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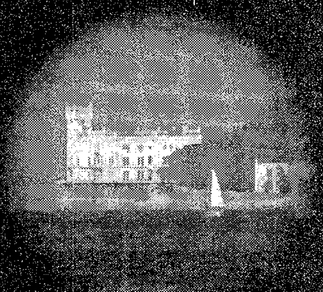

preprint

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Available at: http://www.ictp.trieste.it/~pub_off IC/2001/112

United Nations Educational Scientific and Cultural Organization
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THE ABDUS SALAM INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

NONCOMMUTATIVE CHERN-CONNES CHARACTERS OF SOME NONCOMPACT QUANTUM ALGEBRAS

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Abstract

We prove in this paper that the periodic cyclic homology of the quantized algebras of functions on coadjoint orbits of connected and simply connected Lie group, are isomorphic to the periodic cyclic homology of the quantized algebras of functions on coadjoint orbits of compact maximal subgroups, without localization. Some noncompact quantum groups and algebras were constructed and their irreducible representations were classified in recent works of Do Ngoc Diep and Nguyen Viet Hai [DH1]-[DH2] by using deformation quantization. In this paper we compute their K-groups, periodic cyclic homology groups and their Chern characters.

MIRAMARE – TRIESTE
September 2001

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Introduction

Let G be a connected Lie group, K a fixed maximal compact subgroup of G and A a locally convex algebra over the complex numbers. One of the major results in of V. Nistor in [N1] is Theorem 1.1 of [N1], saying that up to localization at some maximal ideal $\mathfrak{m}$ of the algebra $C^\infty_{inv}(G)$ of bi-invariant functions f, $f(\gamma^{-1}x\gamma) \equiv f(x), \forall g, x \in G$, the periodic cyclic homology of the crossed product $A \rtimes G$ and that for $A \rtimes K$, are isomorphic, i.e.

$$\mathrm{PHC}_*(A \rtimes G)_{\mathfrak{m}} \cong \mathrm{PHC}_{*+q}(A \rtimes K)_{\mathfrak{m}}. \qquad (*)$$

In the case where $A = \mathbb{C}$ and the action of G on $\mathbb{C}$ is trivial, the crossed product becomes convolution and the restrictions of elements of $C^\infty_c(G)$, as functions on G, to the coadjoint orbits give algebras of quantized functions on the coadjoint orbits. We prove the isomorphism (*) in that case without any localization. The main reason is that on coadjoint orbits, the bi-invariant functions correspond to constants and their localization are the same constants. This will be done in the first section. It is interesting that with this isomorphism (without any localization) for the quantized algebras of functions on coadjoint orbits, we can reduce the computation of the noncommutative Chern-Connes characters to some more easily computed ones for maximal compact subgroups. Our main observation is that the conjugacy in G corresponds to the adjoint action on $\mathfrak{g}$ and to the coadjoint action on $\mathfrak{g}^*$. It is especially important in the concrete cases in the last two sections, involving the Lie groups $\mathrm{Aff}(\mathbb{R})$ and $\mathrm{Aff}(\mathbb{C})$, (see §3 and §4).

Indeed, the homogeneous classical mechanical systems with fixed Lie groups of symmetry were classified as coadjoint orbits of the Lie groups of symmetry or their central extension by $\mathbb{R}$, in the vector space dual to the Lie algebras, see [K1]. For some special cases, where all the nontrivial orbits are of dimension equal to the dimension of the group (class $\overline{\mathrm{MD}}$), all such Lie groups have been classified and all the orbits explicitly computed. They reduce to the cases of the groups of all affine transformations of the real or complex lines, see [D1].

The group of affine transformations of the real line has two 2-dimensional coadjoint orbits: the upper and lower half-planes, see [D1]. Using deformation quantization, a quantum analogue of the half-planes was constructed in [DH1]. The group of affine transformations of the complex line has one orbit of complex dimension 2: namely, the punctured (withdraw a complex line through the origin) complex plane. Its quantum algebra was constructed in [DH2]. We compute the K-groups, the periodic cyclic homology of these quantum algebras and the corresponding Chern-Connes characters.

In order to obtain these results, we use the methods from [DKT1]-[DKT2] and the methods and results from [N1]: We first construct some diffeomorphims realizing canonical coordinates on coadjoint orbits and then reduce the quantum algebras to the ones related to the corresponding quantum groups. We then reduce these quantum algebras of quantum function on coadjoint orbits to the quantum algebras related to the maximal compact subgroups, that are more easily computed. This computation is realized in the last two sections.

Notes on Notation: As usual we denote by capital letters some Lie groups, namely G, K, etc. Their corresponding Lie algebras will be denoted by the corresponding Gothic letters, namely $\mathfrak{g}$, $\mathfrak{k}$, etc. The dual space to a Lie algebra or a vector space will be denoted by the same letter with $*$, e.g. $\mathfrak{g}^*$ or V^* is the dual space of the Lie algebra $\mathfrak{g}$ or the vector space V. PHC_* will denote the periodic cyclic homology, following [N1]. If A is a locally convex $\mathbb{C}$-algebra on which a Lie group G acts smoothly, then $A \rtimes G$ denotes the crossed product of A with G. $\Im(z)$ and $\Re(z)$ will mean the imaginary and real parts of the complex number z, while as $[a]$ and $\{a\}$ we denote the integral and fractional parts of a. $\mathbb{R}$ and $\mathbb{C}$ means the field of real or complex numbers, respectively. For any $\mathbb{C}$-algebra A, denote $A^\natural = \{A_n = A^{\otimes n+1}\}$ the well-known Connes-Tsygan complex.

1 Localization and coadjoint orbits

Let G be a connected and simply connected Lie group and $C^\infty_{inv}(G)$ the convolution algebra of bi-invariant functions on G, see [N1]. We prove in this section that localization of the convolution algebra $C^\infty_{inv}(G)$ at a maximal ideal corresponds to the (quantized) convolution algebra of functions with compact support on the corresponding orbit. We first modify some results obtained in the work [N1] of V. Nistor.

1.1 Preparation

Recall that a quasi-cyclic object in an Abelian category $\mathcal{M}$ is a graded object $(X_n)_{n\geq 0}$, $X_n \in Ob(\mathcal{M})$ together with morphisms $d_i : X_n \to X_{n-1}$, for $i = 0, \dots, n$ and $T_{n+1} : X_n \to X_n$ satisfying the following two axioms:

$$(S1) \qquad d_i d_j = d_{j+1} d_i, \text{ for } i < j$$

$$(C1) \qquad d_i T_{n+1} = \begin{cases} T_n d_{i-1} & \text{for } 1 \leq i \leq n \\ d_n & \text{for } i = 0 \end{cases}$$

see [N1] for more details. V. Nistor pointed out the examples of quasi-cyclic objects like:

(i) the cyclic objects,

(ii) $\mathfrak{A}^\natural$, where

$$\mathfrak{A} := A \rtimes G = C^\infty_c(G, A) = \{\varphi \in C^\infty(G, A); \varphi \text{ is of compact support }\},$$

G a Lie group, A a locally convex algebra on which G acts smoothly, with the twisted convolution product

$$\varphi * \psi(g) := \int_G \varphi(h)\alpha_h(\psi(h^{-1}g)dh,$$

where dh is a fixed left invariant Haar measure on G, $\alpha : G \to \mathrm{Aut}(A)$ a smooth action of G on A in the sense that the map $g \mapsto \alpha_g$ is continuous and unital and the map $g \mapsto \alpha_g(a)$, for all $a \in A$ is smooth. In that case, we have $\mathfrak{A}^\natural_n = (A \rtimes G)^\natural_n = C^\infty_c(G^{n+1}, A^{\otimes n+1})$ with the operations

$$(d_j\varphi)(g_0, g_1, \dots, g_n) :=$$

$$\int_G d_j \circ (1 \otimes \ldots \otimes 1 \otimes \alpha_h \otimes 1 \otimes \ldots \otimes 1)(\varphi(g_0, \ldots, g_{j-1}, h, h^{-1}g_j, \ldots, g_{n-1}))dh,$$

$j = 1, \ldots, n-1,$

$$(d_n\varphi)(g_0, g_1, \ldots, g_n) :=$$

$$\int_G d_n \circ (\alpha_n \otimes 1 \ldots \otimes 1)(\varphi(h^{-1}g_0, g_1, \ldots, g_{n-1}), h)dh$$

and

$$(t_{n+1}\varphi)(g_0, \ldots, g_n) := t_{n+1}(\varphi(g_1, \ldots, g_n, g_0))$$

As remarked in [N1], the operators d_j, t_{n+1} on the right hand side are the ones of the cyclic Connes-Tsygan complex $A^\natural$ (see [C].

In particular, if $A = \mathbb{C}$ and the action of G on $\mathbb{C}$ is trivial, we have the algebra of convolution.

(iii) Let A, G be as in (ii) above. Then $\{L_n(U, G_1)\}$ is a quasi-cyclic object , where $U \subset G$ is an open set, G_1 is some other Lie group and for any group homomorphism $\rho : G \to G_1$, $L_n(G, G_1) := C_c^\infty(G \times G_1^{n+1}, A^{\otimes n+1})$ with similar operations d_j, T_{n+1} (see [N1] for more details).

1.2 An H-relative cohomology complex

Let us now introduce some new examples of quasi-cyclic objects, related with some quotient maps. In (ii) above, G_1 is a group, ρ a homomorphism. However in what follows, G_1 is replaced by the homogeneous space $H \setminus G$ and ρ is just the quotient map.

Consider the canonical quotient map $\rho : G \to H \setminus G$, for some subgroup H. Consider an open set $U \subset G$. Define

$$L_n(U, H \setminus G) := C_c^\infty(U \times (H \setminus G)^{n+1}, A^{\otimes n+1})$$

and define also

$$(d_j\varphi)(\gamma, Hg_0, \ldots, \hat{g_j}, \ldots, Hg_n) := \int_G d_j(\varphi(\gamma, Hg_0, \ldots, Hg_n))d\mu(g_j),$$

$j = 1, \ldots, n$, $d\mu()$ is the quotient measure on the quotient space $H \setminus G$,

$$(d_0\varphi)(\gamma, \hat{g_0}, Hg_1, \ldots, Hg_n) := \int_G d_0(\varphi(\gamma, Hg_0, Hg_1, \ldots, Hg_n))d\mu(g_0)$$

and

$$(T_{n+1}\varphi)(\gamma, Hg_0, \ldots, Hg_n) := (1 \otimes \alpha_\gamma^{-1} \otimes \ldots, \otimes 1)t_{n+1}(\varphi(\gamma, Hg_1, \ldots, Hg_n, Hg_0))$$

Proposition 1.1 *$(\{L_n(U, H \setminus G)\}, d_j, d_0, T_{n+1})$ is a quasi-cyclic object.*

Proof. By similar arguments to those in the work of V. Nistor, see [N1], it is easy to see that we have also a quasi-cyclic object. □

This quasi-cyclic object is related to the quantum algebras of functions of orbits, as we shall see later.

Let us now define an action of G on the quasi-cyclic object $(\{L_n(U, H \setminus G)\}, d_j, d_0, T_{n+1})$. For a fixed action $\alpha : G \to \mathrm{Aut}(A)$, define $\beta : G \to GL(L_n(U, H \setminus G))$ by

$$\beta_\gamma(\varphi)(\gamma, Hg_0, Hg_1, \dots, Hg_n) := \alpha_\gamma^{\otimes n+1}(\varphi(\gamma^{-1}\gamma_1\gamma, Hg_0\gamma, \dots, Hg_n\gamma)),$$

for all γ, γ_1 in G, $g_0, \dots, g_n$ in G, φ in $L_n(U, H \setminus G)$.

In particular, if $H = \{e\}$ and $U = G$, we have $H \setminus G \cong G$ and a map $p : L(G, G) \to (A \rtimes G)^\natural$, defined by

$$p\varphi(h_0, \dots, h_n) := \int_G \Psi(\beta_\gamma\varphi)(g_n, g_0, \dots, g_{n-1}),$$

where $g_0 := h_0$, $g_1 := h_0h_1$, $g_2 := h_0h_1h_2$, ... $g_n := h_0h_1h_2 \dots h_n$, $\Psi := (\alpha_{g_n} \otimes \alpha_{g_0} \otimes \dots \alpha_{g_{n-1}})^{-1}$.

We now define the map J which gives rise to an isomorphism in Hochschild homology, (see [N1]).

Consider again an open set U which is Ad_G-invariant in G, and define $J : L_n(U, H \setminus G) \to L(U, \{e\})$ by

$$(J\varphi)(\gamma) := \int_{(H\setminus G)^{n+1}} \varphi(\gamma, Hg_0, \dots, Hg_n)d\mu(g_0) \dots d\mu(g_n)$$

Note that we here use $d(g)$ to denote the relative quasi-invariant measure on the quotient space $H \setminus G$.

Lemma 1.2 *J is a morphism of quasi-cyclic objects and $HH(J)$ is an isomorphism of the corresponding Hochschild homology groups.*

Proof. In [N1] the similar assertion was proven for the absolute case. In the relative case, we have a similar argument, which we omit here. □.

Suppose we have some continuous homomorphism of a compact group K into G. We have then

Proposition 1.3 $\mathrm{HC}_*(L(K, H \setminus G)^K) \cong \mathrm{HC}_*(((A \rtimes K)^K)^H)$

Proof. We have from [N1] the isomorphism $\mathrm{HC}_*(L(K, G)^K) \cong \mathrm{HC}_*(((A \rtimes K)^K))$. From the definition of complexes defining $\mathrm{HC}_*(L(K, G)^K)$ and $\mathrm{HC}_*(((A \rtimes K)^K))$, we have isomorphic H-invariant homology group $\mathrm{HC}_*((L(K, G)^K)^H) \cong \mathrm{HC}_*(((A \rtimes K)^K)^H)$. But $\mathrm{HC}_*((L(K, G)^K)^H) \cong \mathrm{HC}_* * (L(K, H \setminus G)^K)$ □

Let G be a connected Lie group, $\mathcal{F}$ a smooth G-module, $q = \dim(G/K)$. We now define the G-equivariant homology: Consider the complex of relative Lie algebra homology

$$0 \to (\wedge^q(\mathfrak{g}/\mathfrak{k}) \otimes_H \mathcal{F}) \otimes_K \mathbb{C} \xrightarrow{\delta} \dots \xrightarrow{\delta} (\wedge^0(\mathfrak{g}/\mathfrak{k}) \otimes_H \mathcal{F}) \otimes_K \mathbb{C} \longrightarrow \mathcal{F} \otimes_G \mathbb{C} \longrightarrow 0, \qquad (I)$$

where $\delta : (\wedge^j(\mathfrak{g}/\mathfrak{k}) \otimes_H \mathcal{F}) \otimes_K \mathbb{C} \longrightarrow (\wedge^{j-1}(\mathfrak{g}/\mathfrak{k}) \otimes_H \mathcal{F}) \otimes_K \mathbb{C}$ is defined as

$$\delta(\dot{X}_1 \wedge \dots \wedge \dot{X}_j \otimes \xi) := \sum_{i=1}^{j} \dot{X}_1 \wedge \dots \wedge \hat{X}_i \wedge \dots \dot{X}_j \otimes X_i(\xi) -$$

$$- \sum_{i<j} (-1)^{i+k}[X_i, X_k] \wedge \dot{X}_1 \wedge \dots \wedge \hat{X}_i \wedge \dots \wedge \hat{X}_k \wedge \dots \wedge \dot{X}_j \otimes \xi,$$

and for $X_i \in \mathfrak{g}$, $\dot{X}_i$ is the class of X_i in $\mathfrak{g}/\mathfrak{k}$, and $\hat{X}_i$ indicates that $\dot{X}_i$ is omitted. In the complex (I) $\mathcal{F}$ is regarded as a smooth H-module. It is not hard to prove

Proposition 1.4 *This complex is acyclic.*

For acyclicity of a similar complex with $\mathbb{C}$ instead of H, see [N1](Prop. 3.6). In our relative case this is true also, as can be verified.

1.3 A relative complex and the corresponding Main Lemma

In the main body of the paper of V. Nistor is the construction of complexes computing the mentioned isomorphism,

$$\mathrm{PHC}_*(A \rtimes G)_{\mathrm{m}} \cong \mathrm{PHC}_{*+q}(A \rtimes K)_{\mathrm{m}}.$$

We now introduce a relative complex satisfying all the conditions of the Main Lemma of Nistor.

1.3.1 The Main Lemma

Following [N1], let us consider the following **data**:

- Two exact sequences of quasi-cyclic objects of complete locally convex spaces

$$(E1) \quad 0 \longleftarrow \mathcal{Y} \longleftarrow \mathcal{X}^{(0)} \xleftarrow{\delta} \dots \xleftarrow{\delta} \mathcal{X}^{(q-1)} \xleftarrow{\delta} \xleftarrow{\delta} \mathcal{X}^{(q)} \longleftarrow 0$$

$$(E2) \quad 0 \longrightarrow \mathcal{X}^{(0)} \xrightarrow{\sigma} \dots \xrightarrow{\sigma} \mathcal{X}^{(q-1)} \xrightarrow{\sigma} \mathcal{X}^{(q)} \longrightarrow \longrightarrow \mathcal{Z} \longrightarrow 0$$

- A C^∞-action of $\mathbb{R}$ on $\mathcal{X}^{(j)}$ for any j such that $\eta_1 = T_{n+1}^{n+1}$ and the derivative $\nabla = \frac{d\eta_t}{dt}|_{t=0}$ of η at $t = 0$ is equal to $\nabla = \delta\sigma + \sigma\delta$, with the convention that $\delta(\mathcal{X}^{(0)}) = \sigma(\mathcal{X}^{(q)}) = 0$
- The endomorphisms $1 - T_{n+1}^{n+1}$ are injective on $\mathcal{X}_n^{(j)}$ for all $j = 0, \dots, q$ and for all $n \geq 0$.

Let us recall that a *precyclic object* is a quasi-cyclic object $((X_n)_{n\geq 0}, d_j, T_{n+1})$ such that $T_{n+1}^{n+1} = 1$. Given a precyclic object one constructs the *Connes-Tsygan complex* as in the cyclic case, see [N1]. The homology of the *2-periodic total complex* $\mathrm{Tot}\, \mathcal{C}(X)$ associated to the bi-complex $\mathcal{C}(X)$ is defined as the periodic cyclic homology $\mathrm{PHC}(X)$ of the complex X, (see [N1], Definition 2.2, and the definition thereafter).

We now associate to the data satisfying the above definition, some new objects

$$\tilde{\mathcal{X}}^{(0)} := \mathcal{X}^{(0)}/\nabla\mathcal{X}^{(0)}, \quad \tilde{\mathcal{X}}^{(j)} := \mathcal{X}^{(j)}/(\nabla\mathcal{X}^{(j)} + \sigma\mathcal{X}^{(j-1)}), \forall j = 1, \dots, q.$$

and state the followings Lemma due to V. Nistor (see [N1]

Lemma 1.5 (The Main Lemma) *(i) For any $j = 1, \dots, q$, the object $\tilde{\mathcal{X}}^{(j)}$ is a precyclic object.*

(ii) The complex

$$0 \longleftarrow \mathcal{Y} \longleftarrow \tilde{\mathcal{X}}^{(0)} \xleftarrow{\delta} \tilde{\mathcal{X}}^{(1)} \xleftarrow{\delta} \dots$$
$$\dots \xleftarrow{\delta} \tilde{\mathcal{X}}^{(q-1)} \xleftarrow{\delta} \mathcal{Z} \cong \mathcal{X}^{(q)}/\sigma\mathcal{X}^{(q-1)} \longleftarrow 0$$

is acyclic.

(iii) $\mathrm{PHC}_*(\tilde{\mathcal{X}}^{(j)}) = 0$ *for any* $j = 0, \ldots, q$.

(iv) $\mathrm{PHC}_*(\mathcal{Y}) \cong \mathrm{PHC}_{*+q}(\mathcal{Z})$.

It was shown in [N1] that all the conditions of the Main Lemma are satisfied for the absolute Nistor's complex. We now verify conditions of this lemma for the H-relative complex.

1.3.2 Relative form of $\mathcal{X}^{(j)}$ and δ

For any Lie group G, let $\mathbb{C}_\Delta$ be the one-dimensional representation of G by multiplication with the modular function Δ, $G_x = \{\gamma \in G | \gamma x = x\gamma\}$, $\mathfrak{g}_x = \mathrm{Lie}\, G_x$, K a maximal compact subgroup of G and $U \subset G$, $K_x = K \cap G_x$. Let Δ' denote the modular function for G_x,

$$\mathcal{F}' := \begin{cases} \mathbb{C}_{\Delta'} \otimes L(U', G_x), & \text{if } x' \in K_x, \\ \mathbb{C}_{\Delta'} \otimes L(U', G_x)/(1-x)(C_{\Delta'} \otimes L(U', G_x)), & \text{if } x' \notin K_x \end{cases}$$

$$M := \begin{cases} K_{x'}, & \text{if } x' \in K_x, \\ \text{The maximal compact subgroup in } G_{x'}/(x'), & \text{if } x' \notin K_x \end{cases}$$

and $\mathfrak{m} := \mathrm{Lie}(M)$.

Define the H-relative $\mathfrak{g}_x$-cohomology complex $\{C_j\}$ of $\mathcal{F}'$ by $C_j := (\Lambda^j \mathfrak{g}_x \otimes \mathcal{F}')^H$ and $\delta_0 : C_j \to C_{j-1}$ with

$$\delta_0(X_0 \wedge \ldots \wedge X_j \otimes \xi) = \sum_{j=1}^{j} X_1 \wedge \ldots \wedge \hat{X}_i \wedge \ldots \wedge X_j \otimes X_i(\xi)$$

$$- \sum_{1 \leq i < k \leq j} (-1)^k [X_i, X_k] \wedge X_1 \wedge \ldots \wedge \hat{X}_i \wedge \ldots \wedge X^k \wedge \ldots \wedge X_j \otimes \xi$$

Now we define $\mathcal{X}^{(j)} := C_j/C'_j$, where C'_j is generated by $\mathfrak{m} \wedge C_{j-1}$ and $(\gamma - 1)C_j$ for all $\gamma \in K_x \cup (x)$. It is not hard to see that

$$\delta_0(X \wedge \omega) = X(\omega) - X \wedge \delta_0(\omega), \forall X \in \mathfrak{g}_x, \forall \omega \in C_{j-1},$$

where $X(\omega)$ means the contraction of X and ω with values in C_{j-1}. Define δ to be the quotient map of δ_0 on $\mathcal{X}^{(j)} := C_j/C'_j$.

Define also $d_j : C_k \to C_k, j = 1, \ldots, k$ by

$$d_j(X_1 \wedge \ldots \wedge X_k \otimes \lambda \otimes \xi) := X_1 \wedge \ldots \wedge X_k \otimes \lambda \otimes d_j\xi,$$

$\forall X_1 \wedge \ldots \wedge X_k \in \Lambda^k \mathfrak{g}_x, \forall \lambda \in \mathbb{C}_\Delta, \forall \xi \in F.$

1.3.3 Definition of σ in the relative case

Let C_j σ be as in 1.3.1 and 1.3.2. For every element $Z \in C^\infty(U', \mathfrak{g}_x)$, define $Z(x \exp X) := X$ and

$$\sigma_0 : C_k \to C_{k+1}; \sigma_0(\omega) := Z \wedge \omega,$$

Define $\sigma : \mathcal{X}^{(k)} \to \mathcal{X}^{(k+1)}$ to be the quotient maps of σ_0 mode C'_j.

1.3.4 Relative form of η and the equation $\nabla = \sigma\delta + \delta\sigma$

Let G_x, H be as before, $H' = H \cap G_x$. First define an action η' of $\mathbb{R}$ on $L(U', H \setminus G_x)$ as

$$(\eta'_t\varphi)(x \exp X, H'g_0, \dots, H'g_n) := \beta_{\exp(tX)}(x \exp(X), H'g_0, \dots, H'g_n), \qquad (**)$$

for any $\varphi \in L(U', H' \setminus G_x)$, $t \in \mathbb{R}$. It is natural to extend this action to C_j by

$$t.(X_1 \wedge \dots \wedge X_k \otimes \lambda \otimes \xi) := X_1 \wedge \dots \wedge X_k \otimes \lambda \otimes \eta'_t(\xi),$$

for all $X_1, \dots, X_k \in \mathfrak{g}_x$, $\lambda \in \mathbb{C}_\Delta$ and $\xi \in L(U', H \setminus G_x)$. Let ∇_0 be the derivative of the action of η'_t above at 0. By a similar computation as in [N1], one can see that

$$\nabla_0 = \delta_0\sigma_0 + \sigma_0\delta_0.$$

It is also not hard to see that each C'_j is invariant under this action and therefore we get and action η of $\mathbb{R}$ in $\mathcal{X}^{(j)}$.

Finally for the above data, in a way analogous to that in [N1], we define

$$\mathcal{Y} := (\mathbb{C}_{\Delta'} \otimes_H L(U', H' \setminus G_x)) \otimes_{G_x} \mathbb{C} = ((A \rtimes G_x)^\natural_{U'})^H$$

and

$$\mathcal{Z} := \begin{cases} L(U' \cap K_x, H' \setminus G_{x'}) \otimes_{K_{x'}} \mathbb{C} & \text{if } x' \in K_x \\ L(U' \cap M_0, H' \setminus G_{x'}) \otimes_M \mathbb{C} & \text{if } x' \notin K_x \end{cases},$$

where M_0 is the inverse image of M in $G_{x'}$ of the maximal compact subgroup of $G_{x'}/(x)$.

By a similar argument to that in [N1], we can conclude that for the H-relative complex all the conditions of the main lemma of [N1] are also satisfied.

1.4 Passage to coadjoint orbits

After the statement of his main result (Theorem 1.1, p. 4 in [N1]), V. Nistor stated as follows. "This fits with Mackey's method of orbits, except that now for reasons we do not yet understand, we obtain orbits on $(\operatorname{Lie} G)$ rather than in $(\operatorname{Lie} G)^*$. An interesting feature of the result is worthwhile stressing: there is no γ-obstruction in cyclic cohomology." This means that he didn't work with the coadjoint orbits. We now explain that it is natural to pass to coadjoint orbits and that the localization disappears on coadjoint orbits.

Let G be a connected and simply connected Lie group, $\mathfrak{g} = \operatorname{Lie}(G)$, $\mathfrak{g}^*$ the dual of $\mathfrak{g}$. Let F be a fixed point in $\mathfrak{g}^*$, G_F the stabilizer of F. Let us consider the natural projection $G \twoheadrightarrow G_F \setminus G = \Omega \hookrightarrow \mathfrak{g}^*$ defined as $x_0 \mapsto G_F x_0 = \tilde{F}_0 \in \mathfrak{g} \leftrightarrow F_0 \in \mathfrak{g}^*$. Suppose that $x_0 = \exp(\tilde{F}_0)$, i.e. $\tilde{F}_0 = \ln x_0$, then from the well-known Van Campbell-Hausdorff-Dynkin formula for $\ln(\exp(X)\exp(Y))$, we deduce that $\ln(x x_0 x^{-1}) = \operatorname{Ad}_x \tilde{F}_0$. We have therefore the following

Lemma 1.6 *Under the map $G \twoheadrightarrow G_F \setminus G = \Omega \hookrightarrow \mathfrak{g}^*$, the element $x x_0 x^{-1}$ in G goes to the element $\operatorname{Ad}(x^{-1})F_0$, and the conjugacy orbit of x_0 goes to the coadjoint orbit*

$$\Omega_{F_0} = \{\operatorname{Ad}(x^{-1})F_0 \quad | \quad x \in G\}.$$

1.5 Localization on coadjoint orbits

We apply the construction of the subsection 1.2 to the case of coadjoint orbit $\Omega = G_F \setminus G$.

Lemma 1.7 *Let G be a connected and simply connected Lie group. There is a natural isomorphism $C_c^\infty(\mathfrak{g}) \cong C_c^\infty(U)$, where U is an open set in G.*

Proof. The exponential map $\exp : \mathfrak{g} \to G$ is a local diffeomorphism and the image $U = \exp \mathfrak{g}$ of $\mathfrak{g}$ is open in G. □.

Lemma 1.8 *There is a natural isomorphism between convolution algebras*

$$C_c^\infty(\mathfrak{g}) \cong C_c^\infty(\mathfrak{g}^*)$$

Proof. It is natural to identify $\mathfrak{g}$ with $\mathfrak{g}^*$. □

Lemma 1.9 *There is a natural isomorphism between convolution algebras*

$$C_c^\infty(\mathfrak{g})^G \cong C_c^\infty(\mathfrak{g}^*)^G$$

Proof. Under the isomorphism $\mathfrak{g} \cong \mathfrak{g}^*$, the adjoint action $\mathrm{Ad}_G\, x$ becomes the coadjoint action $\mathrm{Ad}_G^*(x^{-1})$.
□

Lemma 1.10 *There is a one-to-one correspondence between the localization of the algebra $C_{inv}^\infty(G)$ at maximal ideals and the central characters of the Lie group G.*

Proof. On the coadjoint orbits, bi-invariant functions $\varphi \in C_{inv}^\infty(G)$ are constant. Therefore, localization at any maximal ideal at points of the coadjoint orbits are the same as the constants. The value of the constant function on coadjoint orbits are the same as the central character of the representation associate to these orbits, (see for example [K1]). □

The following lemmas 1.11 and 1.12 are implicit in [K1]:

Lemma 1.11 *For almost all connected and simply connected Lie groups, namely "almost algebraic" or solvable Lie groups, there is a one-to-one correspondence between the central characters and the irreducible unitary representations*

Proof. For large classes of connected and simply connected Lie groups, the statement has been verified. It was verified also, (see [K1]) for almost all connected and simply connected Lie groups. □

Lemma 1.12 *For almost all connected and simply connected Lie groups, e.g. almost algebraic or solvable Lie groups, there is a one-to-one correspondence between the irreducible unitary representations of G and the coadjoint orbits or their coverings*

Proof. For large classes of connected and simply connected Lie groups this statement has been verified, see [K1] for more details. □

We now have the main theorem. Let us first fix some notations. For any Lie group G and subgroups $K, \Omega = \Omega_F$ a coadjoint orbit of G, passing through a fixed point F in $\mathfrak{g}^*$, we shall write $\Omega|_K = \Omega_{F|K}$ for the coadjoint K-orbit passing through $F|_{\mathfrak{k}}$. We also write $K_{F|_K}$ for the stabilizer of coadjoint K-action at $F|_{\mathfrak{k}}$.

Theorem 1.13 (Main Theorem) *Let G be a connected and simply connected Lie group, $\Omega = \Omega_F$ a coadjoint orbit of G passing through a fixed point F in $\mathfrak{g}^*$, K a maximal compact subgroup of G, $q = \dim(G/K)$ and Ω_K the coadjoint orbit passing through $F|_{\mathfrak{k}}$. Let $C_c^\infty(\Omega)$ (resp. $C_c^\infty(\Omega|_K)$) be the quantized algebra of functions on Ω (resp., $\Omega|_K$). Then we have an isomorphism*

$$\mathrm{PHC}_*(C_c^\infty(\Omega_G) \cong \mathrm{PHC}_{*+q}(C_c^\infty(\Omega_K)).$$

Proof. First observe that a coadjoint orbit $\Omega_F \subseteq \mathfrak{g}^*$ can be identified with the homogeneous space $H \setminus G$, where $H = G_F$ is the stabilizer of an arbitrary point F in Ω_F. We have therefore

$$C_c^\infty(\Omega_F) \cong C_c^\infty(G_F \setminus G) \cong C_c^\infty(G)^H. \qquad \star$$

Let us indicate with sub-index K the restrictions onto K. We have the same notations for a fixed maximal compact subgroup K, $C_c^\infty(\Omega_{F_K}) \cong C_c^\infty(K_{F_K} \setminus K) \cong C_c^\infty(K)^{H_K}$. From Proposition 1.3, we have

$$\mathrm{HC}_* *(L(K, H \setminus G)^K) \cong \mathrm{HC}_* *((A \rtimes K)^K)^H.$$

Also we have from Proposition 1.4 that the complex

$$0 \to (\wedge^q(\mathfrak{g}/\mathfrak{k}) \otimes_H \mathcal{F}) \otimes_K \mathbb{C} \xrightarrow{\delta} \dots \xrightarrow{\delta} (\wedge^0(\mathfrak{g}/\mathfrak{k}) \otimes_H \mathcal{F}) \otimes_K \mathbb{C} \longrightarrow \mathcal{F} \otimes_G \mathbb{C} \longrightarrow 0$$

is acyclic. Using Lemma 1.9, we can identify $C_c^\infty(\Omega_F)$ with some algebra of functions on a conjugacy orbit of G in $\mathfrak{g}$ with adjoint action. A function in $C_{inv}^\infty(G)$ means a function which is constant on conjugacy classes of G. So from Lemmas 1.7-1.9, it is the same as a function, which is constant on each coadjoint orbit. From Lemmas 1.10-1.12, a maximal ideal in the space of central functions corresponds exactly to a coadjoint orbit. The localization at the ideal $\mathfrak{m}$ therefore means taking restrictions of functions on the associated orbit. Now we can apply the theorem 1.1 of Nistor to conclude that the homology groups with localization are isomorphic to the groups without localization. □

2 Noncommutative Chern-Connes characters

Let us now briefly recall the construction of noncommutative Chern-Connes characters. We then present an interesting result (Theorem 2.1) for the noncommutative Chern-Connes characters of the quantized algebras of functions on coadjoint orbits.

2.1 K-groups. Connes-Kasparov-Rosenberg Theorem

In the following, K-groups shall mean the $\mathbb{Z}/(2)$-graded algebraic K-groups of algebras over the field of complex numbers. For connected solvable Lie groups, the so called Connes-Kasparov conjecture asserting similar isomorphisms has been proved (see, e.g. [J]) and it is known for large classes of Lie groups as Connes-Kasparov-Rosenberg Theorem, i.e.

$$K_*(C_c^\infty(G)) \cong K_{*+q}(C^\infty(K)),$$

where $q = \dim(G/K)$.

2.2 Noncommutative Chern-Connes characters

For the general notion of Chern-Connes characters, readers are referred to the work of J. Cuntz [Cu]. From the works of J. Cuntz [Cu] and V. Nistor [N1]-[N2] we can deduce that there is a natural noncommutative Chern-Connes character ch with values in the localizations of $\mathrm{PHC}_*(A)$, as

$$ch : K_*(C_c^\infty(G)) \to \mathrm{PHC}_*(C_c^\infty(G))_{\mathfrak{m}}.$$

2.3 The commutative diagrams

There is a natural commutative diagram:

$$\begin{array}{ccc} K_*(C_c^\infty(G)) & \xrightarrow{ch_G} & \mathrm{PHC}_*(C_c^\infty(G)))_{\mathfrak{m}} \\ \downarrow & & \downarrow \\ K_{*+q}(C_c^\infty(K)) & \xrightarrow{ch_K} & \mathrm{PHC}_{*+q}(C_c^\infty(K))_{\mathfrak{m}} \end{array}$$

where the first vertical row is the Connes-Kasparov-Rosenberg isomorphism, and the second vertical one is the isomorphism of V. Nistor. This can be reduced to the maximal torus case

$$\begin{array}{ccc} K_{*+q}(C_c^\infty(K)) & \xrightarrow{ch_K} & \mathrm{PHC}_{*+q}(C_c^\infty(K))_{\mathfrak{m}} \\ \downarrow & & \downarrow \\ K_{*+q}(C_c^\infty(\mathbb{T}))^W & \xrightarrow{ch_{\mathbb{T}}} & \mathrm{PHC}_{*+q}(C_c^\infty(\mathbb{T}))_{\mathfrak{m}}^W \end{array}$$

where $W = W(\mathbb{T})$ is the Weyl group corresponding to the maximal torus $\mathbb{T}$ and the sub-indices of ch indicate the corresponding target groups. In this second commutative diagram the first vertical row is the well-known result of the K-theory of compact Lie groups and the second one is the reduction of V. Nistor in [N1]. The horizontal row on the bottom is an isomorphism, as is well-known in topology. We have therefore the following interesting consequence

Theorem 2.1 *Let G be a connected and simply connected Lie group, $\Omega = \Omega_F$ a coadjoint orbit of G passing through a fixed point F, K a maximal compact subgroup of G, $\Omega|_K$ the coadjoint orbit of K passing through $F|_{\mathfrak{k}}$, $\mathbb{T}$ the maximal torus of G in K and $W = N(\mathbb{T})/\mathbb{T}$ the Weyl group corresponding to $\mathbb{T}$. For any co-adjoint orbit Ω, let $C_c^\infty(\Omega)$ be the quantized algebra of functions on Ω with compact* *support. Then, there is a commutative diagram for the noncommutative Chern-Connes characters of the quantized algebra of functions on coadjoint orbits*

$$\begin{array}{ccc}
K_*(C_c^\infty(\Omega)) & \xrightarrow{ch_\Omega} & \mathrm{PHC}_*(C_c^\infty(\Omega))) \\
I\downarrow & & \downarrow III \\
K_{*+q}(C_c^\infty(\Omega_K)) & \xrightarrow{ch_{\Omega_K}} & \mathrm{PHC}_{*+q}(C_c^\infty(\Omega_K)) \\
II\downarrow & & \downarrow IV \\
K_{*+q}(C_c^\infty(\mathbb{T}))^W & \xrightarrow{ch_\mathbb{T}} & \mathrm{PHC}_{*+q}(C_c^\infty(\mathbb{T}))^W
\end{array}$$

and modulo torsion, the noncommutative Chern-Connes characters are isomorphisms.

Proof. Let us first consider the commutative diagram

$$\begin{array}{ccc}
K_*(C_c^\infty(G)) & \xrightarrow{ch_G} & \mathrm{PHC}_*(C_c^\infty(G)))_\mathfrak{m} \\
\downarrow & & \downarrow \\
K_{*+q}(C_c^\infty(K)) & \xrightarrow{ch_K} & \mathrm{PHC}_{*+q}(C_c^\infty(K))_\mathfrak{m}
\end{array}$$

and

$$\begin{array}{ccc}
K_{*+q}(C_c^\infty(K)) & \xrightarrow{ch_K} & \mathrm{PHC}_{*+q}(C_c^\infty(K))_\mathfrak{m} \\
\downarrow & & \downarrow \\
K_{*+q}(C_c^\infty(\mathbb{T}))^W & \xrightarrow{ch_\mathbb{T}} & \mathrm{PHC}_{*+q}(C_c^\infty(\mathbb{T}))_\mathfrak{m}^W
\end{array}$$

Localizing $C_c^\infty(G)$ at the ideals $\mathfrak{m}$, corresponding to the orbit Ω yields $C_c^\infty(\Omega)$, as was explained in the proof of Theorem 1.13. Then by doing similar computations to what was done in [DKT1]-[DKT2], since the representations of K are defined by their restrictions to maximal tori, we have the isomorphism (I), (II) and (IV). The isomorphism (III) is the main theorem 1.13. □

Note that in this diagram we don't need to take localization because, as explained in the proof of the main theorem 1.13, localization at an ideal $\mathfrak{m}$ in $C_{inv}^\infty(G)$ give us the quantized algebras of functions on coadjoint orbits.

Note that $\mathrm{PHC}_*(.)$ here is formally different from $\mathrm{HP}_*(.)$ in [DKT1]-[DKT2] in the sense that in the definition of $\mathrm{PHC}_*(.)$ products are used in place of direct sums in the definition of the total complexes. However, this does not affect the definition of the noncommutative Chern-Connes characters, because for the entire cyclic homology $\mathrm{HE}_*(.)$, one uses convergent series of finite degree cycles. For the algebraic version $\mathrm{HP}_*(.)$, we used the Cuntz-Quillen $\mathcal{X}$-complexes, which reduced also to products in the bicomplexes.

In the next two sections we shall apply Theorem 1.13 to deduce isomorphisms of cohomologies of quantized algebras of functions on coadjoint orbits. In order to do this, we recall the results from [DH1] about the algebras and then compute the K-theory, periodic cyclic homology and the noncommutative Chern characters.

3 Quantum half-planes

Applying the main theorem 1.13, we compute in this section the noncommutative Chern-Connes characters for the quantum algebras of functions on the half-planes.

3.1 Deformation quantization

Let us recall some results from [DH1]: Recall that the Lie algebra $\mathfrak{g} = \mathfrak{aff}(\mathbb{R})$ of affine transformations of the real straight line is described as follows, see for example [D1]: The Lie group $\mathrm{Aff}(\mathbb{R})$ of affine transformations:

$$x \in \mathbb{R} \mapsto ax + b, \text{ for some parameters } a, b \in \mathbb{R}, a \neq 0.$$

is known to be a two-dimensional Lie group which is isomorphic to the group of matrices

$$\mathrm{Aff}(\mathbb{R}) \cong \{\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} | a, b \in \mathbb{R}, a \neq 0\}.$$

We consider its connected component $\mathrm{Aff}_0(\mathbb{R})$ of the identity element given by

$$G = \mathrm{Aff}_0(\mathbb{R}) = \{\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} | a, b \in \mathbb{R}, a > 0\}.$$

Its Lie algebra

$$\mathfrak{g} = \mathfrak{aff}(\mathbb{R}) \cong \{\begin{pmatrix} \alpha & \beta \\ 0 & 0 \end{pmatrix} | \alpha, \beta \in \mathbb{R}\}$$

admits a basis of two generators X, Y with the only nonzero Lie bracket $[X, Y] = Y$, i.e.

$$\mathfrak{g} = \mathfrak{aff}(\mathbb{R}) \cong \{\alpha X + \beta Y | [X, Y] = Y, \alpha, \beta \in \mathbb{R}\}.$$

The co-adjoint action of G on $\mathfrak{g}^*$ is given (see e.g. [K1]) by

$$\langle K(g)F, Z\rangle = \langle F, \mathrm{Ad}(g^{-1})Z\rangle, \forall F \in \mathfrak{g}^*, g \in G \text{ and } Z \in \mathfrak{g}.$$

Denote the co-adjoint orbit of G in $\mathfrak{g}$, passing through F by

$$\Omega_F = K(G)F := \{K(g)F | g \in G\}.$$

Because the group $G = \mathrm{Aff}_0(\mathbb{R})$ is exponential (see [D1]), then for $F \in \mathfrak{g}^* = \mathfrak{aff}(\mathbb{R})^*$, we have

$$\Omega_F = \{K(\exp(U)F | U \in \mathfrak{aff}(\mathbb{R})\}.$$

and hence that

$$\langle K(\exp U)F, Z\rangle = \langle F, \exp(-\mathrm{ad}_U)Z\rangle.$$

It is easy therefore to see that

$$K(\exp U)F = \langle F, \exp(-\mathrm{ad}_U)X\rangle X^* + \langle F, \exp(-\mathrm{ad}_U)Y\rangle Y^*.$$

For a general element $U = \alpha X + \beta Y \in \mathfrak{g}$, we have

$$\exp(-\operatorname{ad}_U) = \sum_{n=0}^{\infty} \frac{1}{n!} \begin{pmatrix} 0 & 0 \\ \beta & -\alpha \end{pmatrix}^n = \begin{pmatrix} 1 & 0 \\ L & e^{-\alpha} \end{pmatrix},$$

where $L = \alpha + \beta + \frac{\alpha}{\beta}(1 - e^{\beta})$. This means that

$$K(\exp U)F = (\lambda + \mu L)X^* + (\mu e^{-\alpha})Y^*.$$

From this formula one deduces [D1] the following description of all co-adjoint orbits of G in $\mathfrak{g}^*$:

- If $\mu = 0$, each point $(x = \lambda, y = 0)$ on the abscissa ordinate corresponds to a 0-dimensional co-adjoint orbit

$$\Omega_\lambda = \{\lambda X^*\}, \quad \lambda \in \mathbb{R}.$$

- For $\mu \neq 0$, there are two 2-dimensional co-adjoint orbits: the upper half-plane $\{(\lambda, \mu) \mid \lambda, \mu \in \mathbb{R}, \mu > 0\}$ corresponds to the co-adjoint orbit

$$\Omega_+ := \{F = (\lambda + \mu L)X^* + (\mu e^{-\alpha})Y^* \mid \mu > 0\}, \tag{1}$$

and the lower half-plane $\{(\lambda, \mu) \mid \lambda, \mu \in \mathbb{R}, \mu < 0\}$ corresponds to the co-adjoint orbit

$$\Omega_- := \{F = (\lambda + \mu L)X^* + (\mu e^{-\alpha})Y^* \mid \mu < 0\}. \tag{2}$$

We shall work henceforth on the fixed co-adjoint orbit Ω_+. The case of the co-adjoint orbit Ω_- could be similarly treated. First we study the geometry of this orbit and introduce some canonical coordinates in it. It is well-known from the orbit method [K1] that the Lie algebra $\mathfrak{g} = \mathfrak{aff}(\mathbb{R})$ is realized by the complete right-invariant Hamiltonian vector fields on co-adjoint orbits $\Omega_F \cong G_F \setminus G$ with flat (co-adjoint) action of the Lie group $G = \operatorname{Aff}_0(\mathbb{R})$. On the orbit Ω_+ we choose a fix point $F = Y^*$. It is well-known from the orbit method that we can choose an arbitrary point F on Ω_F. It is easy to see that the stabilizer of this (and therefore of any) point is trivial, i.e. $G_F = \{e\}$. We identify therefore G with $G_{Y^*} \setminus G$. There is a natural diffeomorphism $\mathrm{Id}_{\mathbb{R}} \times \exp(.)$ from the standard symplectic space $\mathbb{R}^2$ with symplectic 2-form $dp \wedge dq$ in canonical Darboux (p, q)-coordinates, onto the upper half-plane $\mathbb{H}_+ \cong \mathbb{R} \rtimes \mathbb{R}_+$ with coordinates (p, e^q), which is, from the above coordinate description, also diffeomorphic to the co-adjoint orbit Ω_+. We can use therefore (p, q) as the standard canonical Darboux coordinates in Ω_{Y^*}. There are also non-canonical Darboux coordinates $(x, y) = (p, e^q)$ on Ω_{Y^*}. We show now that in these coordinates (x, y), the Kirillov form looks like $\omega_{Y^*}(x, y) = \frac{1}{y} dx \wedge dy$, but in the canonical Darboux coordinates (p, q), the Kirillov form is just the standard symplectic form $dp \wedge dq$. This means that there are symplectomorphisms between the standard symplectic space $\mathbb{R}^2, dp \wedge dq)$, the upper half-plane $(\mathbb{H}_+, \frac{1}{y} dx \wedge dy)$ and the co-adjoint orbit $(\Omega_{Y^*}, \omega_{Y^*})$. Each element $Z \in \mathfrak{g}$ can be considered as a linear functional $\tilde{Z}$ on co-adjoint orbits, as subsets of $\mathfrak{g}^*$, where $\tilde{Z}(F) := \langle F, Z \rangle$. It is well-known that this linear function is just the Hamiltonian function associated with the Hamiltonian vector field ξ_Z, which represents $Z \in \mathfrak{g}$ following the formula

$$(\xi_Z f)(x) := \frac{d}{dt} f(x \exp(tZ))|_{t=0}, \forall f \in C^\infty(\Omega_+).$$

The Kirillov form ω_F is defined by the formula

$$\omega_F(\xi_Z, \xi_T) = \langle F, [Z,T] \rangle, \forall Z, T \in \mathfrak{g} = \mathrm{aff}(\mathbb{R}). \tag{3}$$

This form defines the symplectic structure and the Poisson brackets on the co-adjoint orbit Ω_+. For the derivative along the direction ξ_Z and the Poisson bracket we have relation $\xi_Z(f) = \{\tilde{Z}, f\}, \forall f \in C^\infty(\Omega_+)$. It is well-known in differential geometry that the correspondence $Z \mapsto \xi_Z, Z \in \mathfrak{g}$ defines a representation of our Lie algebra by vector fields on co-adjoint orbits. If the action of G on Ω_+ is flat [D1], we have the second Lie algebra homomorphism from strictly Hamiltonian right-invariant vector fields into the Lie algebra of smooth functions on the orbit with respect to the associated Poisson brackets.

Denote by ψ the indicated symplectomorphism from $\mathbb{R}^2$ onto Ω_+

$$(p,q) \in \mathbb{R}^2 \mapsto \psi(p,q) := (p, e^q) \in \Omega_+$$

It was proven in [DH1] that:

- Hamiltonian function $f_Z = \tilde{Z}$ in canonical coordinates (p,q) of the orbit Ω_+ is of the form

$$\tilde{Z} \circ \psi(p,q) = \alpha p + \beta e^q, \text{ if } Z = \begin{pmatrix} \alpha & \beta \\ 0 & 0 \end{pmatrix}.$$

- In the canonical coordinates (p,q) of the orbit Ω_+, the Kirillov form ω_{Y^*} is just the standard form $\omega = dp \wedge dq$.

Let us denote by Λ the 2-tensor associated with the canonical Kirillov standard form $\omega = dp \wedge dq$ in the canonical Darboux coordinates. Recall the deformed $\star$-product of two smooth functions $u, v \in C^\infty(\Omega_\pm)$

$$u \star_h v = u.v + \sum_{r \geq 1} \frac{1}{r} \left(\frac{h}{2i} \right)^r P^r(u,v), \tag{$\star\star$}$$

where

$$P^r(u,v) = \Lambda^{i_1,j_1} \Lambda^{i_2,j_2} \dots \Lambda^{i_r,j_r} \partial_{i_1} \dots \partial_{i_r} u \partial_{j_1} \dots \partial_{j_r} v,$$

with the ordinary multi-index notations of the partial derivations. Note that in [DH1], ($\star\star$) was shown for the normalized Planck constant $h = 1$. The situation is the same for an arbitrary nonzero value of h. It was shown that to every element $X \in \mathfrak{g}$ corresponds a function $\tilde{X}$ on $\mathfrak{g}^*$ and therefore on the K-orbits Ω_F.

It was shown in [DH1](Proposition3.1) that

$$\frac{i}{h}\tilde{X} \star_h \frac{i}{h}\tilde{T} - \frac{i}{h}\tilde{T} \star_h \frac{i}{h}\tilde{X} = \frac{i}{h}\widetilde{[X,T]}, \forall Z, T \in \mathrm{aff}(\mathbb{R}).$$

One therefore has a representation

$$X \longrightarrow \frac{i}{h}\tilde{X} \star_h$$

of the Lie algebra $C_c^\infty(\Omega_G)$ by the left $\star_h$-multiplication. On the half-plane with the fixed Darboux (q,p)-coordinates, one fixes the Fourier transformation in p-coordinate

$$\mathcal{F}_p(u)(\eta,q) := \frac{1}{2\pi}\int_{\mathbb{R}} \exp(-ip\eta)u(p,q)dp$$

and obtain that for the element $\tilde{Z} = \alpha p + \beta e^q$, the operator ℓ_Z acting on the dense subspace $L^2(\mathbb{R}^2, \frac{dpdq}{2\pi})^\infty$ of smooth functions by left $\star_h$-multiplication by $i\tilde{Z}\star_h$, i.e. $\ell_Z(u) := \frac{i}{h}\tilde{Z}\star_h u$. It was precisely computed (see Proposition 3.4 in [DH1]) that

$$\hat{\ell}_Z(u) := \mathcal{F}_p \circ \ell_Z \circ \mathcal{F}_p^{-1}(u) = \alpha(\frac{1}{2}\partial_q - \partial_p)u.$$

It was also proven in Theorem 4.2 of [DH1] that:

The representation $\exp(\hat{\ell}_Z)$ of the group $G = \mathrm{Aff}_0(\mathbb{R})$ is exactly the irreducible unitary representation T_{Ω_+} of $G = \mathrm{Aff}_0(\mathbb{R})$ associated, following the orbit method construction, to the orbit Ω_+, which is the upper half-plane $\mathbb{H} \cong \mathbb{R} \rtimes \mathbb{R}^*$, i. e.

$$(\exp(\hat{\ell}_Z)f)(y) = (T_{\Omega_+}(g)f)(y) = e^{\frac{i}{h}by}f(ay), \forall f \in L^2(\mathbb{R}^*, \frac{dy}{y}),$$

where $g = \exp Z = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$.

3.2 K-groups and periodic cyclic homology and Chern-Connes characters

Let us apply now the general notion of Chern-Connes characters to the example of the quantum algebras of functions on the coadjoint orbits of the groups of affine transformation of the real (in this section) and complex (in the next section) lines. Recall that the noncommutative Chern-Connes characters are some homomorphisms ch from the K-groups $K_*(.)$ to the corresponding periodic cyclic homology groups $\mathrm{PHC}_*(.)$.

Lemma 3.1 *In the groups* $\mathrm{Aff}(\mathbb{R})$*, the maximal compact subgroups is* $K \cong \mathbb{Z}_2 = \mathbb{Z}/(2\mathbb{Z})$*. In its connected component of identity* $\mathrm{Aff}_0(\mathbb{R})$*, the maximal compact subgroup is trivial, i.e.* $K \cong \{e\}$*.*

Proof. The proof is clear. □

Proposition 3.2 *Let Ω_+ be the coadjoint orbit which is the upper half-plane. Then*

$$K_*(C_c^\infty(\Omega_+)) \cong \mathrm{PHC}_*((C_c^\infty(\Omega_+)) \cong \{e\}.$$

and therefore the noncommutative Chern-Connes characters are isomorphisms.

Proof. Because the maximal compact subgroups of G are trivial, we can conclude that the K-groups and the PHC_*-groups are also trivial. □

4 Quantum punctured complex plane

In this section we demonstrate another application of the main theorem for the group of affine transformations of the complex line. The deformation of the coadjoint orbits of this group is in some sense more complicated than the one in the real case, see e.g. [DH2].

4.1 Deformation quantization

The group $\mathrm{Aff}(\mathbb{C})$ is defined as

$$\mathrm{Aff}(\mathbb{C}) = \{\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mid \quad a, b \in \mathbb{C}, a \neq 0\}.$$

It is isomorphic to the semi-direct product of the complex line $\mathbb{C}$ and the punctured complex line $\mathbb{C}^* = \mathbb{C} \setminus (0)$. The group is connected but not simply connected. and the exponent map

$$\exp : \mathbb{C} \to \mathbb{C}^*; z \mapsto e^z$$

gives rise to the universal covering

$$\widetilde{\mathrm{Aff}(\mathbb{C})} \cong \mathbb{C} \rtimes \mathbb{C} \cong \{(z,w) | z, w \in \mathbb{C}\}$$

of $\mathrm{Aff}(\mathbb{C})$ with multiplication

$$(z,w)(z',w') := (z+z', w+e^z w').$$

As a real Lie group, it is 4-dimensional and we denote its Lie algebra by $\mathrm{aff}(\mathbb{C}) = \mathrm{Lie}\,\mathrm{Aff}(\mathbb{C})$ The dual space $\mathfrak{g}^*$ of $\mathfrak{g} = \mathrm{Lie}\,\mathrm{Aff}(\mathbb{C})$ can be identify with $\mathbb{R}^4$ with coordinates $(\alpha, \beta, \gamma, \delta)$, see [D1]. The coadjoint orbits of $\widetilde{\mathrm{Aff}(\mathbb{C})}$ in $\mathfrak{g}^*$ passing though a point $F = \alpha X_1^* + \beta X_2^* + \gamma Y_1^* + \delta Y_2^*$ is denote by Ω_F, where $X_1^*, X_2^*, Y_1^*, Y_2^*$ form the basis of $\mathfrak{g}^*$ dual to the basis X_1, X_2, Y_1, Y_2 of $\mathfrak{g}$ with the brackets

$$[X_1, Y_1] = Y_1, [X_1, Y_2] = Y_2, [X_2, Y_1] = Y_2, [X_2, Y_2] = -Y_1.$$

Then

- Each point $(\alpha, 0, 0, \delta)$ is a 0-dimensional coadjoint orbit, denoted $\Omega_{(\alpha,0,0,\delta)}$,
- The open set $\beta^2 + \gamma^2 \neq 0$ is the single 4-dimensional coadjoint orbit $\Omega \approx \mathbb{C} \times \mathbb{C}^*$, the punctured complex plane.

Note that the orbit Ω is not simply connected and there is no diffeomorphism from some symplectic vector space onto it. In [DH2], some system of diffeomorphisms was constructed. Let us recall them. Consider

$$\mathbb{H}_k := \{w = q_1 + iq_2 \in \mathbb{C} | -\infty < q_1 < +\infty, 2k\pi < q_2 < 2k\pi < q_2 < (2k+1)\pi\},$$

for each $k = 0, \pm 1, \ldots$. Let $\mathbb{C}_k := \mathbb{C} \setminus L$, where L is the positive real line

$$L = \{\rho e^{i\varphi} \in \mathbb{C} | 0 < \rho < \infty, \varphi = 0\}.$$

There is a natural map

$$\mathbb{C} \times \mathbb{C} \to \Omega \cong \mathbb{C} \times \mathbb{C}^*; (z, w) \mapsto (z, e^w),$$

whose restriction gives a diffeomorphism

$$\varphi_k : \mathbb{C} \times \mathbb{H}_k \to \mathbb{C} \times \mathbb{C}^*.$$

On $\mathbb{C} \times \mathbb{H}_k$ we have the natural symplectic form

$$\omega_0 := \frac{1}{2}[dz \wedge dw + d\bar{z} \wedge d\bar{w}],$$

induced from the standard symplectic form on $\mathbb{C}^2$ with coordinates (z, w). The corresponding symplectic form matrix is

$$\wedge = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \text{ and } \wedge^{-1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

The corresponding Poisson brackets of functions $f, g \in C^\infty(\Omega)$ is

$$\{f, g\} = \wedge^{ij}\frac{\partial f}{\partial x^i}\frac{\partial g}{\partial x^j} = \frac{\partial f}{\partial p_1}\frac{\partial g}{\partial q_1} - \frac{\partial f}{\partial q_1}\frac{\partial g}{\partial p_1} - \frac{\partial f}{\partial p_2}\frac{\partial g}{\partial q_2} + \frac{\partial f}{\partial q_2}\frac{\partial g}{\partial p_2}.$$

For an arbitrary element $A \in \text{aff}(\mathbb{C})$ it was computed in [DH2](Proposition 2.4) that:

- the corresponding function on Ω is

$$\tilde{A} \circ \varphi_k(z, w) = \frac{1}{2}[\alpha z + \beta e^w + \bar{\alpha}\bar{z} + \bar{\beta}e^{\bar{w}}].$$

- In the local coordinates (z, w) of the orbit Ω, the Kirillov form Ω coincides with the standard form

$$\omega_0 := \frac{1}{2}[dz \wedge dw + d\bar{z} \wedge d\bar{w}].$$

It was also proven in [DH2](Proposition 3.1) that for all $A, B \in \text{aff}(\mathbb{C})$, the Moyal $\star_h$ product satisfies the relation

$$\frac{i}{h}\tilde{A} \star_h \frac{i}{h}\tilde{B} - \frac{i}{h}\tilde{B} \star_h \frac{i}{h}\tilde{A} = \frac{i}{h}[A, B].$$

This means that we have some representation $\ell_A^{(k)} : A \mapsto \frac{i}{h}\tilde{A}\star_h$ of Lie algebra $\text{aff}(\mathbb{C})$ on the space $C^\infty(\Omega)$. Denote by $\mathcal{F}_z$ the Fourier transformation

$$\mathcal{F}_z(f)(\xi, w) := \frac{1}{2\pi}\int_{\mathbb{R}^2} \exp(-i\Re(\xi\bar{z}))f(z, w)dp_1dp_2$$

and the inverse Fourier transformation

$$\mathcal{F}_z^{-1}(\tilde{f})(z, w) := \frac{1}{2\pi}\int_{\mathbb{R}^2} \exp(i\Re(\xi\bar{z}))\tilde{f}(\xi, w)dp_1dp_2$$

By the same computation as in [DH2] we have for each $A = \begin{pmatrix} \alpha & \beta \\ 0 & 0 \end{pmatrix} \in \text{aff}(\mathbb{C})$ and for each compactly supported smooth function $f \in C_c^\infty(\mathbb{C} \times \mathbb{H}_k)$,

$$\hat{\ell}_A f = \mathcal{F}_z \circ \ell_A^{(k)} \circ \mathcal{F}_z^{-1}(f) = [\alpha(\frac{1}{2}\partial_w - \partial_{\bar{\xi}}) + \bar{\alpha}(\frac{1}{2}\partial_{\bar{w}} - \partial_{\xi}) + \frac{i}{2}(\beta e^{w - \frac{1}{2}\bar{\xi}} + \bar{\beta}e^{\bar{w} - \frac{1}{2\xi}})f].$$

It was shown [DH2](Theorem 4.2) that the representation $\exp(\hat{\ell}_A^{(k)})$ of the universal covering r group $\widetilde{\mathrm{Aff}(\mathbb{C})}$ is is coincided with the irreducible unitary representation T_θ of $\widetilde{\mathrm{Aff}(\mathbb{C})}$ associated with Ω by the obit method, i.e.

$$\exp(\hat{\ell}_A^{(k)})f(x) = [T_\theta(\exp A)f](x),$$

realizing the the space $L^2(\mathbb{R} \times \mathbb{S}^1)$ and acting as

$$[T_\theta(z,w)f](x) = \exp(\frac{i}{h}\Re(wx) + 2\pi\theta[\frac{\Im(x+z)}{2\pi}]))f(x \oplus z),$$

where $(z,w) \in \widetilde{\mathrm{Aff}(\mathbb{C})}$, $x \in \mathbb{R}\times\mathbb{S}^1 = \mathbb{C}\setminus(0)$, $f \in L^2(\mathbb{R}\times\mathbb{S}^1)$, $x \oplus z := \Re(x+z) + 2\pi i\{\frac{x+z}{2\pi}\}$, $[a]$ is the integral part of a and $\{a\}$ is the decimal part of a.

4.2 K-groups and periodic cyclic homology

Lemma 4.1 *The maximal compact subgroup* $K = \{\begin{bmatrix} e^{i\varphi} & 0 \\ 0 & 1 \end{bmatrix}\}$ *of* Aff$(\mathbb{C})$ *is isomorphic to* $\mathbb{S}^1$.

Proof is easy and is omitted. □

From this one deduces the following results.

Proposition 4.2 *Let* Ω *b the coadjoint orbit of* Aff$(\mathbb{C})$*, which is the punctured complex plane. Then,*

$$K_0(C_c^\infty(\Omega)) \cong \mathbb{Z} \text{ and } K_1(C_c^\infty(\Omega)) \cong \{0\}.$$

$$\mathrm{PHC}_0(C_c^\infty(\Omega)) \cong \mathbb{Z} \text{ and } \mathrm{PHC}_1(C_c^\infty(\Omega)) \cong \{0\}.$$

Proof. Because of Theorem 1.13 and Lemma 4.1, $K_*(C_c^\infty(\Omega))$ are the same as $K_*(C^\infty(\mathbb{S}^1)) \cong K^*(\mathbb{S}^1)$, and $\mathrm{PHC}_*(C_c^\infty(\Omega))$ are isomorphic to $\mathrm{PHC}_*(C_c^\infty(\mathbb{S}^1))$. The assertions become clear. □

4.3 Chern-Connes characters

Proposition 4.3 *Let* Ω *be the coadjoint orbit of* Aff$(\mathbb{C})$*, which is the punctured complex plane. Then, the Chern-Connes character*

$$ch : K_*(C_c^\infty(\Omega)) \to \mathrm{PHC}_*(C_c^\infty(\Omega))$$

is an isomorphism.

Proof. From Theorem 2.1, we have the commutative diagram:

$$\begin{array}{ccc}
K_*(C_c^\infty(\Omega)) & \xrightarrow{ch_\Omega} & \mathrm{PHC}_*(C_c^\infty(\Omega)) \\
I\downarrow & & \downarrow III \\
K_{*+q}(C_c^\infty(\Omega_K)) & \xrightarrow{ch_{\Omega_K}} & \mathrm{PHC}_{*+q}(C_c^\infty(\Omega_K)) \\
II\downarrow & & \downarrow IV \\
K_{*+q}(C_c^\infty(\mathbb{T}))^W & \xrightarrow{ch_{\mathbb{T}}} & \mathrm{PHC}_{*+q}(C_c^\infty(\mathbb{T}))_{\mathrm{m}}^W
\end{array}$$

It was shown [DKT1]-[DKT2] that the Chern-Connes characters are reduced to the classical Chern characters of commutative tori. For the tori, $ch_{\mathbb{T}}$ is an isomorphism modulo torsions, and therefore ch_{Ω_K} and ch_{Ω} are also isomorphisms modulo torsions. In our case of the quantum punctured complex plane, the groups are either 0 or $\mathbb{Z}$. Hence, Chern-Connes character

$$ch_{\Omega} : K_*(C_c^{\infty}(\Omega)) \to \mathrm{PHC}_*(C_c^{\infty}(\Omega))$$

is an isomorphism. □

Acknowledgments

This work was completed during the time the first author visited Abdus Salam ICTP. The first author expresses his sincere thanks for the hospitality and for the excellent conditions provided.

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