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CHARACTERISTIC CLASSES AND TRANSFER RELATIONS IN COBORDISM

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Abstract

Decompositions of products of the Ray elements and low dimensional free generators of the symplectic cobordism ring are obtained. In particular it is stated that most of the $4n$-dimensional generators, for n small, after multiplication by the Ray elements ϕ_i, $i \geqslant 0$ land in the ideal generated by Ray elements of low dimension.

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1 Introduction

Most of the relations between Ray elements ϕ_i and free generators in the torsion part of the symplectic cobordism ring up to dimension 32 [14, 15] can be conventionally subdivided into three types: to the first type correspond relations which mainly follow from relations in the integral part coming from $M\mathrm{Sp}_{4n}$. In more detail, relations of the first type have the form $(x+y)\phi_i = 0$, and these relations follow from the fact that the sum of free generators $x+y$ is divisible by 2, whereas the Ray elements have order 2. Relations of the second type have the form $z\phi_i = 0$ where z is again a $4n$-dimensional generator from the free part. The aim of the paper is to elucidate origins of the relations of third type as in the Abstract.

Let ζ be the universal $\mathrm{Sp}(1)$-bundle, then $\zeta_1 \otimes_{\mathbb{C}} \zeta_2 \otimes_{\mathbb{C}} \zeta_3$ is a symplectic bundle over $B\mathrm{Sp}(1) \times B\mathrm{Sp}(1) \times B\mathrm{Sp}(1)$; also $\zeta_1 \otimes_{\mathbb{C}} \zeta_2^2$ and $\zeta_1 \otimes_{\mathbb{R}} \zeta_2$ are symplectic bundles over $B\mathrm{Sp}(1) \times B\mathrm{Sp}(1)$.

Section 2 is devoted to the calculation of transfers [1, 5, 2, 11]. In Section 3 we prove the following main result:

Theorem 1.1. *Let $x_i = p_1(\zeta_i)$, i=1,2, be the first Conner-Floyd symplectic Pontyagin class. Let ϕ_j, $j \geqslant 0$, be the Ray elements, and let n be such that $M\mathrm{Sp}_{4m}$ is torsion free for $m \leqslant 2n-1$. Then*

a) The element $\phi_j p_1(\zeta_1 \otimes_{\mathbb{C}} \zeta_2^2)$ is divisible by $\phi_0 x_1 + \phi_1 x_1^2 + \ldots + \phi_{[n/2]} x_1^{2[n/2]}$;

b) $\phi_j p_1(\zeta_1 \otimes_{\mathbb{R}} \zeta_2) = 0$

in the ring $M\mathrm{Sp}^*(\mathbb{H}P(n)^2) = M\mathrm{Sp}^*[[x_1, x_2]]/(x_i^{n+1})$.

In Section 4 we shall see that in terms of the coefficients a_{klm} of the first Conner-Floyd symplectic Pontryagin class

$$p_1(\zeta_1 \otimes_{\mathbb{C}} \zeta_2 \otimes_{\mathbb{C}} \zeta_3) = \sum_{k+l+m \geqslant 1} a_{klm} p_1^k(\zeta_1) p_1^l(\zeta_2) p_1^m(\zeta_3)$$

the structure of $M\mathrm{Sp}_{4k}$, $k \leqslant 4$ can be interpreted as follows:

k	$M\mathrm{Sp}_{4k}$	generators
1	$\mathbb{Z}$	a_{011}
2	$\mathbb{Z}+\mathbb{Z}$	a_{012}, a_{111}
3	$\mathbb{Z}+\mathbb{Z}+\mathbb{Z}$	a_{022}, $a_{011}a_{111}$, a_{211}
4	$\mathbb{Z}+\mathbb{Z}+\mathbb{Z}+\mathbb{Z}+\mathbb{Z}$	a_{014}, $a_{011}a_{211}$, a_{122}, a_{111}^2, $2y_4$.

Then Theorem 1.1 implies

Corollary 1.1. *For $i \geqslant 0$ one has* *a)* $\phi_i a_{001} = \phi_i a_{012} = \phi_i a_{022} = \phi_i a_{014} = 0$;

b) $\phi_i a_{111}$ *and* $\phi_i a_{122}$ *belong to the ideal* $\phi_0 M\mathrm{Sp}^*$;

c) $\phi_i a_{211}$ *belong to the ideal* $\phi_0 M\mathrm{Sp}^* + \phi_1 M\mathrm{Sp}^*$.

Relations of Corollary 1.1 imply that multiplication by the elements ϕ_i, $i \geqslant 0$, carries most of the low dimensional generators from the free part of $M\mathrm{Sp}_{4n}$ to the ideal generated by the elements ϕ_0 and ϕ_1.

2 Preliminaries and calculations with transfer

Let ξ and Λ be, respectively, the universal U(1)-bundle and the universal Spin(3)-bundle. Thus the sphere bundle of Λ is $\pi : B\mathrm{U}(1) \to B\mathrm{Sp}(1)$, and one has

$$\pi^*(\zeta) = \xi + \bar{\xi}; \tag{2.1}$$

$$\pi^*(\Lambda) = \xi^2 + \mathbb{R}; \tag{2.2}$$

$$\zeta \otimes_{\mathbb{H}} \zeta = \Lambda + \mathbb{R}; \tag{2.3}$$

where ζ is the universal Sp(1)-bundle as above. Let N be the normalizer of the torus U(1) in Sp(1). The classifying space BN coincides with the orbit space of the complex projective space $\mathbb{C}P(\infty)$ under the free involution I, which acts via

$$I : [z_0, z_1, ...] \mapsto [-\bar{z}_1, \bar{z}_0, ...]$$

in homogeneous coordinates.

The bundle $p : B\mathrm{N} \to B\mathrm{Sp}(1)$ coincides with the projective bundle of Λ and we have the canonical splitting

$$p^*(\Lambda) = \mu + \nu, \tag{2.4}$$

defined by projectivisation p, where μ and ν are a plane and linear real bundles. Of course for the double covering $q : B\mathrm{U}(1) \to B\mathrm{N}$ we have $q^*(\mu) = \xi^2$ and $q^*(\nu) = \mathbb{R}$.

Let τ_π and τ_p be the transfer maps of the bundles π and p [2, 5, 11]. The following lemma follows from [6].

Lemma 2.1. $\pi^*\tau_\pi^* = 1 + I^*$ *and* $\pi^*\tau_p^* = q^*$

The following lemma follows from definitions

Lemma 2.2. $(\xi_1\xi_2^2 + \bar{\xi}_1\bar{\xi}_2^2)_! = (\xi_1 + \bar{\xi}_1) \otimes_{\mathbb{R}} \mu$, *where '!' is the Atiyah transfer of the double covering* $1_{B\mathrm{U}(1)} \times q$.

Let f be the map $f : B\mathrm{N} \to B\mathbb{Z}/2$ induced by the projection of N onto the Weyl group $\mathbb{Z}/2$ and let $\tau_{1\times q}^*$ be the transfer homomorphism for the above double covering $1_{B\mathrm{U}(1)} \times q$.

Lemma 2.3. *For some elements* $\alpha_i \in \widetilde{M\mathrm{Sp}}^*(B\mathbb{Z}/2)$ *the following formula holds*

$$\tau_{1\times q}^*(p_1(\xi_1\xi_2^2 + \bar{\xi}_1\bar{\xi}_2^2) = p_1((\xi_1 + \bar{\xi}_1) \otimes_{\mathbb{R}} \mu) + \sum_{i\geqslant 0} f^*(\alpha_i)p_2^i((\xi_1 + \bar{\xi}_1) \otimes_{\mathbb{R}} \mu).$$

Proof. Taking into account Lemma 2.2 the proof follows from the following formula [13]:

Let q be the double covering $q : X \to B$, let $\eta \to X$ be the symplectic line bundle, $\eta_! \to B$ the Atiyah transfer bundle, τ_q the transfer map of the covering q and $f : X \to B\mathbb{Z}/2$ the classifying map of the real line bundle associated with q. Then for some elements α_i from $\widetilde{M\mathrm{Sp}}^*(B\mathbb{Z}/2)$ the following formula holds

$$\tau_q^*(p_1(\eta)) = p_1(\eta_!) + \sum_{i\geqslant 0} f^*(\alpha_i)p_2^i(\eta_!).$$

Lemma 2.4. *Let* τ *be the transfer of the sphere bundle of a* Spin(3)*-bundle, then* $\phi_j Im\tau^* = 0$, $j \geqslant 0$.

Proof. Of course it suffices to prove this for the universal Spin(3)-bundle π, that is, $\phi_j\tau_\pi^*(a) = 0$ for all $a \in M\mathrm{Sp}^*(B\mathrm{U}(1))$.

Let δ_π be the Boardman map [3]. Then as it is known from [2],

$$\tau_\pi^*(a) = \delta_\pi(ae(\xi_2^2)).$$

Here $e(\xi_2^2)$ is the Euler class of the bundle ξ_2^2 which is the bundle of tangents along the fibers. Then from [10, 7] $\phi_j e(\xi_2^2) = 0$. This proves Lemma 2.4.

Recall from [10, 7, 8] that the bundle Λ is MSp-orientable and the corresponding Euler class has the form

$$e(\Lambda) = \phi_0 p_1(\zeta) + \sum_{j\geqslant 1} \phi_j p_1(\zeta)^j. \tag{2.5}$$

The restrictions of π and p to the symplectic projective space $\mathbb{H}P(n)$ will be denoted by the same symbols. Total spaces of these bundles coincide, respectively, with the complex projective space $\mathbb{C}P(2n+1)$ and with the orbit space $\mathbb{C}P(2n+1)/I$ under the free involution I which acts via

$$[z_0, z_1, ..., z_{2n}, z_{2n+1}] \mapsto [-\bar{z}_1, \bar{z}_0, ..., -\bar{z}_{2n+1}, \bar{z}_{2n}]$$

in homogeneous coordinates.

Proposition 2.1. $\phi_j \tau^*_{\pi\times 1}(p_i(r\xi\otimes_{\mathbb{R}}\zeta) = 0$ *for* $\pi\times 1 = \pi\times 1_{BSp(1)} : BU(1)\times BSp(1) \to BSp(1)^2$, $j \geqslant 0$ *and* $i = 1, 2$.

Proof. We have $p_i(r\xi\otimes_{\mathbb{R}}\zeta) = \sum_{k\geqslant 0}\omega_k^{(i)}p_1^k(\zeta)$ in $MSp^*(BU(1)\times BSp(1)) = MSp^*(BU(1))[[p_1(\zeta)]]$. Then it follows from Lemma 2.4, that

$$\phi_j\tau^*(\sum_{k\geqslant 0}\omega_k^{(i)}p_1^k(\zeta)) = \sum_{k\geqslant 0}\phi_j\tau^*(\omega_k^{(i)}p_1^k(\zeta)) = 0.$$

3 Proof of Theorem 1.1

The bundle $\pi\times 1 : \mathbb{C}P(2n+1)\times\mathbb{H}P(n) \to \mathbb{H}P(n)\times\mathbb{H}P(n)$ coincides with the sphere bundle of the pullback of $\Lambda \to \mathbb{H}P(n)$ along the projection on the first factor $\mathbb{H}P(n)\times\mathbb{H}P(n) \to \mathbb{H}P(n)$. So taking into account the formula (2.5) we have to prove that

$$(\pi\times 1_{\mathbb{H}P(n)})^*(\phi_j p_1(\zeta_1\otimes_{\mathbb{C}}\zeta_2^2)) = 0$$

in $MSp^*(\mathbb{C}P(2n+1)\times\mathbb{H}P(n))$. The transfer $\tau^* = \tau^*_{1\times\pi}$ of the bundle $1_{\mathbb{C}P(2n+1)}\times\pi$ is a composite of two transfers, namely

$$\tau^* = \tau_1^*(\tau_2^*)$$

where τ_1 is the transfer of the bundle $1_{\mathbb{C}P(2n+1)}\times q$ and τ_2 is the transfer of $1_{\mathbb{C}P(2n+1)}\times p$, where the bundles p, π and q are the bundles defined above, that is

$1\times q : \mathbb{C}P(2n+1)\times\mathbb{C}P(2n+1) \to \mathbb{C}P(2n+1)\times\mathbb{C}P(2n+1)/I$;

$1\times p : \mathbb{C}P(2n+1)\times\mathbb{C}P(2n+1)/I \to \mathbb{C}P(2n+1)\times\mathbb{H}P(n)$.

Using the formula (2.2) we have $(\xi_1+\bar{\xi_1})\otimes_{\mathbb{C}}\zeta_2^2) = (\xi_1+\bar{\xi_1})\otimes_{\mathbb{R}}(\Lambda+\mathbb{R})$, hence we have

$$p_1((\xi_1+\bar{\xi_1})\otimes_{\mathbb{C}}\zeta_2^2) = p_1((\xi_1+\bar{\xi_1})\otimes_{\mathbb{R}}\Lambda) + p_1(\xi_1+\bar{\xi_1}). \tag{3.1}$$

Applying Lemma 2.2 and Lemma 2.3 we have

$$\tau_1^*(p_1(\xi_1\xi_2^2+\bar{\xi_1}\bar{\xi_2^2})) = p_1((\xi_1+\bar{\xi_1})\otimes_{\mathbb{R}}\mu) + \sum_{i\geqslant 0} f^*(\alpha_i)p_2^i((\xi_1+\bar{\xi_1})\otimes_{\mathbb{R}}\mu),$$

Then by the formula (2.4)

$$p_1((\xi_1+\bar{\xi_1})\otimes_{\mathbb{R}}\mu) = (1\times p)^* p_1((\xi_1+\bar{\xi_1})\otimes_{\mathbb{R}}\Lambda) - p_1((\xi_1+\bar{\xi_1})\otimes_{\mathbb{R}}\nu),$$

hence

$\tau^*(p_1(\xi_1\xi_2^2+\bar{\xi_1}\bar{\xi_2^2})) =$

$\tau_2^*((p_1(\xi_1+\bar{\xi_1})\otimes_{\mathbb{R}}\mu)) + \tau_2^*(\sum_{i\geqslant 0} f^*(\alpha_i)p_2^i((\xi_1+\bar{\xi_1})\otimes_{\mathbb{R}}\mu)) =$

$\tau_2^*(1\times p)^*(p_1((\xi_1+\bar{\xi_1})\otimes_{\mathbb{R}}\Lambda)) - \tau_2^*(p_1(\xi_1+\bar{\xi_1})\otimes_{\mathbb{R}}\nu) +$

$\tau_2^*(\sum_{i\geqslant 0} f^*(\alpha_i)p_2^i(\xi_1+\bar{\xi_1})\otimes_{\mathbb{R}}\mu) =$

$p_1((\xi_1+\bar{\xi_1})\otimes_{\mathbb{R}}\Lambda)\tau_2^*(1) - \tau_2^*(p_1(\xi_1+\bar{\xi_1})\otimes_{\mathbb{R}}\nu) +$

$$\tau_2^*(\sum_{i\geqslant 0} f^*(\alpha_i)p_2^i(\xi_1+\bar{\xi}_1)\otimes_{\mathbb{R}}\mu). \tag{3.2}$$

Now we have to prove that $\tau_2^*(1)=1$, the second summand in (3.2) coincides with $x_1=p_1(\zeta_1)$ and the third summand is zero.

Note that the bundle $(\xi_1+\bar{\xi}_1)\otimes_{\mathbb{R}}\nu$ is pullback of the bundle $\zeta\otimes\eta\to B\mathrm{Sp}(1)\times B\mathbb{Z}/2$ along the map (π,f). Then $p_1(\zeta\otimes\eta)$ is an element from $M\mathrm{Sp}^*(B\mathbb{Z}/2)[[p_1(\zeta)]]$, hence we have

$$p_1(\zeta\otimes\eta)=p_1(\zeta)+\Sigma_{i\geqslant 0}\beta_i p_1^i(\zeta)$$

for some elements $\beta_i\in\widetilde{M\mathrm{Sp}}^*(B\mathbb{Z}/2)$. This implies

$$p_1((\xi_1+\bar{\xi}_1)\otimes_{\mathbb{R}}\nu)=p_1(\xi_1+\bar{\xi}_1)+\sum_{i\geqslant 0} f^*(\beta_i)p_1^i(\xi_1+\bar{\xi}_1)$$

Similarly the bundle $(\xi_1+\bar{\xi}_1)\otimes_{\mathbb{R}}\mu$ is pullback of the bundle $\zeta\otimes\eta(2)\to B\mathrm{Sp}(1)\times B\mathrm{O}(2)$, where $\eta(2)\to B\mathrm{O}(2)$ is the universal $\mathrm{O}(2)$-bundle. Hence

$$p_2((\xi_1+\bar{\xi}_1)\otimes_{\mathbb{R}}\mu)\in M\mathrm{Sp}^*(B\mathrm{N})[[p_1(\xi_1+\bar{\xi}_1)]]$$

and we have for the third summand of the formula (3.2)

$$\sum_{i\geqslant 0} f^*(\alpha_i)p_2^i((\xi_1+\bar{\xi}_1)\otimes_{\mathbb{R}}\mu)=\Sigma_{i\geqslant 0}\gamma_i p_1^i(\xi_1+\bar{\xi}_1)$$

for some $\gamma_i\in\widetilde{M\mathrm{Sp}}^*(B\mathrm{N})$.

So using (3.1) we have

$$\tau^*(p_1(\xi_1\xi_2^2+\bar{\xi}_1\bar{\xi}_2^2))=$$
$$p_1((\xi_1+\bar{\xi}_1)\otimes_{\mathbb{R}}\zeta^2)\tau_2^*(1)-x_1(\tau_2^*(1)+1)+\sum_{i=0}^{n}\tau_2^*(\delta_i)x_1^i, \tag{3.3}$$

for some $\delta_i\in M\mathrm{Sp}^*(\mathbb{C}P(2n+1)/I)$.

It is known from [14, 15] that up to dimension 32, $M\mathrm{Sp}_{4n}$ is torsion free. Motivated by this fact let us assume that $M\mathrm{Sp}_{4m}$ is torsion free when $m\leqslant 2n-1$. Then it follows that $M\mathrm{Sp}^{4k}(\mathbb{H}P(n))$ is torsion free, when $k\geqslant 1-n$. Then since the minimal dimension of the elements γ_i from the formula (3.3) is $4-4n$, it follows from Lemma 2.1 that the third summand of (3.3) restricts to zero in $M\mathrm{Sp}^*(\mathbb{H}P(n))$. Also $\tau^*(p)(1)=1$ and $\tau^*(\pi)(1)=2$.

Thus we obtain from (3.3) and then Lemma 2.4

$$\phi_j p_1(\zeta\otimes_{\mathbb{C}}\zeta_2^2)=\phi_j\tau^*(p_1(\xi_1\xi_2+\bar{\xi}_1\bar{\xi}_2))=0.$$

This proves Theorem 1.1 a).

For the proof of b) note that it follows from Lemma 2.1 that for the bundle $\pi \times 1 = \pi \times 1_{\mathbb{H}P(n)})$ we have

$$(\pi \times 1)^* \tau^*_{\pi\times 1}(p_1(r\xi_1 \otimes_{\mathbb{R}} \zeta_2)) = (1+I)^*(p_1((\xi_1 + \bar{\xi}_1) \otimes_{\mathbb{C}} \zeta_2)) =$$

$$2p_1((\xi_1 + \bar{\xi}_1) \otimes_{\mathbb{C}} \zeta_2) = (\pi \times 1)^* p_1((\zeta_1 \otimes_{\mathbb{R}} \zeta_2)).$$

Then by (2.5) any element from $\ker(1 \times \pi)^*$ is divisible by $e(\Lambda)$. On the other hand by hypothesis $M\mathrm{Sp}^{4k}(\mathbb{H}P(n))$ is torsion free for $k \geqslant 1-n$. Hence we conclude that restriction of the homomorphism $(\pi \times 1)^*$ to $M\mathrm{Sp}(\mathbb{H}P(n)^2)$ is a monomorphism, thus in $M\mathrm{Sp}^4(\mathbb{H}P(n)^2)$ one has $p_1(\zeta_1 \otimes_{\mathbb{R}} \zeta_2) = \tau^*_{\pi\times 1}(p_1(r\xi_1 \otimes_{\mathbb{R}} \zeta_2))$

Now since Proposition 2.1 says that the right-hand side is zero after multiplication by ϕ_j, this completes the proof of Theorem 1.1.

4 Proof of Corollary 1.1

Let $h : \pi_*(M\mathrm{Sp}) \to H_*(M\mathrm{Sp}) = \mathbb{Z}[q_1, q_2, ...]$ be the Hurevicz homomorphism. Since $\pi_{4n}(M\mathrm{Sp})$ is torsion free for small n (see [12, 14, 15]), the Hurevicz homomorphism is a monomorphism in these dimensions. So in low dimensions $4n$ the Hurevicz homomorhism determines all relations. Our aim here is to express the coefficients a_{klm} from the introduction through the generators x-es.

Values of the Hurevicz homomorphism on these a_{klm} are calculated in [9]. In low dimensions we have

$h(a_{100}) = h(a_{010}) = h(a_{001}) = 4;$
$h(a_{200}) = h(a_{020}) = h(a_{002}) = 0;$
$h(a_{110}) = h(a_{101}) = h(a_{011}) = 24q_1;$
$h(a_{111}) = 360q_2$
$h(a_{210}) = ... = h(a_{012}) = 60q_2 - 24q_1^2;$
$h(a_{300}) = ... = h(a_{003}) = 0;$
$h(a_{220}) = ... = h(a_{022}) = 280q_3 - 120q_1q_2 + 24q_1^3;$
$h(a_{310}) = ... = h(a_{013}) = 112q_3 - 96q_1q_2 + 48q_1^3;$
$h(a_{211}) = ... = h(a_{112}) = 1680q_3 - 360q_1q_2;$
$h(a_{122}) = ... = h(a_{221}) = 75600q_4 - 3360q_1q_3 + 360q_1^2q_2;$
$h(a_{410}) = ... = h(a_{014}) = 180q_4 - 360q_1q_3 + 420q_1^2q_2 - 120q_2^2 - 120q_1^4;$

Further the Hurevicz images of generators of $M\mathrm{Sp}$ are calculated in [12]. Namely $h(M\mathrm{Sp}_{4k}) \subset H_{4k}(M\mathrm{Sp})$ has the following generators

$k=1$: $24q_1;$
$k=2$: $20q_2 - 8q_1^2;\ 144q_1^2;$
$k=3$: $56q_3 - 72q_1q_2 + 24q_1^3;\ 120q_1q_2 - 48q_1^3;\ 3456q_1^3$;
$k=4$: $12q_4 - 24q_1q_3 - 8q_2^2 + 28q_1^2q_2 - 8q_1^4;$
$50q_2^2 + 168q_1q_3 - 256q_1^2q_2 + 80q_1^4;\ 100q_2^2 - 80q_1^2q_2 + 16q_1^4;$
$2880q_1^2q_2 - 1152q_1^4;\ 20736q_1^4.$

So we have that the elements $a_{011}, a_{111}, a_{022}, a_{122}, a_{112}, a_{120}, a_{140}$ are generators as in the introduction.

Remark 4.1. In terms of $2x_i$, the generators of $M\mathrm{Sp}_{4n}$ from [12], we have modulo $2M\mathrm{Sp}_*$: $2x_1 = a_{011}$, $2x_2 = a_{012}$, $2x_3 = a_{022}$, $x_4 = a_{014}$, $x_1^2 = a_{111}$, etc.

Remark 4.2. Alternatively images of the elements a_{ijk} in complex cobordism $M\mathrm{U}_*$ can be calculated in terms of two-valued formal groups:

$$\mu^*(p_1(\zeta_1 \otimes_{\mathbb{C}} \zeta_2 \otimes_{\mathbb{C}} \zeta_3)) = \Theta_1(x_1, Y^+) + \Theta_1(x_1, Y^-),$$

where $Y^+ + Y^- = \Theta_1(x_2, x_3)$, $Y^+Y^- = \Theta_2(x_2, x_3)$; Θ_1 and Θ_2 are the coefficient of the two-valued formal group [4] and μ is the obvious map from the symplectic cobordism theory to the complex cobordism theory.

Let us consider now Corollary 1.1. From Theorem 1.1 a) we have in $M\mathrm{Sp}^*(\mathbb{H}P(4) \times \mathbb{H}P(4)$ the relation of the form

$$\theta_j(a_{011}x_2^2 + a_{111}x_1x_2^2 + (a_{022})x_2^4 + (a_{211})x_1^2x_2^2 + (a_{122})x_1x_2^4 + ...)$$

$$= (\phi_0 x_1 + \sum_{1 \leqslant i \leqslant n} \phi_i x_1^i) b(x_1, x_2)$$

for some element $b(x_1, x_2) =\in M\mathrm{Sp} * (\mathbb{H}P(n)^2)$. Then since $\theta_{2i+1} = 0$ [13] by the equality of the coefficients at the monomials $x_1x_2^2$, $x_1^2x_2^2$ and $x_1x_2^4$ we obtain the assertions b) and c) of Corollary 1.1.

Similarly from Proposition 2.1 we have

$$\phi_j(a_{110}x_2^2 + a_{120}x_1x_2^2 + a_{220}x_1^2x_2^2 + a_{140}x_1x_2^4 + ...) = 0,$$

and hence the assertion a) of Corollary 1.1 is valid.

Proposition 4.1. *In dimension* 32 *there is an element* y_4^2 *such that* $\phi_2 y_4^2$ *does not belong to the ideal generated by* ϕ_0 *and* ϕ_1. *Moreover* $\phi_{2^i} y_4^2$ *does not belong to the ideal generated by* ϕ_0, ϕ_1, ... , ϕ_{2^i-1}, $i \geqslant 1$.
Proof. It follows from the calculations of the symplectic cobordism ring made in [14, 15].

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