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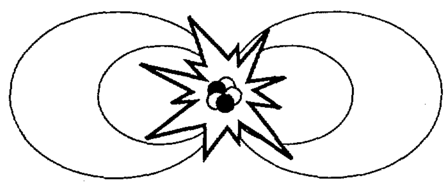
J. Scheffel



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FUSION PLASMA PHYSICS
ALFVÉN LABORATORY
ROYAL INSTITUTE OF TECHNOLOGY
SE-100 44 STOCKHOLM SWEDEN

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On Analytical Solution of the Navier-Stokes Equations

J. Scheffel

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The Alfvén Laboratory
Division of Fusion Plasma Physics
Royal Institute of Technology
SE-100 44 Stockholm, Sweden
(Association EURATOM/NFR)

ON ANALYTICAL SOLUTION OF THE NAVIER-STOKES EQUATIONS

J. Scheffel

Fusion Plasma Physics, Alfvén Laboratory (Association EURATOM-NFR)
Royal Institute of Technology, SE-100 44 Stockholm, Sweden
http://www.fusion.kth.se

ABSTRACT

An analythical method for solving the dissipative, nonlinear and non-stationary Navier-Stokes equations is presented. Velocity and pressure is expanded in power series of cartesian coordinates and time. The method is applied to 2-D incompressible gravitational flow in a bounded, rectangular domain.

Keywords: Navier-Stokes equations, incompressible flow, viscous flow.

1. A LONG STANDING PROBLEM

The Navier-Stokes (NS) equations have since long been identified to model such diverse phenomena as weather changes, ocean flow, airplane motion and thermonuclear plasmas. They are central in the study of naturally occuring turbulence and have thus been intensively studied.

So far only numerical solutions have been attainable [1]. Intense theoretical study of the equations has, however, been carried out since their early formulation by Louis Navier and George Gabriel Stokes. The following theoretical results are presently established [2].

In two dimensions (plane-parallel flow) there exists a unique solution to Eqs.(1) at all instants of time independently of the size of the gravitation g, the initial condition or the domain S. The proof holds for generalized (or weak) solutions, but it has also been shown that the generalized solution becomes a classical (smooth) solution if g (or more generally, the external force) is Hölder continuous inside S.

Since this is the case here, we can assume the solution to be real and analytic. It is, in the present method, expanded as a power series in x, y and t within the domains $x \in [0,1]$, $y \in [0,1]$ and $t \in [0,\infty[$. Since the solutions to Eqs. (1) are unique, convergence can be determined from the requirement that the error should remain less than some number ε when further expansion terms are added.

From a physical point of view, smooth solutions are both desired and expected. The Navier-Stokes equations are indeed routinely solved numerically in applications of hydrodynamics and magnetohydrodynamics (MHD), since they appear in the one-fluid equations obtained by taking velocity moments of the more fundamental Boltzmann equation.

Dissipation in the form of viscosity is central in the study of the NS equations and is retained (in scalar form) throughout this study. We will assume that the fluid is incompressible.

The success of the present expansion method for a bounded domain depends on three factors. First, the established existence and uniqueness of solutions to the initial-value problem with the boundary condition $\mathbf{v} = \mathbf{0}$ (no slip or normal flow at the boundary). This removes any problem associated with bifurcation points in the solution algorithm. Second, it was essential to determine solutions of div$\mathbf{v}$ = 0 that satisfy the boundary conditions for all orders of the expansion. Third, modern symbolic computer math programs (like MAPLE), running on fast, high-memory computers are required; the analytic method suggested here certainly is intractable for "hand" calculations.

2. THE 2-D, VISCOUS INCOMPRESSIBLE NAVIER-STOKES EQUATIONS

We wish to solve the equations

$$\begin{aligned} &\rho(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}) = -\nabla p + \nu\Delta\mathbf{v} - g\mathbf{e}_y \\ &\nabla\cdot\mathbf{v} = 0 \end{aligned} \qquad (1)$$

in the region V of Fig. (1) with boundary S, with ν denoting viscosity, g gravitational acceleration, Δ the Laplace operator and $\mathbf{e}_y$ a unit vector in the positive y-direction. The boundary condition is that $\mathbf{v} = \mathbf{0}$ on S (no normal flow, no slip at boundary due to viscosity). The initial condition $\mathbf{v} = \mathbf{v}_0$ must satisfy $\mathbf{v}_0 = \mathbf{0}$ on S and $\nabla \cdot \mathbf{v}_0 = 0$ in V. Possible forms for $\mathbf{v}_0$ are discussed in Section 4.

The gravitational term is of potential form, enabling the substitution $p = h\text{-}gy$. After normalization we obtain the following scalar equations, implicitly containing gravitation;

$$\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = -\frac{\partial h}{\partial x} + \nu(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2})$$
$$\frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} = -\frac{\partial h}{\partial y} + \nu(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2}) \quad (2)$$
$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$$

3. EXPANSION METHOD

The velocity $\mathbf{v} = (v_x, v_y)$ and potential h are expanded as power series in cartesian coordinates x, y and time t:

$$v_x = \sum_{k,m,n} a_{kmn} x^k y^m t^n$$
$$v_y = \sum_{k,m,n} b_{kmn} x^k y^m t^n \quad (3)$$
$$h = \sum_{k,m,n} d_{kmn} x^k y^m t^n$$

The desired order of expansion must be given. The expansion order (K,M,N) is here defined as the order in x, y, and t up to which Eqs. (2) hold identically. From Eqs. (2) it is apparent that low order coefficients couple to higher order coefficients because of space and time derivatives. Thus the expansions (3) must be carried out to higher order than (K,M,N), this order being given by the degree of the derivatives operating on v_x, v_y and h, respectively.

When a solution is obtained, there will thus remain free higher order coefficients in (3). This makes the solution multi-valued, although exact to the given order. If the method converges, the remaing coefficients will have diminishing influence at higher orders.

4. NO SIMPLE SOLUTIONS

We will here show that there are no simple solutions to Eq.(1). This will become clear as we discuss the possible forms for the initial velocity field.

Let us, for clarity, introduce the time series

$$
\begin{aligned}
v_x &= \sum_i t^i f_i(x,y) \\
v_y &= \sum_i t^i g_i(x,y) \\
h &= \sum_i t^i h_i(x,y)
\end{aligned}
\tag{4}
$$

Using Eq. (2), there results

$$
\begin{aligned}
& f_1 + f_0 \frac{\partial f_0}{\partial x} + g_0 \frac{\partial f_0}{\partial y} = -\frac{\partial h_0}{\partial x} + \nu\left(\frac{\partial^2 f_0}{\partial x^2} + \frac{\partial^2 f_0}{\partial y^2}\right) \\
& g_1 + f_0 \frac{\partial g_0}{\partial x} + g_0 \frac{\partial g_0}{\partial y} = -\frac{\partial h_0}{\partial y} + \nu\left(\frac{\partial^2 g_0}{\partial x^2} + \frac{\partial^2 g_0}{\partial y^2}\right) \\
& \frac{\partial f_0}{\partial x} + \frac{\partial g_0}{\partial y} = 0 \\
& \frac{\partial f_1}{\partial x} + \frac{\partial g_1}{\partial y} = 0
\end{aligned}
\tag{5}
$$

We now wish to determine possible initial conditions. We immediately exclude the cases $f_0 = 0$, $g_0 \neq 0$ and $f_0 \neq 0$, $g_0 = 0$, since they do not satisfy both the zero order incompressibility condition in (5) and the boundary conditions.

A. Zero initial flow

Now assume that there is no initial flow; $\mathbf{v}(t=0) = \mathbf{0}$. Thus $f_0 = g_0 = 0$. We obtain

$$
\begin{aligned}
& \frac{\partial f_1}{\partial y} - \frac{\partial g_1}{\partial x} = 0 \\
& \frac{\partial f_1}{\partial x} + \frac{\partial g_1}{\partial y} = 0
\end{aligned}
\tag{6}
$$

and, using the Laplace operator,

$$
\begin{aligned}
\Delta f_1 &= 0 \\
\Delta g_1 &= 0\,.
\end{aligned}
\tag{7}
$$

Since $\mathbf{v} = \mathbf{0}$ on S, we must have $f_1 = g_1 = 0$ on S. A well known theorem of potential theory then states that Eq.(7) implies that $f_1 \equiv g_1 \equiv 0$ in V. Thus, if the initial flow is zero, the flow must be zero everywhere for all times.

This result may seem surprising, since one can imagine the physical situation that the system is started from a static state with a spatially varying pressure gradient, which thus would drive a flow in the fluid.

B. Finite initial flow

What classes of initial flow satisfy Eq.(5) and the boundary conditions? In order that the boundary conditions should be satisfied, we must require that

$$f_0 \propto x^{k_1}(1-x)^{k_2}\, y^{k_3}(1-y)^{k_4} \tag{8}$$

where k_i, $i = 1..4$, are integers ≥ 1. The ansatz (8) is, however too simple to satisfy the incompressibility condition $\partial f_0/\partial x + \partial g_0/\partial y = 0$. The reason is that

$$g_0 = -\int \frac{\partial f_0}{\partial x} dy, \tag{9}$$

which expression does not satisfy the boundary conditions at $y = 1$. However, on differentiating (8) w.r.t. x, there results

$$\frac{\partial f_0}{\partial x} = -x^{(k_1-1)}(1-x)^{(k_2-1)}\, y^{k_3}(1-y)^{k_4}(k_1(1-x)-k_2 x)\ . \tag{10}$$

This expression can be used for g_0, since then $f_0 = -\int(\partial g_0/\partial y)\mathrm{d}x$ is exactly integrable. Hence, the simplest possible initial conditions that satisfy both the incompressibility and the boundary conditions, are

$$\begin{aligned} f_0 &= x^{K_1}(1-x)^{K_2}\, y^{(K_3-1)}(1-y)^{(K_4-1)}(K_3(1-y)-K_4 y) \\ g_0 &= -x^{(K_1-1)}(1-x)^{(K_2-1)}\, y^{K_3}(1-y)^{K_4}(K_1(1-x)-K_2 x), \end{aligned} \tag{11}$$

where K_i, $i = 1..4$, are integers ≥ 2 in order to satisfy the boundary conditions. Any linear combination of these initial conditions is also a possible initial condition.

C. Higher orders

Using the result (11) in the first two of Eqs. (5), we observe that it is required that $K_i \geq 4$ for the viscous terms to satisfy the boundary conditions. The simplest relevant initial condition for first order (in time) accuracy, omitting multiplying constants, is thus

$$f_{00} = x^4(1-x)^4 y^3(1-y)^3(1-2y)$$
$$g_{00} = -x^3(1-x)^3 y^4(1-y)^4(1-2x)\,. \tag{12}$$

This in turn forces the nonlinear convective terms to include terms of order $(14,15,N)$ and $(15,14,N)$ in Eqs. (5). We thus see, that already f_1 and g_1 must feature a considerable complexity. By similar reasoning, higher orders of f_i and g_i feature even higher degrees of complexity and detail. This sheds some light on the difficulty to obtain closed analytical solutions to the NS equations.

5. ANALYTICAL METHOD

In essence, the analytical method consists in substituting the expansions (3) in Eqs.(2), whereafter the coefficients are determined to desired order.

A problem of the present complexity (three variables, nonlinear, high orders) requires a computer math program. Large resources of primary memory and high performance processors are also essential. In this work MAPLE is used on Mac G3 and Sun Blade 1000 platforms.

First, boundary and initial conditions are imposed. These are $\mathbf{v}(0,y,t) = \mathbf{v}(1,y,t) = \mathbf{v}(x,0,t) = \mathbf{v}(x,1,t) = 0$, and $\mathbf{v}(x,y,0) = \mathbf{v}_0 = (f_0,g_0)$, respectively. A method for successive solution of the associated linear equations for the coefficients in (3) is used.

An algorithm has been written to determine the coefficients of (3) from each power of the NS equations. The nonlinearity does not generate spurious solutions due to the proven uniqueness of the problem and the design of the algorithm. The solution contains viscosity ν and coefficients in $\mathbf{v}_0$ as free parameters.

The combined Eqs.(2) and (3) can also be solved as a system of recurrence relations for the coefficients. Since this system may be nonlinear and implicit, it was found to lead to tedious and operations requiring high memory-consumption.

6. RESULTS

An analytical solution to Eqs. (2), being correct to order (8,8,1), has been determined using the simple initial condition (12) multiplied with an arbitrary magnitude C. The resulting orders of v_x, v_y and h are thus (10,10,2), (10,10,2) and (9,9,1), respectively. We have further imposed the condition $h(1,y,t) = 0$. A solution, where the remaining coefficients have been set to zero, is given in Appendix. It is clearly seen that the boundary conditions are satisfied for all times. Viscosity becomes included up to second order. It should be noted that the present "solution" is only correct upp to order (8,8,1) and is probably not representative for an exact solution of the NS equations with initial condition (12). It serves to show, however, that the present method works algorithmically and it also gives some insight into the structure of solutions to the NS equations.

In Figs. 2-4 the solution with $C = 0.001$ and $\nu = 0.1$ is displayed for various times. It may be noted that the changes in velocity appear on a faster time scale than changes in pressure. This may also be seen by comparing the global potential energy of the system $W = \iint h dxdy$ with its global kinetic energy $Q = 1/2\iint (v_x^2 + v_x^2)dxdy$. We find

$$W = const + 0.001351\nu C - 0.02846C^2 - C\nu(0.8700C + 0.2232\nu)t \qquad (13)$$

$$\begin{aligned} Q = C^2[&2.537\cdot 10^{-11} + (2.549\cdot 10^{-7}C - 1.185\cdot 10^{-6}\nu)t \\ &+ (0.1350\nu^2 + 0.03608C\nu + 0.1823C^2)t^2 + O(t^3)] \end{aligned} \qquad (14)$$

The potential energy is continuously decreasing in time, as expected. The kinetic energy may be linearly decreasing in time but nonlinearly increasing in time. For $C = 0.001$ and $\nu = 0.1$, Q increases for all times except very early.

One should note that energy is not conserved in the system because of the dissipation brought about by viscosity. The sum of W and Q is thus not a constant.

7. DISCUSSION

A method for obtaining analytical solutions to the viscous, nonlinear and non-stationary Navier-Stokes equations has been demonstrated. Using power expansions in x, y, and t, the two-dimensional motion of a fluid under the influence of gravity is computed analytically, satisfying the Navier-Stokes equations up to eighth order in space and first order in time.

The algorithm is fairly efficient, but need more development to avoid computer memory consumption much above 100 Mb. It may then be used for solving the three-dimensional Navier-Stokes equations, for which less is known analytically.

8. REFERENCES

The literature in this field is enormous. Every week 20-30 articles, related to the NS equations, are published. These two references summarises numerical and theoretical aspects of the problem:

[1] C. A. J. Fletcher, **Computational Techniques for Fluid Dynamics**, Vols 1 &2, Springer, 2000.

[2] O. A. Ladyzhenskaya, **The Mathematical Theory of Viscous Incompressible Flow**, Gordon and Breach, 1969.

9. APPENDIX

$$
\begin{aligned}
v_x = \frac{1}{11760}\, x\, y\,(-1+y)\,(x-1)\,C\,(& -47040\, x^3 y^3 + 211680\, v\, x^5 t + 70560\, x^3 t\, v \\
& + 1270080\, x\, t^2 v^2 - 70560\, x^6 t\, v - 6511680\, x^6 t^2 v^2 + 30764160\, y^6 t^2 v^2 - 24978240\, x^4 t^2 v^2 \\
& + 30764160\, y^5 t^2 v^2 - 11760\, x^6 y^2 - 176400\, x^4 y^4 - 4480\, x^8 t\, v + 40320\, x^7 t^2 v^2 \\
& - 7922880\, y^7 t^2 v^2 + 47040\, x^6 y^3 - 7197120\, x^2 t^2 v^2 - 2634240\, x^3 y^5 t\, v + 1599360\, x^2 y^5 t\, v \\
& + 53760\, x^2 y^6 t\, v + 32034240\, x\, y^5 t^2 v^2 + 388080\, x^8 y^5 t\, C + 635040\, x^6 y\, t\, v \\
& - 3405696\, x^4 y^4 t\, v - 14978880\, x^6 y^2 t^2 v^2 - 235536\, x^7 y^5 t\, v - 868864\, x^6 y^8 t\, v \\
& + 1905120\, v\, x^4 y\, t + 1962128\, x^8 y^5 t\, v - 35280\, x^4 y^2 + 625737\, x^8 y^8 t\, C - 2040384\, x^6 y^7 t\, v \\
& - 236928\, x^6 y^6 t\, v + 298224612\, x^8 y^7 t^2 v\, C + 6437235\, x^7 y^8 t\, C - 211680\, v\, x^4 t \\
& - 156408\, x^5 y^8 t\, C + 298224612\, x^8 y^8 t^2 v\, C + 167233248\, x^7 y^4 t^2 v^2 + 23567040\, x^2 y^6 t^2 v^2 \\
& - 32901120\, x^4 y^7 t^2 v^2 + 40320\, x^8 t^2 v^2 + 48507498\, x^8 y^7 t^2 v^2 + 1270080\, x\, y\, t^2 v^2 \\
& - 196224\, x^4 y^6 t\, v - 67200\, x^3 y^6 t\, v + 4445280\, x^5 y^2 t\, v + 2798880\, t\, x^3 y^2 v \\
& - 33445440\, x^4 y^2 t^2 v^2 - 1607920\, x^8 y^7 t\, v + 1715168\, x^5 y^8 t\, v - 29211840\, x^5 y^4 t^2 v^2 \\
& + 48507498\, x^8 y^8 t^2 v^2 - 14434560\, x^6 y^7 t^2 v^2 + 61824\, x^5 y^6 t\, v + 35138880\, x\, y^3 t^2 v^2 \\
& - 498960\, x^6 y^7 t\, C + 89376\, x^7 y^5 t\, C - 498960\, x^6 y^8 t\, C - 15120000\, x^2 y^8 t^2 v^2 \\
& - 564480\, x^8 y\, t^2 v^2 + 1171520\, x\, y^8 t\, v + 120344\, x^8 y^4 t\, C - 15680\, x^8 y^3 t\, C - 3684303\, x^8 y^7 t\, C \\
& - 3147564\, x^8 y^6 t\, C - 6511680\, x^6 y\, t^2 v^2 + 1219008\, x^6 y^5 t\, v + 1938048\, x^4 y^5 t\, v \\
& - 26880\, x^7 y^4 t\, v - 6720\, x^7 y^2 t\, v + 58800\, x^7 y^3 t\, v - 208320\, x^2 y^7 t\, v - 806064\, x^4 y^7 t\, v \\
& + 260400\, x^3 y^7 t\, v + 543648\, x^5 y^7 t\, v + 336000\, x^6 y^4 t\, v - 6348416\, x^8 y^8 t\, v \\
& + 5785920\, x^4 y^6 t^2 v^2 + 29967840\, x^7 y^2 t^2 v^2 - 6652800\, x\, y^7 t^2 v^2 + 19051200\, x^5 y\, t^2 v^2 \\
& - 54774720\, x^6 y^4 t^2 v^2 + 356028\, x^7 y^7 t\, v + 52920000\, x^5 y^3 t^2 v^2 + 52073280\, x^3 y^3 t^2 v^2 \\
& - 132160\, x^8 y^2 t\, v + 393344\, x^8 y^4 t\, v + 1431920\, x^3 y^8 t\, v + 48968640\, x^3 y^6 t^2 v^2 \\
& + 32034240\, x\, y^6 t^2 v^2 - 587440\, x^8 y^3 t\, v - 125340768\, x^7 y^5 t^2 v^2 - 564480\, x^7 y\, t^2 v^2 \\
& + 167233248\, x^8 y^4 t^2 v^2 - 14434560\, x^6 y^8 t^2 v^2 + 18204480\, x^3 y\, t^2 v^2 - 24978240\, x^4 y\, t^2 v^2 \\
& + 23567040\, x^2 y^5 t^2 v^2 + 33868800\, y^3 t^2 v^2 - 46992960\, x\, y^4 t^2 v^2 - 73241280\, x^4 y^4 t^2 v^2 \\
& - 114095520\, x^8 y^3 t^2 v^2 + 705600\, v\, x\, y^4 t + 6585600\, x^3 y^4 t\, v - 6256320\, x^3 y^3 t\, v \\
& - 564480\, v\, x\, y^3 t + 25872\, x^6 y^5 t\, C + 2127195\, x^7 y^7 t\, C + 11128320\, x^5 y^7 t^2 v^2 \\
& - 156408\, x^5 y^7 t\, C + 26671680\, x^2 y^3 t^2 v^2 + 19051200\, x^5 t^2 v^2 - 3998400\, x^2 y^4 t\, v \\
& - 7197120\, x\, y^2 t^2 v^2 + 57696\, x^7 y^6 t\, v + 1737540\, x^7 y^3 t^2 v\, C + 1737540\, x^8 y^3 t^2 v\, C \\
& + 92117844\, x^8 y^5 t^2 v\, C + 298224612\, x^7 y^7 t^2 v\, C + 3198720\, t\, x^2 y^3 v + 141120\, v\, x\, y^2 t \\
& - 282240\, x\, y^5 t\, v - 7922880\, y^8 t^2 v^2 + 48968640\, x^3 y^5 t^2 v^2 + 9737280\, x^3 y^2 t^2 v^2 \\
& - 114095520\, x^7 y^3 t^2 v^2 - 15120000\, x^2 y^7 t^2 v^2 + 48507498\, x^7 y^8 t^2 v^2 + 29967840\, x^8 y^2 t^2 v^2 \\
& - 18878328\, x^8 y^4 t^2 v\, C + 298224612\, x^7 y^8 t^2 v\, C + 92117844\, x^8 y^6 t^2 v\, C \\
& + 92117844\, x^7 y^6 t^2 v\, C - 125340768\, x^7 y^6 t^2 v^2 + 18204480\, x^3 t^2 v^2 + 10584000\, x^5 y^2 t^2 v^2 \\
& + 517440\, x^6 y^3 t\, v + 1171520\, y^8 t\, v + 27357120\, x^6 y^3 t^2 v^2 - 125340768\, x^8 y^6 t^2 v^2 \\
& + 11128320\, x^5 y^8 t^2 v^2 + 49815360\, x^5 y^6 t^2 v^2 + 92117844\, x^7 y^5 t^2 v\, C - 1428000\, x^6 y^2 t\, v \\
& + 11760\, x^3 y^2 - 634116\, x^7 y^6 t\, C - 113232\, x^6 y^6 t\, C - 8467200\, y^2 t^2 v^2 - 310464\, x^5 y^5 t\, v \\
& - 32901120\, x^4 y^8 t^2 v^2 + 10281600\, x^3 y^8 t^2 v^2 - 1905120\, x^5 y\, t\, v + 56448\, x^5 y^4 t\, v \\
& - 7197120\, x^2 y\, t^2 v^2 + 8890560\, x^4 y^3 t^2 v^2 + 49815360\, x^5 y^5 t^2 v^2 - 125340768\, x^8 y^5 t^2 v^2 \\
& - 48263040\, y^4 t^2 v^2 - 328672\, x^8 y^6 t\, v - 30058560\, x^3 y^4 t^2 v^2 - 4384468\, x^7 y^8 t\, v \\
& - 15664320\, x^2 y^2 t^2 v^2 + 5785920\, x^4 y^5 t^2 v^2 - 6652800\, x\, y^8 t^2 v^2 + 82880\, x^8 y\, t\, v
\end{aligned}
$$

$$
\begin{aligned}
&- 799680\, t\, x^2 y^2 \nu + 963200\, x^2 y^8 t \nu + 365456\, x^4 y^8 t \nu + 24252480\, x^6 y^6 t^2 \nu^2 \\
&+ 48507498\, x^7 y^7 t^2 \nu^2 + 10281600\, x^3 y^7 t^2 \nu^2 - 10584\, x^4 y^7 t C - 10584\, x^4 y^8 t C \\
&- 55460160\, x^2 y^4 t^2 \nu^2 - 635040\, x^3 y t \nu + 5597760\, x^4 y^3 t \nu - 18878328\, x^7 y^4 t^2 \nu C \\
&- 2963520\, x^5 y^3 t \nu - 5103840\, x^4 y^2 t \nu + 35280\, x^5 y^2 + 141120\, x^4 y^3 + 58800\, x^3 y^4 \\
&+ 23520\, x^6 y^5 - 23520\, x^3 y^5 - 70560\, x^5 y^5 - 58800\, x^6 y^4 + 176400\, x^5 y^4 + 70560\, x^4 y^5 \\
&- 141120\, x^5 y^3 + 24252480\, x^6 y^5 t^2 \nu^2)
\end{aligned}
$$

$$
\begin{aligned}
\nu_y = \frac{1}{11760}\, & x\, y\, (-1 + y)\, (x - 1)\, C (7197120\, y^2 t^2 \nu^2 + 211680\, y^4 t \nu + 70560\, y^6 t \nu \\
&+ 8467200\, x^2 t^2 \nu^2 + 181440\, y^8 t^2 \nu^2 - 211680\, y^5 t \nu - 1270080\, y t^2 \nu^2 - 18204480\, y^3 t^2 \nu^2 \\
&- 1176000\, x^8 t \nu + 3366195\, x^7 y^7 t C + 18404064\, x^5 y^7 t^2 \nu^2 - 70560\, y^3 t \nu + 24978240\, y^4 t^2 \nu^2 \\
&- 30764160\, x^6 t^2 \nu^2 + 181440\, y^7 t^2 \nu^2 - 4445280\, x^2 y^5 t \nu - 25595136\, x^4 y^7 t^2 \nu^2 \\
&- 134769600\, x^4 y^3 t^2 \nu^2 + 67200\, x y^8 t \nu - 383705280\, x^2 y^3 t^2 \nu^2 - 14055552\, x^5 y^5 t^2 \nu^2 \\
&+ 7599543\, x^8 y^7 t C + 47370960\, x^8 y^4 t^2 \nu^2 + 167308596\, x^7 y^7 t^2 \nu C - 1270080\, x y t^2 \nu^2 \\
&+ 11340000\, x^3 y^8 t^2 \nu^2 + 26880\, x^6 y^3 t \nu - 420672\, x^4 y^8 t \nu - 2782080\, x^2 y^7 t^2 \nu^2 \\
&- 25492320\, x^7 y t^2 \nu^2 - 75264\, x^4 y^5 t \nu - 310464\, x^5 y^5 t \nu + 910968\, x^8 y^7 t \nu \\
&- 14597226\, x^7 y^8 t^2 \nu^2 - 134769600\, x^6 y^3 t^2 \nu^2 - 25492320\, x^8 y t^2 \nu^2 + 117321876\, x^6 y^7 t^2 \nu C \\
&- 2798880\, t\, x^2 y^3 \nu + 46569600\, x^5 y t^2 \nu^2 - 38976\, x^4 y^7 t \nu - 2247264\, x^7 y^6 t \nu \\
&+ 21611520\, x^3 y^6 t^2 \nu^2 + 46569600\, x^6 y t^2 \nu^2 + 12700800\, x^5 y^2 t^2 \nu^2 - 677376\, x^5 y^4 t \nu \\
&- 503580\, x^7 y^6 t C + 75600\, x^5 y^7 t \nu - 98448\, x^6 y^6 t C + 1481760\, x^2 y^6 t \nu - 1417920\, x^3 y^6 t \nu \\
&+ 253680\, x^7 y^3 t \nu - 25872\, C y^6 t x^5 + 282240\, x^5 y t \nu + 28224\, t C x^7 y^4 - 20160\, x y^7 t \nu \\
&- 635040\, x y^6 t \nu + 981456\, x^7 y^4 t \nu - 11760\, x^2 y^3 + 388080\, x^7 y^2 t^2 \nu C + 70560\, x^5 y^5 \\
&+ 117321876\, x^5 y^8 t^2 \nu C - 70560\, x^5 y^4 + 117321876\, x^6 y^8 t^2 \nu C - 176400\, x^4 y^5 \\
&+ 5880\, x^3 y^8 t C + 388080\, x^8 y^2 t^2 \nu C + 23520\, x^5 y^3 + 3292800\, x^4 y^4 t \nu + 11760\, x^2 y^6 \\
&+ 6560316\, x^4 y^8 t^2 \nu C - 47040\, x^3 y^6 + 58800\, x^4 y^6 - 23520\, x^5 y^6 - 182784\, x^6 y^4 t \nu \\
&- 141120\, x^2 y t \nu + 20160\, x^7 y t \nu + 6039936\, x^4 y^6 t^2 \nu C + 6560316\, x^4 y^7 t^2 \nu C + 176400\, x^4 y^4 \\
&+ 1640520\, x^3 y^7 t^2 \nu C - 58800\, x^4 y^3 + 1640520\, x^3 y^8 t^2 \nu C + 47040\, x^3 y^3 \\
&+ 7472304\, x^5 y^6 t^2 \nu C - 141120\, x^3 y^4 + 4148333\, x^7 y^8 t C - 19051200\, y^5 t^2 \nu^2 \\
&+ 98189280\, x^8 y^2 t^2 \nu^2 - 147364560\, x^7 y^3 t^2 \nu^2 + 89959968\, x^8 y^6 t^2 \nu^2 + 26931744\, x^7 y^5 t^2 \nu^2 \\
&+ 1646400\, x^8 y t \nu + 6350400\, x y^6 t^2 \nu^2 + 18404064\, x^6 y^8 t^2 \nu^2 - 14597226\, x^7 y^7 t^2 \nu^2 \\
&+ 220147200\, x^4 y^2 t^2 \nu^2 + 24978240\, x y^4 t^2 \nu^2 + 1471344\, x^5 y^8 t \nu - 19051200\, x y^5 t^2 \nu^2 \\
&+ 12700800\, x^6 y^2 t^2 \nu^2 + 48263040\, x^4 t^2 \nu^2 + 18404064\, x^6 y^7 t^2 \nu^2 - 14597226\, x^8 y^8 t^2 \nu^2 \\
&+ 107956800\, x^5 y^4 t^2 \nu^2 - 5597760\, x^3 y^4 t \nu + 635040\, \nu x y^3 t - 1905120\, \nu x y^4 t \\
&- 1646400\, x^8 y^2 t \nu + 1360464\, x^8 y^5 t \nu + 1300560\, x^8 y^6 t \nu - 1089564\, x^8 y^5 t C \\
&+ 31187520\, x^3 y^3 t^2 \nu^2 + 26931744\, x^8 y^5 t^2 \nu^2 + 95844\, x^5 y^7 t C - 2782080\, x^2 y^8 t^2 \nu^2 \\
&- 1908256\, x^6 y^8 t C + 5103840\, x^2 y^4 t \nu + 103096\, x^4 y^8 t C + 377300\, x^5 y^8 t C \\
&+ 607656\, x^6 y^7 t C - 18204480\, x y^3 t^2 \nu^2 - 1141056\, x^5 y^6 t \nu - 175812\, x^7 y^5 t C \\
&- 705600\, \nu x^4 y t + 344064\, x^4 y^6 t \nu + 3057600\, x^3 y^5 t \nu + 6256320\, x^3 y^3 t \nu - 241920\, x^7 y^2 t \nu \\
&+ 40320\, x^2 y^7 t \nu + 78960\, x^3 y^7 t \nu + 3844764\, x^8 y^6 t C - 33868800\, x^3 t^2 \nu^2 + 56448\, x^6 y^5 t \nu \\
&+ 6350400\, y^6 t^2 \nu^2 + 89959968\, x^7 y^6 t^2 \nu^2 + 6552000\, x^7 t^2 \nu^2 - 379008\, x^7 y^5 t \nu - 35280\, x^2 y^5
\end{aligned}
$$

$$- 160876800\, x^4\, y\, t^2\, \nu^2 - 25595136\, x^4\, y^8\, t^2\, \nu^2 - 14055552\, x^6\, y^5\, t^2\, \nu^2 + 35280\, x^2\, y^4$$
$$+ 181440\, x\, y^7\, t^2\, \nu^2 + 98189280\, x^7\, y^2\, t^2\, \nu^2 + 7197120\, x\, y^2\, t^2\, \nu^2 - 134769600\, x^5\, y^3\, t^2\, \nu^2$$
$$+ 127260\, x^7\, y^7\, t\, \nu + 107956800\, x^6\, y^4\, t^2\, \nu^2 - 59040\, x^6\, y^7\, t\, \nu + 4181184\, x^4\, y^6\, t^2\, \nu^2$$
$$+ 47370960\, x^7\, y^4\, t^2\, \nu^2 - 531552\, x^6\, y^8\, t\, \nu + 18404064\, x^5\, y^8\, t^2\, \nu^2 - 147364560\, x^8\, y^3\, t^2\, \nu^2$$
$$+ 107956800\, x^4\, y^4\, t^2\, \nu^2 - 46992960\, x^2\, y^5\, t^2\, \nu^2 - 1599360\, x^5\, y^2\, t\, \nu + 564480\, x^3\, y\, t\, \nu$$
$$+ 205851744\, x^8\, y^5\, t^2\, \nu\, C + 167308596\, x^8\, y^8\, t^2\, \nu\, C - 4260060\, x^8\, y^3\, t^2\, \nu\, C$$
$$+ 167308596\, x^8\, y^7\, t^2\, \nu\, C - 5794740\, x^7\, y^4\, t^2\, \nu\, C - 3198720\, t\, x^3\, y^2\, \nu - 6585600\, x^4\, y^3\, t\, \nu$$
$$+ 211891680\, x^7\, y^6\, t^2\, \nu\, C - 5794740\, x^8\, y^4\, t^2\, \nu\, C + 167308596\, x^7\, y^8\, t^2\, \nu\, C$$
$$- 4260060\, x^7\, y^3\, t^2\, \nu\, C + 211891680\, x^8\, y^6\, t^2\, \nu\, C - 14597226\, x^8\, y^7\, t^2\, \nu^2 + 2540160\, x^5\, y^3\, t\, \nu$$
$$+ 3998400\, x^4\, y^2\, t\, \nu + 205851744\, x^7\, y^5\, t^2\, \nu\, C + 24978240\, x^3\, y^4\, t^2\, \nu^2 + 232424640\, x^2\, y^4\, t^2\, \nu^2$$
$$+ 48972672\, x^5\, y^6\, t^2\, \nu^2 - 33868800\, x^3\, y^5\, t^2\, \nu^2 - 58847040\, x^4\, y^5\, t^2\, \nu^2 + 48972672\, x^6\, y^6\, t^2\, \nu^2$$
$$+ 799680\, t\, x^2\, y^2\, \nu + 1905120\, x\, y^5\, t\, \nu - 147329280\, x^3\, y^2\, t^2\, \nu^2 + 8487360\, x^2\, y^6\, t^2\, \nu^2$$
$$+ 11340000\, x^3\, y^7\, t^2\, \nu^2 - 69007680\, x^2\, y\, t^2\, \nu^2 + 473472\, x^6\, y^6\, t\, \nu + 123621120\, x^3\, y\, t^2\, \nu^2$$
$$+ 252745920\, x^2\, y^2\, t^2\, \nu^2 + 425040\, x^3\, y^8\, t\, \nu + 1232328\, x^7\, y^8\, t\, \nu + 2016036\, x^8\, y^8\, t\, \nu$$
$$- 30764160\, x^5\, t^2\, \nu^2 - 134400\, x^2\, y^8\, t\, \nu + 181440\, x\, y^8\, t^2\, \nu^2 + 6552000\, x^8\, t^2\, \nu^2$$
$$+ 1432368\, x^6\, y^5\, t^2\, \nu\, C + 117321876\, x^5\, y^7\, t^2\, \nu\, C + 1432368\, x^5\, y^5\, t^2\, \nu\, C + 7472304\, x^6\, y^6\, t^2\, \nu\, C$$
$$+ 141120\, x^3\, y^5 + 8381681\, x^8\, y^8\, t\, C + 23520\, x^8\, y^4\, t\, C)$$

$$P = -\frac{411}{560} x^9\, y^8\, C^2 + Cd - \frac{347}{140} y^7\, x^9\, C^2 - \frac{2}{5} C^2\, x^8\, y^6 - g\, y + \frac{233}{16} C^2\, x^9\, y^9 - \frac{3}{20} y^9\, x^6\, C^2$$
$$- \frac{1893}{140} y^9\, x^7\, C^2 + \frac{2}{5} y^6\, x^9\, C^2 + \frac{62}{35} C^2\, x^7\, y^8 + \frac{3}{20} C^2\, x^6\, y^8 - \frac{19}{16} C^2\, x^8\, y^8 + \frac{347}{140} C^2\, x^8\, y^7$$
$$- \frac{499}{560} y^9\, x^8\, C^2 + \Big(\frac{72}{35} C\, x^5\, y^7 + \frac{4}{21} y^9\, x^2\, C + \frac{59}{7} y^9\, x^5\, C + \frac{59}{7} y^5\, x^9\, C - \frac{279}{35} x^8\, y^5\, C - \frac{529}{35} y^9\, x^6\, C$$
$$- \frac{4}{7} y^9\, x^3\, C - \frac{72}{35} x^7\, y^5\, C + \frac{65959}{2940} C\, x^9\, y^9 + \frac{468}{35} x^7\, y^6\, C - \frac{39}{7} y^4\, x^9\, C - \frac{708}{49} x^7\, y^7\, C + \frac{39}{7} x^8\, y^4\, C$$
$$- \frac{26917}{980} y^9\, x^8\, C - \frac{31}{7} y^9\, x^4\, C + \frac{4}{7} y^3\, x^9\, C + \frac{361}{35} y^6\, x^9\, C - \frac{4}{7} x^8\, y^3\, C - \frac{93}{7} x^8\, y^6\, C - \frac{19461}{980} y^8\, x^9\, C$$
$$+ \frac{4761}{245} y^7\, x^8\, C + \frac{4761}{196} y^8\, x^8\, C + \frac{93}{7} x^6\, y^8\, C - \frac{3249}{245} x^7\, y^8\, C - \frac{4049}{245} y^7\, x^9\, C - \frac{351}{35} x^5\, y^8\, C$$
$$+ \frac{4049}{245} y^9\, x^7\, C - \frac{52}{5} x^6\, y^6\, C + \frac{8}{5} x^6\, y^5\, C + \frac{39}{7} x^4\, y^8\, C + \frac{372}{35} C\, x^6\, y^7 - \frac{8}{7} C\, x^4\, y^7 \Big) \nu + \Big(\Big($$
$$- \frac{9}{2} x^8\, y^4\, C^2 + 54\, x^8\, y^5\, C^2 + \frac{9}{2} y^9\, x^4\, C^2 - \frac{372}{5} x^5\, y^8\, C^2 + \frac{372}{5} y^9\, x^5\, C^2 - \frac{22874}{35} x^7\, y^7\, C^2$$
$$+ \frac{3045}{2} y^7\, x^9\, C^2 + \frac{225979}{70} C^2\, x^7\, y^8 - \frac{46876}{7} C^2\, x^8\, y^8 + 4\, C^2\, y^6\, x^6 - \frac{294}{5} x^6\, y^7\, C^2 - 54\, x^9\, y^5\, C^2$$
$$+ \frac{9}{2} x^9\, y^4\, C^2 - 210\, C^2\, x^6\, y^8 + \frac{18786}{5} x^9\, y^8\, C^2 + 230\, y^6\, x^9\, C^2 + \frac{1324}{5} y^9\, x^6\, C^2 - \frac{178551}{70} y^9\, x^7\, C^2$$
$$- \frac{56711}{70} C^2\, x^8\, y^7 + \frac{268353}{35} y^9\, x^8\, C^2 - 24\, x^7\, y^6\, C^2 - \frac{9}{2} y^8\, x^4\, C^2 - \frac{27301}{5} C^2\, x^9\, y^9 - 210\, C^2\, x^8\, y^6 \Big)$$
$$\nu + \Big(480\, x^2\, y^3\, C + 168\, x^6\, y\, C + 4824\, x^2\, y^5\, C - 5040\, x^2\, y^6\, C + \frac{272}{21} y^9\, C + \frac{4}{21} C + 24\, y^5\, C$$

$$+\,5040\,x^6\,y^2\,C+11760\,x^6\,y^4\,C-168\,x\,y^6\,C-10920\,x^4\,y^5\,C-4824\,x^5\,y^2\,C+9072\,x^5\,y^5\,C$$
$$-\,120\,x\,y^4\,C+48\,x\,y^7\,C+216\,x\,y^5\,C+\frac{9792}{7}\,x^3\,y^7\,C-480\,x^3\,y^2\,C-5040\,x^3\,y^6\,C$$
$$-\,2400\,x^2\,y^4\,C+3120\,x^2\,y^7\,C+24\,C\,x\,y^3+7152\,x^3\,y^5\,C+1260\,x^8\,y^2\,C+4320\,x^7\,y^3\,C$$
$$-\,5280\,x^3\,y^4\,C+\frac{1404}{7}\,x^3\,y^8\,C-24\,x^3\,y\,C-3120\,x^7\,y^2\,C-276\,x^9\,y^2\,C-\frac{24}{7}\,x^9\,y\,C$$
$$-\,1260\,x^2\,y^8\,C+2400\,x^4\,y^2\,C+12600\,x^4\,y^4\,C-8040\,x^4\,y^3\,C+\frac{24}{7}\,x^8\,y\,C+5880\,x^4\,y^6\,C$$
$$+\,2160\,x^3\,y^3\,C+120\,x^4\,y\,C-48\,x^7\,y\,C-16800\,x^5\,y^4\,C-3360\,x^7\,y^4\,C-2352\,x^5\,y^6\,C$$
$$-\,11200\,x^6\,y^3\,C-216\,x^5\,y\,C+13632\,x^5\,y^3\,C-80\,y^6\,C-60\,y^8\,C+60\,x^8\,C-\frac{40}{3}\,x^9\,C$$
$$-\,24\,x^5\,C+\frac{8952}{7}\,C\,x^5\,y^7+276\,y^9\,x^2\,C+\frac{1968}{5}\,y^9\,x^5\,C-7008\,y^5\,x^9\,C-\frac{1952}{35}\,y^9\,x^6\,C$$
$$-\,\frac{612}{7}\,y^9\,x^3\,C+1344\,x^7\,y^5\,C-\frac{1618501}{2940}\,C\,x^9\,y^9+3600\,y^4\,x^9\,C+\frac{291528}{245}\,x^7\,y^7\,C$$
$$+\,\frac{224919}{980}\,y^9\,x^8\,C-\frac{288}{7}\,y^9\,x^4\,C-536\,y^3\,x^9\,C+6800\,y^6\,x^9\,C-840\,x^8\,y^3\,C+\frac{8553}{140}\,y^8\,x^9\,C$$
$$-\,\frac{484506}{245}\,y^7\,x^8\,C+\frac{247869}{196}\,y^8\,x^8\,C-\frac{11232}{245}\,x^7\,y^8\,C-\frac{508086}{245}\,y^7\,x^9\,C-\frac{5616}{35}\,x^5\,y^8\,C$$
$$-\,\frac{43416}{245}\,y^9\,x^7\,C-4704\,x^6\,y^5\,C-\frac{38088}{35}\,C\,x^6\,y^7-\frac{13992}{7}\,C\,x^4\,y^7+\frac{720}{7}\,y^7\,C-\frac{720}{7}\,x^7\,C$$
$$+\,80\,x^6\,C\Big)\,v^2\Big)\,t$$

10. FIGURE CAPTIONS

Fig. 1 Fluid domain V with boundary S.

Fig. 2 Pressure, shown as $h(x,y,t)$ with $h(x,0,0) = 0$. Here $C = 1.0 \cdot 10^{-3}$ and $\nu = 0.1$.
a) $t = 0$, b) $t = 0.01$, c) $t = 0.1$, d) $t = 0.5$.

Fig. 3 Velocity $v_x(x,y,t)$ for $C = 1.0 \cdot 10^{-3}$ and $\nu = 0.1$.
a) $t = 0$, b) $t = 0.0001$, c) $t = 0.001$, d) $t = 0.01$.

Fig. 4 Velocity $v_y(x,y,t)$ for $C = 1.0 \cdot 10^{-3}$ and $\nu = 0.1$.
a) $t = 0$, b) $t = 0.0001$, c) $t = 0.001$, d) $t = 0.01$.

Fig. 1

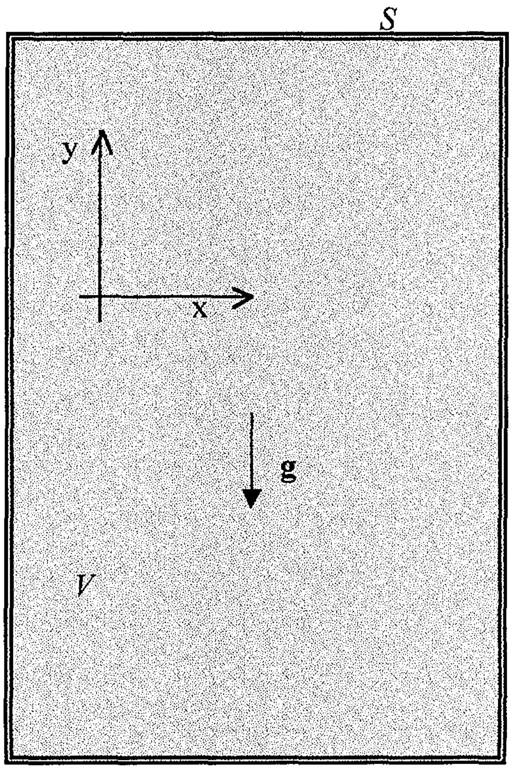
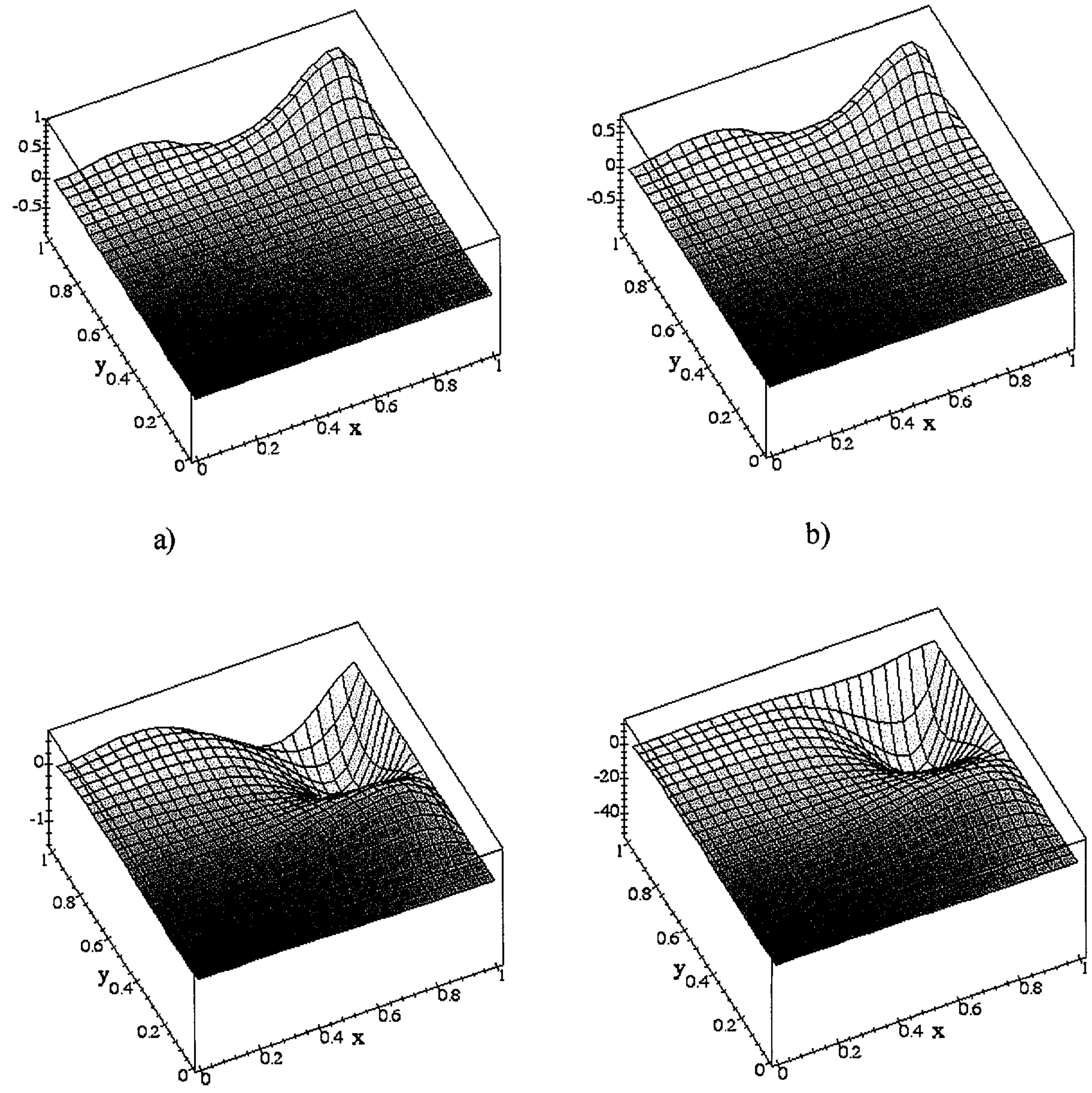


Fig. 2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0.5
0
-0.5
1
0.8
0.6
y 0.4
0.2
0
0
0.2
0.4 x
0.6
0.8
1
0.5
0
-0.5
1
0.8
0.6
y 0.4
0.2
0
0
0.2
0.4 x
0.6
0.8
1
0
-1
1
0.8
0.6
y 0.4
0.2
0
0
0.2
0.4 x
0.6
0.8
1
0
-20
-40
1
0.8
0.6
y 0.4
0.2
0
0
0.2
0.4 x
0.6
0.8
1

a) b) c) d)

Fig. 3

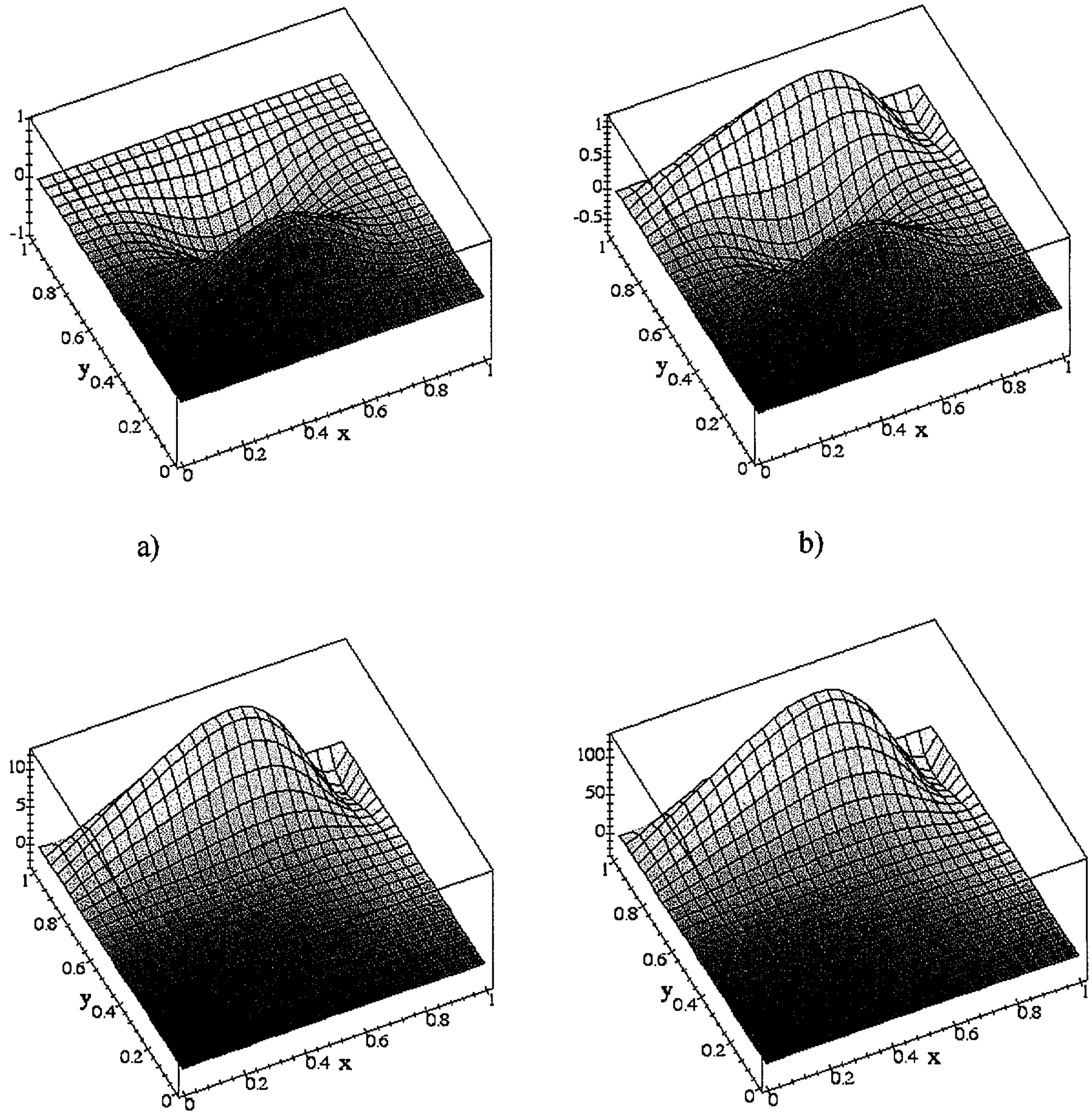
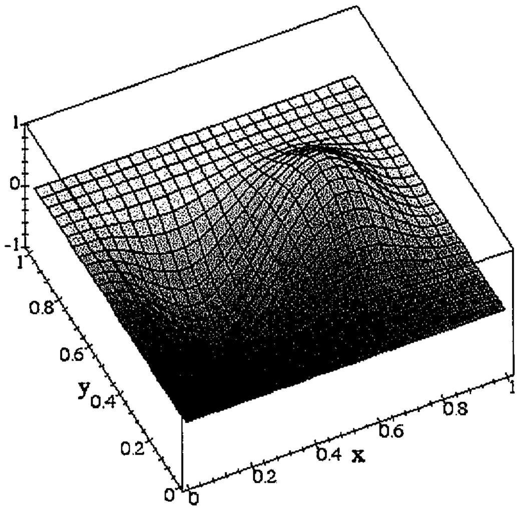
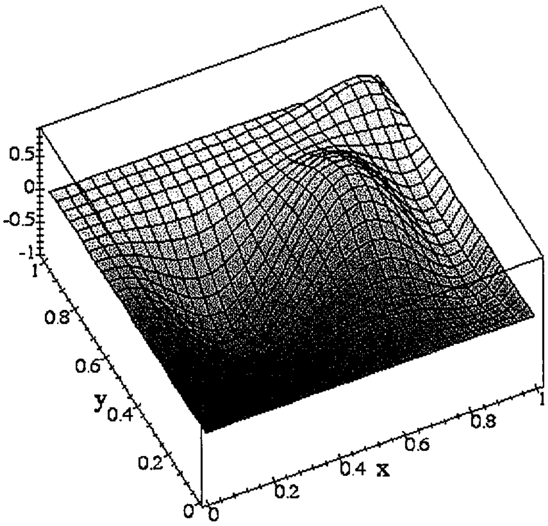


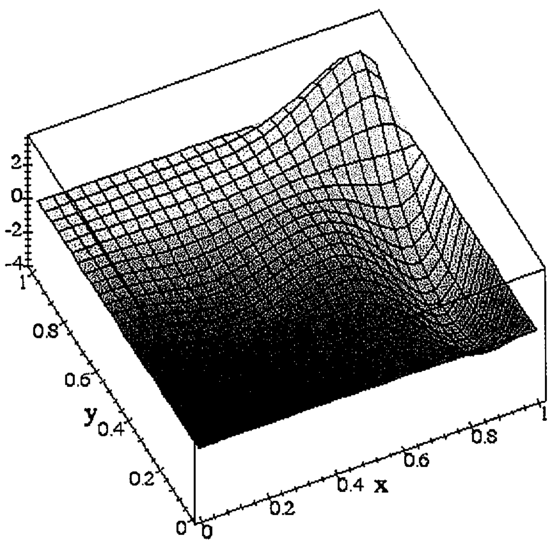
Fig. 4


1
0
-1
1
0.8
0.6
y 0.4
0.2
0
0
0.2
0.4 x
0.6
0.8
1

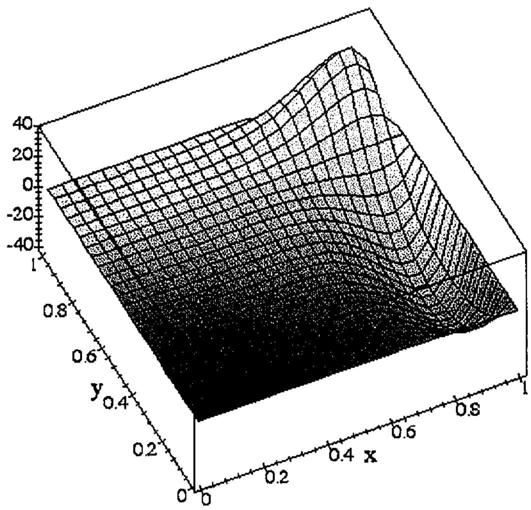
a)


0.5
0
-0.5
-1
1
0.8
0.6
y 0.4
0.2
0
0
0.2
0.4 x
0.6
0.8
1

b)


2
0
-2
-4
1
0.8
0.6
y 0.4
0.2
0
0
0.2
0.4 x
0.6
0.8
1

c)


40
20
0
-20
-40
1
0.8
0.6
y 0.4
0.2
0
0
0.2
0.4 x
0.6
0.8
1

d)

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ON ANALYTICAL SOLUTION OF THE NAVIER-STOKES EQUATIONS

J. Scheffel

ABSTRACT

An analythical method for solving the dissipative, nonlinear and non-stationary Navier-Stokes equations is presented. Velocity and pressure is expanded in power series of cartesian coordinates and time. The method is applied to 2-D incompressible gravitational flow in a bounded, rectangular domain.

Keywords: Navier-Stokes equations, incompressible flow, viscous flow.