EXACT ANALYTIC EXPRESSIONS FOR THE EVOLUTION OF POLARIZATION FOR RADIATION PROPAGATING IN A PLASMA WITH NONUNIFORMLY SHEARED MAGNETIC FIELD

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SHEARED MAGNETIC FIELD

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Riassunto
Le espressioni, già note, per l'evoluzione della polarizzazione di onde elettromagnetiche propaganti in un plasma magnetizzato con shear costante vengono estese a casi in cui questo non è costante. Si trovano soluzioni analitiche esatte per il caso in cui le variazioni spaziali del mezzo sono tali da soddisfare una particolare condizione (eq. 13), eventualmente in un opportuno sistema di riferimento nello spazio della polarizzazione (lo spazio di Poincaré).

Abstract
The known analytic expressions for the evolution of the polarization of electromagnetic waves propagating in a plasma with uniformly sheared magnetic field are extended to the case where the shear is not constant. Exact analytic expressions are found for the case when the space variations of the medium are such that the magnetic field components and the plasma density satisfy a particular condition (eq.13), possibly in a convenient reference frame of polarization space.

Keywords: Polarization Evolution, Magnetized Plasma, Electromagnetic Waves
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1. Let us first consider electromagnetic waves propagating in a plasma with uniformly sheared magnetic field, that is, specifically, a time-independent plasma having a constant density $n$ and constant components of the magnetic field $B$ along and transverse to the propagation direction, $z$, but with the direction of the transverse component of $B$ rotating uniformly with $z$. Then, neglecting particle collisions and thermal effects, it has been found independently by various authors [1-5] that there exist two exact solutions of the propagation equation which represent two e.m. waves which have orthogonal polarizations and propagate with constant phase velocities and ellipticities and with an azimuth which rotates in space following the rotation of the transverse field. These waves have been called [6] helical waves. Since the two helical waves are orthogonal, any fully polarized wave incident on the medium can be resolved into a linear combination of the two helical waves with convenient amplitudes and phases; after propagation the output wave can be obtained by adding together again the two helical waves, including the difference in phase accumulated during propagation due to the difference in phase velocities.

Actually the helical waves for transverse propagation have also been found independently in the past in different fields of research in optics: for stacks of birefringent plates [7], for liquid crystals [8-11] and for microwave twisted-strip polarizers [12].
It is the purpose of this Note to show that exact analytic solutions of the
propagation equation can be found also for plasmas with more general space
dependencies of \( n(z) \) and \( B(z) \) than in the uniformly sheared case. For these
more general configurations again one finds the existence of two waves having
constant ellipticities and azimuths which rotate, in general non-uniformly, in
space. This generalization was briefly discussed previously (see Ref.13, p.68-
70): here we will give a simpler and more complete treatment.

2. Let us consider a fully polarized electromagnetic wave of frequency
\( \omega \) propagating in the \( z \) direction in a magnetized plasma. We call \( \psi \) the
azimuth of the polarization ellipse \( (0 \leq \psi \leq \pi) \) and \( \chi \) the ellipticity given by
\[
\tan \chi = \pm b/a \quad (-\pi/4 \leq \chi \leq \pi/4)
\]
where \( a \) (or \( b \)) is the major (or minor) semiaxis of
the ellipse and the plus (or minus) sign is taken for clockwise (or
anticlockwise) rotation of the wave electric field, looking towards the radiation
source. Then in order to describe the state of polarization, instead of using
\((\psi, \chi)\), it is convenient to use the reduced, 3-component Stokes vector [13],
\[ s = (s_1, s_2, s_3), \]
developed by
\[
s_1 = \cos 2\chi \cos 2\psi, \quad s_2 = \cos 2\chi \sin 2\psi, \quad s_3 = \sin 2\chi,
\]
(1)
Thus \( |s| = 1 \), while \( \psi \) and \( \chi \) are spherical coordinates in polarization space
\((s_1, s_2, s_3)\), the so-called Poincaré space, where the Stokes vector is defined. This
space is different from physical space \((x, y, z)\). For wave propagation in the \( z \)
direction \( \psi, \chi \) and \( s \) are all functions of \( z \). When collisions and thermal effects
are negligible \( s \) satisfies the evolution equation (see Ref.13) given by
\[
\frac{ds(z)}{dz} = \Omega(z) \times s(z)
\]
(2)
where the vector \( \Omega = (\Omega_1, \Omega_2, \Omega_3) \), characteristic of the medium, is given by
\[
\Omega = \begin{bmatrix}
\Omega_1 \\
\Omega_2 \\
\Omega_3
\end{bmatrix}
= \frac{\omega_p^2}{2\omega^3} \begin{bmatrix}
(B_x^2 - B_y^2)(e/mc)^2 \\
2B_x B_y (e/mc)^2 \\
2\omega B_z (e/mc)
\end{bmatrix}
\]
(3)
Here \( e \) and \( m \) are the electron charge and mass, \( \omega_p = (4\pi ne^2/m)^{1/2} \) is the plasma
frequency, \( n(z) \) is the plasma density, \( B_x(z), B_y(z) \) and \( B_z(z) \) are the
components of $B(z)$. This expression for $\Omega$ is valid when $\omega^2 >> \omega_p^2$ and $\omega >> \omega_c = eB/mc$ but, if required, a more general expression can be found in Ref.13. It should be noted that $\Omega$ is defined in Poincaré space and indeed eq.2 describes a precession (in Poincaré space) of $s$ around the "instantaneous" axis $\Omega$, with an "angular velocity" $\Omega = |\Omega|$, as z (taking the place of "time") changes.

If we call $\theta(z)$ the angle between $B$ and the z direction and $\alpha(z)$ the angle between the transverse component of $B$ and the x direction, then the components of $B$ are $B = B(\sin\theta\cos\alpha, \sin\theta\sin\alpha, \cos\theta)$ and $\Omega$ can be written as

$$\Omega = \frac{\omega_p^2 \omega_c}{2c \omega^3} \begin{bmatrix} \omega_c \sin^2 \theta \cos 2\alpha \\ \omega_c \sin^2 \theta \sin 2\alpha \\ 2 \omega \cos \theta \end{bmatrix}$$

(4)

For particular solutions of eq.2 it is convenient to introduce the spherical coordinates of $\Omega$ in Poincaré space, $\eta(z)$ and $\phi(z)$, so that

$$\Omega = \Omega(z) \begin{bmatrix} \cos \eta(z) \cos \phi(z) \\ \cos \eta(z) \sin \phi(z) \\ \sin \eta(z) \end{bmatrix}$$

(5)

From eqs.4 and 5 one has

$$\phi = 2\alpha, \quad \tan \eta = \frac{2\omega \cos \theta}{\omega_c \sin^2 \theta}$$

(6)

$$\Omega = \frac{\omega_p^2 \omega_c}{2c \omega^3} (\omega_c^2 \sin^4 \theta + 4\omega^2 \cos^2 \theta)^{1/2}$$

(7)

and eqs.6 show how the orientation $(\phi, \eta)$ of $\Omega$ in Poincaré space changes as the orientation $(\alpha, \theta)$ of $B$ encountered by the wave in physical space changes.

In the uniformly sheared plasma, defined in Section 1, $\omega_p, \omega_c, \theta$ and $d\alpha/dz$ (and hence also $\Omega, \eta$ and $d\phi/dz$) are all constants. In this case, as one moves along the z direction, $B$ precesses in physical space around the z axis while $\Omega$ performs in Poincaré space a uniform precession ($d\phi/dz=$const.) around the $s_3$ axis. The combination of this precession of $\Omega$ together with the precession of $s$ about $\Omega$ gives the trajectory of the tip of $s$ which is a cycloid on the sphere $|s|=1$ (the Poincaré sphere) and has been illustrated in Ref.3. For
two particular choices of the initial \( s \), the cycloids reduce to parallels on the Poincaré sphere and they correspond to the two helical waves which propagate with constant ellipticity and with an azimuth \( \psi = \phi = 2 \alpha \) which is linear in \( z \) and follows the orientation of the transverse component of \( \mathbf{B} \). This special result has been obtained geometrically [3] and algebraically [1,2,5,7-13]. We will give an algebrical analysis following a procedure which allows an immediate extension to families of configurations where the magnetic field shear is not necessarily uniform and the uniformly sheared configuration appears as a special case.

Let us define two new independent variables \( u(z) \) and \( v(z) \) by

\[
\begin{align*}
    u &= \cos \phi \, s_1 + \sin \phi \, s_2 \\
    v &= -\sin \phi \, s_1 + \cos \phi \, s_2
\end{align*}
\]

(8)

to be used in place of \( s_1 \) and \( s_2 \). This corresponds to using a new reference frame in Poincaré space which is rotating about the \( s_3 \) direction with the same angular velocity, \( d\phi/dz \) (in general not a constant), as the precession of \( \Omega \). Then eq.2 can be written as

\[
\begin{align*}
    \frac{du}{dz} &= - \left( \Omega \sin \eta - \frac{d\phi}{dz} \right) v \\
    \frac{dv}{dz} &= - \Omega \cos \eta \, s_3 + \left( \Omega \sin \eta - \frac{d\phi}{dz} \right) u \\
    \frac{ds_3}{dz} &= \Omega \cos \eta \, v
\end{align*}
\]

(9)

Putting now

\[
    w = \int_{z_0}^{z} \Omega \cos \eta \, dz
\]

(10)

and
eques. 9 become

\[ \frac{du}{dw} = -hv \]

\[ \frac{dv}{dw} = -s_3 + hu \]  \hspace{1cm} (12)

\[ \frac{ds_3}{dw} = v \]

When \( h \) is a constant, namely when

\[ \tan \eta - \frac{1}{\omega \cos \eta} \frac{d\phi}{dz} = \text{constant} \]  \hspace{1cm} (13)

the system of eqs. 12 can be integrated and, calling the constant \( k=(1+h^2)^{1/2} \), one obtains

\[ u(z) = hK_1 \cos(kw) - hK_2 \sin(kw) + K_3 \]

\[ v(z) = kK_2 \cos(kw) + kK_1 \sin(kw) \]  \hspace{1cm} (14)

\[ s_3(z) = -K_1 \cos(kw) + K_2 \sin(kw) + hK_3 \]

where \( K_1, K_2 \) and \( K_3 \) are constants of integration which can be related to the initial values of \( u, v \) and \( s_3 \) (see later) and are not independent. Indeed from |\( s | = 1 \) one obtains \( u^2 + v^2 + s_3^2 = 1 \) and hence \( K_1^2 + K_2^2 + K_3^2 = k^{-2} \). Let us define the vector \( p(z) \) and the matrix \( F(z) \) by

\[ p(z) = \begin{bmatrix} u(z) \\ v(z) \\ s_3(z) \end{bmatrix}, \quad F(z) = \begin{bmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \]  \hspace{1cm} (15)

so that, from eq. 8,
\[ p = F \cdot s \quad \text{and} \quad s = F^{-1} \cdot p \]  \hspace{1cm} (16)

and we have \( F^{-1}(\phi) = F(-\phi) \). We use the subscript \( o \) to denote initial values, i.e. quantities evaluated at the initial point \( z = z_o \), so that

\[ p_o = p(z_o), \quad s_o = s(z_o), \quad \phi_o = \phi(z_o), \quad F_o = F(z_o), \quad p_o = F_o \cdot s_o. \]  \hspace{1cm} (17)

If we express \( K_1, K_2 \) and \( K_3 \) in terms of \( p_o \), then eqs.14 can be written as the vector equation

\[ p(z) = [\cos(kw)A + \sin(kw)B + C] \cdot p_o \]  \hspace{1cm} (18)

where the matrices \( A, B \) and \( C \) are

\[ A = \frac{1}{(1+h^2)} \begin{vmatrix} h^2 & 0 & -h \\ 0 & (1+h^2) & 0 \\ -h & 0 & 1 \end{vmatrix} \]  \hspace{1cm} (19)

\[ B = \frac{1}{k(1+h^2)} \begin{vmatrix} 0 & -h(1+h^2) & 0 \\ hk^2 & 0 & -k^2 \\ 0 & (1+h^2) & 0 \end{vmatrix} \]  \hspace{1cm} (20)

\[ C = \frac{1}{(1+h^2)} \begin{vmatrix} 1 & 0 & h \\ 0 & 0 & 0 \\ h & 0 & h^2 \end{vmatrix} \]  \hspace{1cm} (21)

and do not depend on \( z \). Finally using eqs.16 and 17 we have

\[ s(z) = F^{-1}(z) \cdot [\cos(kw)A + \sin(kw)B + C] \cdot F_o \cdot s_o \]  \hspace{1cm} (22)

where the \( z \) dependence appears only in \( F^{-1} \) and in \( w \). Eq.22 gives \( s(z) \) for any given initial (at \( z = z_o \)) Stokes vector \( s_o \) for waves in a magnetized plasma which satisfies the condition of eq.13.

Let us now see if in this medium there can be propagation of waves having a constant ellipticity, i.e. with \( s_3 \) constant. From eq.22 we have
Thus indeed one has $s_3$ constant when

$$[A \cdot F_o \cdot s_o]_3 = 0 \quad \text{and} \quad [B \cdot F_o \cdot s_o]_3 = 0$$

which, putting $s_o = (s_{o1}, s_{o2}, s_{o3})$, imply

$$s_{o3} = h(\cos \phi_0 s_{o1} + \sin \phi_0 s_{o2})$$

$$\sin \phi_0 s_{o1} = \cos \phi_0 s_{o2}$$

Recalling that $s_{o1}^2 + s_{o2}^2 + s_{o3}^2 = 1$ one finds that these equations are satisfied for two values of $s_o$, say $s_o^+$ and $s_o^-$, with

$$s_o^+ = \frac{1}{(1+h^2)^{1/2}} \begin{bmatrix} \cos \phi_0 \\ \sin \phi_0 \\ h \end{bmatrix} ; \quad s_o^- = -s_o^+$$

Therefore the two waves are orthogonal and it can be verified that $A \cdot F_o \cdot s_o^+ = B \cdot F_o \cdot s_o^+ = 0$ and so from eqs.22 and 26 we have

$$s^+(z) = F^{-1} \cdot C \cdot F_o \cdot s_o^+ = \frac{1}{(1+h^2)^{1/2}} \begin{bmatrix} \cos \phi(z) \\ \sin \phi(z) \\ h \end{bmatrix} , \quad s^-(z) = -s^+(z)$$

Hence we see that for these generalized helical waves, again the azimuth follows the (nonuniform) rotation of the transverse magnetic field and the ellipticity is constant.

The condition $h = \text{const}$, for the existence of the exact solutions found above, can be expressed in terms of the physical quantities to give a condition for $n(z)$ and $B(z)$. Indeed using eqs.6 and 7 the condition of eq.13 becomes

$$\frac{2\omega \cos \theta}{\omega_c \sin^2 \theta} \left[ 1 - \frac{2\omega_2}{\omega_p^2 \omega_c \cos \theta} \frac{d \alpha}{dz} \right] = \text{const}$$

(28)
where $\omega_p$, $\omega_c$, $\theta$ and $d\alpha/dz$ can all be functions of $z$. The case of uniformly sheared plasma examined previously [1-5] is the special case where all the quantities appearing in eq.28, namely $n$, $B$, $\theta$, and $d\alpha/dz$, are all separately constants and then the usual helical waves are found.

The helical waves have been taken as the basis of the analysis of polarization evolution in a general nonuniform magnetized plasma using the method of coupled wave-equations [2]. The same treatment could be repeated taking instead as basis the generalized helical waves described here (eqs.27).

3. It should be noted that the present exact analytic treatment can be extended to configurations more general than those described by eq.5 with the condition of eq.13. Indeed let us consider in the Poincaré space, where $s(z)$ and $\Omega(z)$ are defined, a rotation of the reference axes represented by the transformation matrix $T$, so that $s(z)$ and $\Omega(z)$ change respectively into $s^*(z)=T \cdot s(z)$ and $\Omega^*(z)=T \cdot \Omega(z)$. Then, if in the new reference frame $\Omega^*(z)$ can be represented in the form of eq.5 with parameters $\Omega^*$, $\eta^*$ and $\phi^*$ satisfying eq.13, we can repeat the the integration given above, starting with $s^*(z_o)=T \cdot s(z_o)$ at $z=z_o$ and obtaining $s^*(z)$ from an expression similar to eq.22, in the new reference frame. Finally $s(z)$ is given by $s(z)=T^{-1} \cdot s^*(z)$ and so we have

$$s(z) = T^{-1} \cdot (F^*)^{-1} \cdot [\cos(k*w^*)A^*+\sin(k*w^*)B^*+C^*] \cdot F_{o^*} \cdot T^{-1} \cdot s_0$$

(29)

where all the quantities with the index * correspond to the quantities without the index, but in the new Poincaré-space reference frame. The waves corresponding to the generalized helical waves, namely $[T^{-1} \cdot s^*(z)]$ and $[T^{-1} \cdot s^{-*}(z)]$, in general do not have a constant ellipticity nor an azimuth whose orientation follows that of the transverse field. However eq.29 is an exact solution for propagation in this plasma configuration.

A case of some interest is the one where

$$\Omega = \begin{bmatrix} \cos \eta(z) \cos \phi(z) \\ -\sin \eta(z) \\ \cos \eta(z) \sin \phi(z) \end{bmatrix}$$

(30)
and $T$ is simply a rotation of $\pi/2$ around the first reference axis in Poincaré space. The interest is due to the fact that it is related to models of regions of quasi-transverse propagation which have been considered in astrophysical context [6]. Indeed the treatment given above provides the solutions for a medium where, using the form of eq.30, the quantities $\Omega$, $\eta$ and $d\phi/dz$ are all constants and specifically we can take $\eta=0$ and $\phi(z)=(\pi/2)-g(z-z_o)$, with $g$ constant, in the range $z_o<z<z_o+\pi/g$. Then, comparing eqs.4 and 30, one finds that the physical parameters satisfy the following relations: $\alpha=0$,

$$\frac{\omega_p^2}{2c\omega^3} (\omega_c^2 \sin^4 \theta + 4\omega^2 \cos^2 \theta)^{1/2} = \text{const} \quad \text{and} \quad 2\omega \cos \theta = \omega_c \tan \phi \sin^2 \theta,$$

which implies $\cos \theta = (f^2+1)^{1/2} - f$, with $f(z) = \omega/(\omega_c \tan \phi)$. Thus one has respectively $\theta=0$, $\pi/2$ and $\pi$, for $z=z_o$, $(z_o+\pi/2g)$ and $(z_o+\pi/g)$. A configuration similar to this, across which $B_z$ changes sign while $B_x$ is constant and $B_y=0$, has been called a quasi-transverse transition layer [6] and it has been used to model propagation of radio waves in the solar corona.

A particular configuration with nonuniform shear having an exact analytic solution has also been found (see Ref.2) where $n$ and $B$ are constant, $\theta=\pi/2$ and $\alpha(z) = (2\alpha_o/\pi)\arctan(z/L)$ with $\alpha_o$ and $L$ constants. This configuration does not belong to any of the families of configurations considered above and indeed the solution found in [2] was expressed in terms of hypergeometric functions rather than trigonometric functions as in the present Note.
References
