into the following forms, resembling the formulas for the CFT:

$$S_H = \frac{2\pi}{n} a\sqrt{E_{BH}(2E - kE_{BH})}, \tag{13}$$

$$kE_{BH} = n(E + pV - T_H S_H), \tag{14}$$

in terms of the Hubble entropy $S_H$ and the Bekenstein-Hawking energy $E_{BH}$, where

$$S_H \equiv (n-1)\frac{HV}{4G}, \qquad E_{BH} \equiv n(n-1)\frac{V}{8\pi G a^2}, \qquad T_H \equiv -\frac{\dot{H}}{2\pi H}. \tag{15}$$

The first Friedmann equation (11) can be also rewritten as the following relation among the Bekenstein entropy $S_B = \frac{2\pi}{n}Ea$, the Bekenstein-Hawking entropy $S_{BH} = (n-1)\frac{V}{4Ga}$ and the Hubble entropy $S_H$:

$$S_H^2 = 2S_B S_{BH} - kS_{BH}^2. \tag{16}$$

We consider the moment at which the brane crosses the black hole horizon $a = a_H$, defined as the largest root of $h(a) = 0$, i.e.,

$$\frac{a_H^2}{L^2} + k - \frac{w_{n+1}M}{a_H^{n-1}} = 0. \tag{17}$$

From Eqs. (7,17), we see that

$$H^2 = \frac{1}{L^2} \qquad \text{at} \qquad a = a_H. \tag{18}$$

The entropy $S$ remains constant during the cosmological evolution, but the entropy density,

$$s = \frac{S}{V} = (n-1)\frac{a_H^n}{4GLa^n}, \tag{19}$$

varies with time. From Eqs. (18,19), we see that the entropy density at $a = a_H$ is given in terms of $H$ at $a = a_H$ in the following form:

$$s = (n-1)\frac{H}{4G} \qquad \text{at} \qquad a = a_H, \tag{20}$$

which implies

$$S = S_H \qquad \text{at} \qquad a = a_H. \tag{21}$$

From the temperature expression $T = h'L/(4\pi a_H)$ at $a = a_H$ along with the formula $H^2 = \sigma^2 - h(a)/a^2$ (which follows from Eqs. (4,6)), we see that the CFT temperature at $a = a_H$ can be expressed in terms of $H$ and $\dot{H}$ in the following way:

$$T = -\frac{\dot{H}}{2\pi H} \qquad \text{at} \qquad a = a_H. \tag{22}$$

Eq. (14) along with Eqs. (21,22) implies

$$E_C = kE_{BH} \qquad \text{at} \qquad a = a_H, \tag{23}$$

where $E_C$ is the Casimir energy defined as

$$E_C \equiv n(E + pV - TS). \tag{24}$$

So, for any values of $k$, the thermodynamic quantities of the CFT take the forms simply expressed in terms of the Hubble parameter and its time derivative when the brane crosses the black hole horizon.

The above thermodynamic quantities of the CFT satisfy the first law of thermodynamics,

$$TdS = dE + pdV, \tag{25}$$

which can be expressed in terms of the densities as

$$Tds = d\rho + n(\rho + p - Ts)\frac{da}{a}, \tag{26}$$

making use of $dV = nVda/a$. If the entropy and energy are assumed to be purely extensive, then the combination $\rho + p - Ts$ is always zero. For the conformal system under consideration, the combination is not always zero due to the subextensive contribution. To find the expression for the combination, we express the energy density of the CFT in the following way:

$$\rho = \frac{na_H^n}{16\pi G_{n+2}a^{n+1}}\left(\frac{a_H}{L} + k\frac{L}{a_H}\right), \tag{27}$$

and make use of the equation of state $p = \rho/n$, which is valid for CFTs. The resulting expression is

$$\frac{n}{2}(\rho + p - Ts) = k\frac{\gamma}{a^2}, \tag{28}$$

where the Casimir quantity $\gamma$ is given by

$$\gamma = \frac{n(n-1)a_H^{n-1}}{16\pi Ga^{n-1}}. \tag{29}$$

In other words, the Casimir energy of the CFT is given by

$$E_C = \frac{kn(n-1)a_H^{n-1}\text{Vol}(M^n)}{8\pi Ga}. \tag{30}$$

So, the Casimir energy is positive [negative] for $k = 1$ [$k = -1$] and zero for $k = 0$. The entropy density (19) of the CFT can be expressed in terms of $\gamma$ and $\rho$ as

$$s^2 = \left(\frac{4\pi}{n}\right)^2 \gamma\left(\rho - k\frac{\gamma}{a^2}\right). \tag{31}$$

By making use of Eq. (20), we can show that the entropy density expression (31) at $a = a_H$ (i.e., when the brane crosses the black hole horizon) exactly reproduces the first Friedmann equation

(11). Furthermore, by making use of Eqs. (20,22), we can show that Eq. (28) reproduces the second Friedmann equation (12) when $a = a_H$. This result implies that for *any* values of $k$ the Friedmann equations know about the thermodynamic properties of the CFT.

Since the Casimir energy (30) is negative and zero respectively for the $k = -1, 0$ cases, the Cardy-Verlinde formula proposed in Ref. [1] is not valid for these cases. We can nevertheless infer the modified Cardy-Verlinde formula which is valid for any $k$ from cosmological formulas (13,14). We have shown that $S = S_H$ and $E_C = kE_{BH}$ when $a = a_H$. So, from the cosmological Cardy formula (13) we can infer the following modified form of the Cardy-Verlinde formula, valid for any $k$:

$$S = \sqrt{\frac{2\pi a}{n} S_C (2E - E_C)}, \tag{32}$$

which is just Eq. (13) with $a = a_H$ reexpressed in terms of $S_C$ and $E_C$. Since $E_C$ is zero and negative respectively for $k = 0, -1$, we here choose to define the Casimir entropy $S_C$ first and then define $E_C$ in terms of $S_C$ in the following way:

$$S_C \equiv \frac{2\pi}{n} E_{BH} a \Big|_{a=a_H}, \qquad E_C \equiv \frac{kn}{2\pi a} S_C = kE_{BH}|_{a=a_H} \frac{a_H}{a}, \tag{33}$$

where $E_{BH}$ is defined in Eq. (15). So, the Casimir entropy of the CFT is given by

$$S_C = (n-1)\frac{a_H^{n-1}\mathrm{Vol}(M^n)}{4G}, \tag{34}$$

which is positive for any $k$, in accordance with an interpretation of $S_C$ as a generalization of the central charge to arbitrary spacetime dimensions. Note, the definition of $E_C$ in Eq. (33) is compatible with another definition (24) of $E_C$. This is due to the fact that Eq. (14) and Eq. (24) coincide when $a = a_H$, since $S = S_H$ and $T = T_H$ when $a = a_H$. The modified formula (32) expresses that the entropy $S$ has negative [positive] contribution from the Casimir effect for the $k = 1$ [$k = -1$] case and no Casimir effect contribution for the $k = 0$ case. The modified Cardy-Verlinde formula (32) can be rewritten as the following relation among $S$, $S_C$ and $S_B$:

$$S^2 = 2S_B S_C - kS_C^2. \tag{35}$$

This relation has the same form as the relation (16) among the cosmological entropy bounds, except that the roles of $S_H$ and $S_{BH}$ are respectively taken over by $S$ and $S_C$. These two relations coincide when $a = a_H$.

We have seen that for any values of $k$ the thermodynamic quantities of the CFT take the simple forms that saturate the cosmological bounds, which are originally conjectured [1] for the $k = 1$ case, only. We argue that such conjectured cosmological bounds hold even for the $k = -1, 0$ cases. First of all, the criterion for distinguishing between a weakly and a strongly self-gravitating universe becomes modified when $k \neq 1$. If we choose to define the universe to be weakly [strongly] self-gravitating when the total energy $E$ is less [greater] than $E_{BH}$ (defined as the energy required to form a black hole with the size of the entire universe), then from the first Friedmann equation (11) we see that the criterion on $H$ is modified to

$$\begin{aligned} E \le E_{BH} \quad &\Leftrightarrow \quad S_B \le S_{BH} \qquad \text{for} \qquad Ha \le \sqrt{2-k} \\ E \ge E_{BH} \quad &\Leftrightarrow \quad S_B \ge S_{BH} \qquad \text{for} \qquad Ha \ge \sqrt{2-k}. \end{aligned} \tag{36}$$

We propose that for general values of $k$ the cosmological bound on $S_C$ conjectured in Ref. [1] continues to hold:

$$S_C \le S_{BH}. \tag{37}$$

For the strongly self-gravitating case, from Eqs. (36,37) we have $S_C \le S_{BH} \le S_B$. From Eq. (35) we see that $S$ is a monotonically increasing function of $S_C$ in the interval $S_C \le S_B$ for any values of $k$. So, $S$ reaches its maximum value when $S_C = S_B$ and therefore $S_C = S_{BH}$, for which case $S = S_H$ as can be seen from Eqs. (16,35). The conjectured cosmological bound (37) on $S_C$ for the strongly self-gravitating case therefore implies the Hubble entropy bound for any values of $k$:

$$S \le S_H \qquad \text{for} \qquad Ha \ge \sqrt{2-k}. \tag{38}$$

The criterion (36) on $H$ for the strongly self-gravitating universe along with the first Friedmann equation (7) implies $a^{n-1} \le \omega_{n+1} M/2$. So, from the explicit expression (10) for $T$, we infer the following cosmological bound on the temperature of the CFT:

$$T \ge T_H \qquad \text{for} \qquad Ha \ge \sqrt{2-k}. \tag{39}$$

These cosmological bounds (38,39) are saturated at the moment when the brane crosses the black hole horizon, as shown in Eqs. (21,22), respectively.

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