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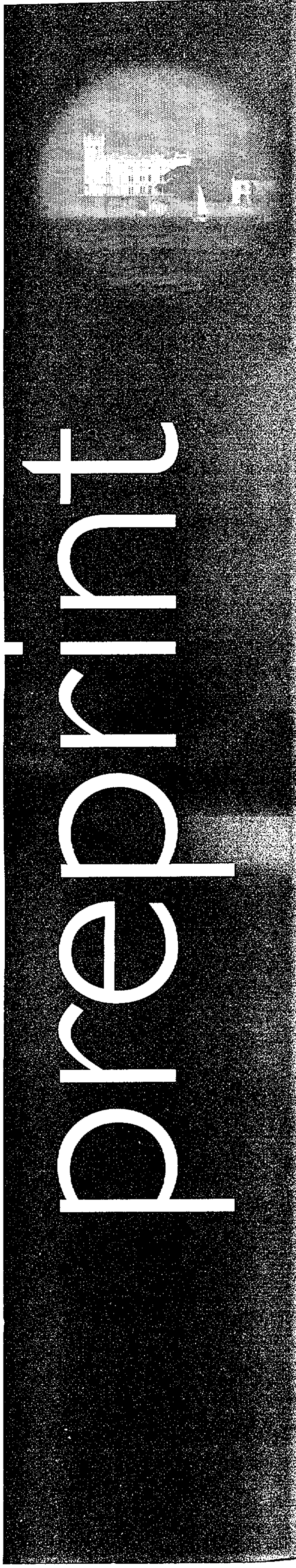

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WEIGHTED SEMICONVEX SPACES OF MEASURABLE FUNCTIONS

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Abstract

Semiconvex spaces are intermediates between locally convex spaces and the non locally convex topological vector spaces. They include all locally convex spaces; hence it is a generalization of locally convex spaces[5]. In this article, we make a study of weighted semiconvex spaces parallel to weighted locally convex spaces where continuous functions are replaced with measurable functions and N_p family replaces Nachbin family on a locally compact space X. Among others, we examine the Hausdorffness, completenes, inductive limits, barrelledness and countably barrelledness of weighted semiconvex spaces. New results are obtained while we have a more elegant proofs of old results. Furthermore, we get extensions of some of the old results. It is observed that the technique of proving theorems in weighted locally convex spaces can be adapted to that of weighted semicovex spaces of measurable functions in most cases.

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1. Preliminaries

Throughout this paper (except otherwise stated), X will denote;

(i) a locally compact Hausdorff space.

(ii) a measure space, with measure μ on a σ-algebra $\aleph$ such that $\aleph$ contains all Borel sets in X.

With this definition, closed sets and open sets are measurable.

An N_p family, $0 < p \leq 1$, V^p on X is defined as a set of non-negative measurable functions $v : X \to [0, \infty]$ satisfying the following condition: if u and $v \in V^p$ and $\lambda \geq 0$, there is $w \in V^p$ such that $\lambda u, \lambda v \leq w$ (pointwise) on X.

Members of V^p are called weights. The set of all continuous p-seminorms(called k-pseudometrics in [5] & [6]) on X is an N_p family on X. It should be noted that a Nachbin family on X is an N_p family.

Note that p is redundant in the definition of the N_p family. Its importance, as we will see later, is that it is relevant in characterizing the semiconvex topology.

$M(X)$ shall denote the space of all complex valued measurable functions on X and $M_b(X)$ shall be the space of all measurable functions that are bounded almost everywhere(a.e) on X. For $f \in M_b(X)$, define $\|f\|_p = \int_X |f|^p d\mu$ and hence $\|\lambda f\|_p = |\lambda|^p \|f\|_p$. We will refer to the topology induced by $\|.\|_p$ on $M_b(X)$ as the p-normed topology .

$M_E(X)$ shall denote the space of all measurable functions on X that are identically zero outside some measurable subset of X. It is easy to see that $C_c(X)$-the space of all complex valued continuous functions on X with compact supports is a subspace of $M_E(X)$.

In fact $M_E(X)$ contains all the characteristic functions of measurable sets $E_n, n \in N$, of X which we will henceforth denote by χ_{E_n}.

We shall assume that $0 < p < 1$ and $\mu(X) < \infty$ throughout this paper.

The relationship between $C_c(X)$ and $M_E(X)$ is set forth in the following proposition.

PROPOSITION 1.1: *$C_c(X)$ is $L^p(\mu)$ dense in $M_E(X)$.*

PROOF: It is clear that $C_c(X) \subset M_E(X)$. Let $f \in M_E(X)$, then there is a measurable subset A of X such that $f(x) = 0$ if $x \in A$. Since $\mu(X) < \infty$, then $\mu(A) < \infty$ and hence by [10, Theorem 2.23], there is a $g \in C_c(X)$ such that $g = f$ except on a set $B \subset X$ with $\mu(B) < \epsilon$ and $\sup_{x \in X} |g(x)| \leq \sup_{x \in X} |f(x)|$. Therefore,

$$|g - f| \leq |g| + |f| \leq 2 \sup_{x \in X} |f(x)|$$

hence,

$$\int_X |g - f|^p d\mu = \leq 2 \sup_{x \in X} |f(x)|^p \epsilon$$

and thus $C_c(X)$ is $L^p(\mu)$ dense in $M_E(X)$.

Let $M_o(X)$ denotes the space of all measurable functions on X such that for any $\epsilon > 0$, there is a measurable set $E \subset X$, $\mu(E) = 0$ and $|f(x)| < \epsilon$ for all $x \in X \setminus E$. This is an analogue of $C_o(X)$ -the space of all continuous complex valued functions on X that vanish at infinity. $M_o(X)$ is not empty. In fact $C_c(X) \subset M_E(X) \subset M_o(X)$.

We shall let $M_E^+(X)$ denote the set of all positive valued functions in $M_E(X)$. If V^p and U^p are two N_p families on X such that for every $v \in V^p$ there is u in V^p such that $v \leq u$ *a.e.*, we write $V^p \leq U^p$. We write $V^p \sim U^p$ if and only if $V^p \leq U^p$ and $U^p \leq V^p$.

2. Weighted semiconvex spaces

If V^p is an N_p family on X, then the corresponding weighted space is

$$MV_o^p(X) = \{f \in M(X) : fv \in M_o(X) \ \forall \ v \in V^p\}$$

If $MV_o^p(X)$ is endowed with the weighted topology W_v^p generated by the family of p-seminorms $\{p_v, v \in V^p\}$ where $p_v(f) = \|fv\|_p$ for each f in $MV_o^p(X)$ and for a fixed $v \in V^p$, then it is a semiconvex space [6]. A base of balanced semiconvex neighbourhoods of zero is formed by the sets

$$V_v^p = \{f \in MV_o^p(X) : \|fv\|_p \leq 1, \ v \in V^p\}$$

V_v^p is a semiconvex set since $\|fv\|$ is a p- seminorm by construction.

EXAMPLES

(i). Consider $L^p[\mu]$, the space of all equivalent classes of measurable functions on X such that $\int_X |f(x)|^p d\mu < \infty$ with a metric defined by
$d(f,g) = \int_X |f(x) - g(x)|^p d\mu$.
$L^p[\mu] = MV_o^p(X)$ algebraically and topologically whenever $V^p = M_o^+(X)$
(ii). $L^p[0,1]$, the space of all equivalence classes of Lebesque measurable functions on [0,1] such that $\int_0^1 |f(x)|^p d\mu < \infty$ with the metric defined by d(f,g) $= \int_0^1 |f(x) - g(x)|^p dx$ where $V^p = M_o^+[0,1]$. $L^p[0,1]$ is a particular case of Example 1.
(iii). ℓ^p sequence spaces where V^p is the set of all sequences (with positive coordinates) converging to zero.
For more examples see [7].

The following result will be needed in this paper.

PROPOSITION 2.1: *$M_E(X)$ is W_v^p dense in $MV_o^p(X)$.*

PROOF: It is clear that $M_E(X) \subset MV_o^p(X)$. Let $f \in MV_o^p(X)$, then f is measurable. By [10, Theorm 1.17] there are simple measurable functions s_n on X such that $0 \leq s_1 \leq s_2 \leq ..... \leq f$ and $s_n \to f$ as $n \to \infty$. Clearly each $s_n \in MV_o^p(X)$. $|(f - s_n)v|^p \leq |fv|^p$ for each $v \in V$. $|fv|^p$ is integrable on X since $fv \in M_o(X)$ and hence by the Lebesque dominated convergence Theorem, $\|(f - s_n)v\|_p \to 0$ as $n \to \infty$. $f - s_n \in V_v^p$ and so, each W_v^p neighbourhood of f intersects some $s_n \in M_E(X)$. Thus f is in the W_v^p closure of $M_E(X)$ and hence $M_E(X)$ is W_v^p dense in $MV_o^p(X)$.

COROLLARY 2.2: *$C_c(X)$ is $L^p[\mu]$ dense in $L^p[\mu]$.*

PROOF: In view of Example 1, Propositions 1.1 and 2.1 give the proof.

We then have a result extending the previously known result for $1 \leq p < \infty$.

COROLLARY 2.3: *For $0 < p < \infty$, $C_c(X)$ is $L^p[\mu]$ dense in $L^p[X]$.*

PROOF: In view of [10, Theorem 3.4], the proof follows immediately from Corollary 2.2.

We now find conditions under which $MV_o^p(X)$ is Hausdorff, which is an analogue of the locally convex spaces[11].

PROPOSITION 2.4: *$MV_o^p(X)$ is Hausdorff under W_v^p if and only if for every measurable set E in X for which there is an $f \in MV_o^p(X)$ such that $f/E \neq 0$, we can find $v \in V^p$ such that $v/E \neq 0$ a.e.*

PROOF: Assume $MV_o^p(X)$ is Hausdorff, then for each $0 \neq f \in MV_o^p(X)$ there is a p- seminorm p_v such that $p_v(f) \neq 0$ i.e. there is a $v \in V^p$ such that $\|fv\|_p \neq 0$. Set $f = \chi_E \in M_E(X)$, then $|fv|^p \neq 0$ a.e. on E. But $f = 1$ on E, hence, $v/E \neq 0$ a.e.

Conversely, suppose there is $0 \neq f \in MV_o^p(X)$ such that $f/E \neq 0$. If we can find $v \in V^p$ such that $v/E \neq 0$ a.e., then $\int |fv|^p d\mu \neq 0$ for some $v \in V^p$ i.e. $p_v(f) \neq 0$ and so $MV_o^p(X)$ is Hausdorff.

The following Theorem is a partial analogue of [12, Theorem 2.1].

PROPOSITION 2.5: *Let V^p be an N_p family on X such that X has a $\sigma - finite$ measure and $M_E^+(X) \leq V^p$, then $MV_0^p(X)$ is Hausdorff under W_v^p if X contains a dense measurable subset $Y (Y \neq X)$.*

PROOF: Let Y be a dense measurable subset of X and suppose $f \in MV_o^p(X), f \neq 0$. Then since Y is dense in X there is an $x_o \in X$ such that $\chi_Y(x_o) = 1 > 0$. $\chi_Y \subseteq M_E^+(X)$, hence since

$M_E^+(X) \leq V^p$ we can find $v \in V^p$ such that $\chi_Y \leq v$ a.e. and hence $fv \geq f\chi_Y$ a.e. Therefore $p_v(f) = \int_X |f|^p v^p d\mu > 0$ and so $MV_o^p(X)$ is Hausdorff.

It should be observed that in the characterization of Hausdorffness in the weighted convex spaces, Y must be locally compact and $V \leq C_o^+(X)$ [12, Theorem 2.1]. Those restrictions are not necessary in the semiconvex weighted spaces.

The relationship between two Nachbin families is set forth in the following Theorem which will be useful in the next section.

THEOREM 2.6: *If U^p and V^p be N_p families on X, then $U^p \leq V^p$ if and only if (1) $MV_o^p(X) \subseteq MU_o^p(X)$ (2) $W_u^p/MV_o^p(X) \leq W_v^p$.*

PROOF: Assume $U^p \leq V^p$, it is clear that $MV_o^p(X) \subseteq MU_o^p(X)$. To prove (2), let U_u^p be an arbitrary neighbourhood of zero in W_u^p, we want to show that there is $v \in V^p$ such that $V_v^p \subseteq U_u^p \cap MV_o^p(X)$. If $f \in V_v^p$, then $f \in MV_o^p(X)$ and $\|fv\|_p \leq 1$. Pick v such that $u \leq v$, then $\|fu\|_p \leq 1$ and hence, $f \in U_u^p$ and the proof is complete. Conversely assume conditions (1) and (2) of the Theorem hold. If $u \in U^p$, then there is a $v \in V^p$ such that $V_v^p \subseteq U_u^p \cap MV_o^p(X)$. If we set $A = \{x \in X : (u-v)(x) > 0\}$, we will show that A is empty. Assume A is non empty and $x_o \in A$, let B $= \{x \in X : v(x) < \frac{v(x_o)+u(x_o)}{2}\}$. Clearly $x_o \in B$. Also $\chi_B \in MV_o(X)$. Choose $f = 2(v(x_o)+u(x_o))^{-1}\chi_B$, then $f \in MV_o^p(X)$. Also

$$|f(x)|v(x) = 2\chi_B(x)v(x)v(x_o) + u(x_o)^{-}1 \leq 2v(x)v(x_o) + u(x_o)^{-}1 \leq 1 \;\forall\; x \in X$$

. If $\mu(X)$ is assumed to be b, $b > 0$, then $\|fv\|_p leqb$. This implies that $f \in \frac{1}{b}V_v^p \subseteq \frac{1}{b}U_u^p$. But $|f(x_o)|u(x_o) > 1$ and thus $f \notin \frac{1}{b}U_u^p$ which contradicts $f \in \frac{1}{b}U_u^p$. Hence A is empty and so $U^p \leq V^p$.

The technique used here is that used to prove the analogue in locally convex spaces by Summers[11,Theorem 3.1. and 3.3.]. But it should be observed that in proving the converse in the semiconvex setting, no assumption is necessary.

COROLLARY 2.7: *If U^p and V^p are N_p families on X with $U^p \sim V^p$, then $MV_o^p(X) = MU_o^p(X)$ i.e. they are the same sets with the same topologies.*

3. Countably hyperbarrelledness in $MV_o^p(X)$

We now consider the question as to when $MV_o^p(X)$ is hyperbarrelled. We will adopt the approach in [12] used for the locally convex weighted spaces. We shall recall the following definitions from [5] and [6].

DEFINITION 1: A closed, balanced and absorbent semiconvex set is called an *hyperbarrel.*

DEFINITION 2: A semiconvex topological vector space E is *hyperbarrelled* if every hyperbarrel in E is a neighbourhood of zero.

DEFINITION 3: A semiconvex space E is called *countably hyperbarrelled* if each hyperbarrel which is a countable union of balanced, semiconvex and closed neighbourhoods of zero in E is a neighbourhood of zero.

First we show when $MV_o^p(X)$ is countably hyperbarrelled. We assume that $\mu(X) = b,\ b \in R^+$ throughout this section. The technique used in the two Lemmas below is due to [3] although under a different setting.

LEMMA 3.1: *Let V^p be an N_p family on X with v and w as members of V^p, then $V_v^p \subseteq V_w^p$ if and only if $w(x) \leq v(x)$ for all $x \in X$.*

PROOF: If $w(x) \leq v(x)$ for all $x \in X$, then $V_v^p \leq V_w^p$. Conversely assume there is a $x_o \in X$ such that $w(x_o) > v(x_o)$. Take $\lambda > 0$ with $v(x_o)^p < \lambda^p b^{p-1} < w(x_o)^p$. The set $G = \{x \in X : v(x) < \lambda b^{\frac{p-1}{p}}\}$ is measurable, since v is a measurable function [10,Theorem 1.2]. Define $f : X \to R$ by $|f(x)| = (b\lambda)^{-1}\chi_G(x)$ for all $x \in X$.

$$|f(x)|v(x) = (b\lambda)^{-1}\chi_G(x)v(x) = \begin{cases} 0 & \text{for all } \ x \in X/G \\ \frac{v(x)}{b\lambda} & \text{for all } \ x \in G \end{cases}$$

It is clear that $fv \in M_E(X) \subseteq M_o(X)$ for all $v \in V^p$ and hence $f \in MV_o^p(X)$. We also claim that $f \in V_v^p$. $|f(x)|v(x) = 0$ for all $x \in X/G$. Also, $|f(x)|v(x) = \frac{v(x)}{b\lambda} \leq (\frac{1}{b})^{\frac{1}{p}}$ for all $x \in G$. Therefore $\int_X |f(x)|^p v(x)^p d\mu \leq \mu(X)(\frac{1}{b}) = 1$ and hence $f \in V_v^p$. Also $\frac{w(x_o)^p}{(\lambda b)^p} > \frac{1}{b}$ and $x_o \in G$, and so $|f(x_o)|^p w(x_o)^p = \frac{w(x_o)^p}{(\lambda b)^p} > \frac{1}{b}$, $x_o \in G$. Thus $\int_X |f(x)|^p w(x)^p d\mu > \frac{1}{b}\mu(X)$, hence $f \notin V_w^p$, i.e. $V_v^p \supseteq V_w^p$ and the proof is complete.

In the sequel, we put $\frac{1}{\infty} = 0$, and $\frac{1}{0} = \infty$.

LEMMA 3.2: *Let s be an arbitrary function from X to $[0, \infty]$ such that the set $N = \{f \in MV_o^p(X) : |f| < s$ a.e.$\}$ is absorbent in $MV_o^p(X)$, then there is a smallest weight v_s greater than $\frac{1}{s}$ a.e. and $N = V_{v_s}$.*

PROOF: Since N is absorbent, then for every $f \in MV_o^p(X)$, there is a $\lambda > 0$ such that $f \in \lambda N$. Since $MV_o^p(X) \neq 0$, $\sup_{f \in N} |f(x)| \neq 0$. Denote $\sup_{f \in N} |f(x)| = n(x)$. It is easy to check that $\frac{1}{n}$ is a weight on X. Since $\sup_{f \in N} |f(x)| \leq s(x)$ a.e. by the definition of N, then $n \leq s$ a.e. and hence $\frac{1}{n} \geq \frac{1}{s}$ a.e. Since $n \neq 0$, $\frac{1}{n}$ is finite and so $\frac{1}{s}$ is finite a.e. Define $v_s = \{inf\ v : v$ is a weight on

X, $\frac{1}{s} \le v$ $a.$ $e\}$. It is easy to see that v_s is a weight on X. Also, $V^p_{\frac{1}{n}} \subseteq V^p_{v_s} \subseteq N \subseteq V^p_{\frac{1}{n}}$. Therefore $N = V^p_{v_s} = V^p_{\frac{1}{n}}$.

DEFINITION 4: A subset N of $MV^p_o(X)$ is called a *full* subset if N = $\{f \in MV^p_o(X) : |f| < n \ a.e.\}$ where n(x) = $\sup_{f \in N} |f(x)|$ (see [3]).

LEMMA 3.3: *Let $\{N_i, i \in N\}$ be a sequence of full subsets of $MV^p_o(X)$ and $n_i = \sup_{f \in N_i} |f(x)|$. Then $N = \bigcap N_i$ is a full subset of $MV^p_o(X)$ and*
$N = \{f \in MV^p_o(X) : |f(x)| \le \inf_i n_i(x)\}$.
PROOF: Put $n(x) = \sup_{f \in N} |f(x)|$. It is easy to see that N = $\{f \in MV^p_o(X) : |f| < n \ a.e.\} = \{f \in MV^p_o(X) : |f| < \inf_i n_i \ a.e.\}$.

DEFINITION 5: Let V^p be an N_p family on X, we define a new N_p family $(V_t)^p$ on X by adjoining the smallest weights(including thier multiples) greater than the suprema of all countable weights in V^p to V^p.

LEMMA 3.4: *Let V^p be an N_p family on X and $(v_n)_n$ be the set of all sequences (v_n) in V^p. Let $v_{s(n)}$ be the smallest weight greater than $\sup_n v_n$ for each $(v_n) \in V^p$. Then $(V_t)^p$ is the system of all positive multiples of $v_{s(n)}$ as (v_n) runs through all sequences in V^p.*

REMARK 3.5: Let $\{U_n, n \in n\}$ be a sequence of balanced, semiconvex and closed neighbourhood of zero in $MV^p_o(X)$ such that U = $\bigcap\{U_n, n \in N\}$ is an hyperbarrel. Then for each U_n, there is an $v_n \in V^p$ such that $V^p_{v_n} \subseteq U_n$ and thus $\bigcap V^p_{v_n} \subseteq \bigcap U_n = U$. Clearly $\bigcap V^p_{v_n}$ is absorbent. Assume $\mu(X) \le 1$, then $V^p_{v_n} = \{f \in MV^p_o(X) : |f(x)| \le \frac{1}{v_n(x)} \ a.e.\}$ hence by Lemma 3.3 $\bigcap\{V^p_{v_n}, v_n \in V^p\} = \{f \in MV^p_o(X) : |f(x)| \le \inf_n \frac{1}{v_n(x)}$ a.e.$\}$. Let $s = \inf_i \frac{1}{v_n}$. By Lemma 3.2, there exists a smallest weight greater than $\frac{1}{s}$ and $\bigcap V^p_{v_n} = V^p_{v_s} = \{f \in MV^p_o(X) : \|fv_s\|_p \le 1\}$. Clearly $(V_t)^p$ is the system of all positive multiples v_s of weights obtained in this way as (v_n) runs through all sequences V^p. Assume $V^p \sim (V_t)^p$, then $V^p_{v_s}$ and thus U is a neighbourhood of zero in $MV^p_o(X) = M(V_t)^p_o(X)$. Therefore $M(V_t)^p_o(X) = MV^p_o(X)$ is countably hyperbarrelled. Conversely assume $MV^p_o(X)$ is countably hyperbarrelled, then U is a neighbourhood of zero in $MV^p_o(X)$ and hence there is $v \in V^p$ such that $V^p_v \subseteq U$. Also, $V_{v_s} \subseteq U$. Since U is arbitrary and v_s is the smallest weight greater than $\frac{1}{s}$, then v can be chosen such that $v_s \le v$. Hence $(V_t)^p \le V^p$. It is easy to see that $V^p \le (V_t)^p$ from the construction of $(V_t)^p$. Therefore $V^p \sim (V_t)^p$. Thus we have shown the following.

THEOREM 3.6: *Let $\mu(X)$ be a probability measure, then $MV^p_o(X)$ is countably hyperbarrelled if and only if $V^p \sim \overline{V}^p$.*

COROLLARY 3.7: *Assume $\mu(X)$ is a probaility measure and $MV_o^p(X)$ is metrizable, then $MV_o^p(X)$ is hyperbarrelled if and only if $V^p \sim \overline{V}^p$.*

PROOF: A metrizable countably hyperbarrelled space is hyperbarrelled.

COROLLARY 3.8: *If $\mu(X)$ be a probability measure, then $L^p(\mu)$ is hyperbarrelled.*

PROOF: Since $L^p(\mu)$ is metrizable, it is sufficient to show that $M_o^+(X) = V^p \sim (V_t)^p$ in view of the fact that $L^p(\mu) = MV_o^p(X)$ whenever $V^p = M_o(X)$. It is clear that if each v_n is in $M_o(X)$, then $\sup_n v_n \in M_o^+(X)$. Set $s(x) = \inf_n \frac{1}{v_n(x)}$, then $\frac{1}{s(x)} = \sup_n v_n(x) \subseteq M_o^+(X)$. Since $\frac{1}{s}$ is a weight in V^p, by setting $v_s = \frac{1}{s}$ it is clear that $(V_t)^p \subseteq M_o^+(X) = V^p$ i.e. $(V_t)^p \leq V^p$. Since $V^p \leq \overline{V}^p$ (in general) the proof is complete.

4. Completeness and Inductive limits of weighted semiconvex spaces

We now examine the completeness of $MV_o^p(X)$.

LEMMA 4.1: *$M_o(X)$ is complete in the p-normed topology $\|.\|_p$.*

PROOF: Let (f_n) be a Cauchy net in $M_o(X)$. For each neighbourhood of zero U in $\|.\|_p$, there is a k such that $f_n \in f_m + U$ for all $m, n > k$. Thus the net (f_n) uniformly converges to f say. Since each (f_n) is measurable and the limit of every pointwise convergent sequence of measurable functions is measurable, then f is measurable. Hence there is an n such that $f_n \in f + U$ for each neighbourhood U of zero in $\|.\|_p$ and hence $f \in f_n + U$. Thus, $\int |f_n - f|^p d\mu < \epsilon, \epsilon > 0$ i.e. there is a $b > 0$ such that $|(f_n - f)(x)|^p < \epsilon b$ a.e. Thus $|(f_n - f)(x)|^p < \epsilon b$ for all $x \in X/A$ where $\mu(A) = 0$. Also, since $f_n \in M_o(X)$,then there is a measurable set B of X such that $|f_n(x)| < \epsilon$ for all $x \in X \setminus B$. Hence $|f(x)| \leq |(f - f_n)(x)| + |f_n(x)| \leq (\epsilon b)^{\frac{1}{p}} + \epsilon$ for all $x \in X \setminus B \cup A$ and so, since $B \setminus cupA$ is measurable, $f \in M_o(X)$. Thus we have shown that $(M_o(X), \|.\|_p)$ is complete.

THEOREM 4.2: *Let U^p be an N_p family on X such that for each $x \in X$, there exists a $u \in U^p$ such that $u(x) > 0$ a.e. If $MU_o^p(X)$ is complete and V^p is an N_p family on X with $U^p \leq V^p$, then $MV_o^p(X)$ is complete.*

PROOF: Let (f_i) be a W_v^p-Cauchy net in $MV_o^p(X)$. It follows from Theorem 2.6 that (f_i) is W_u^p-Cauchy and hence we can find $f \in MU_o^p(X)$ such that $f_i \to f(W_u^p)$. Since (f_i) is W_v^p-Cauchy net in $MV_o^p(X)$, then for any $V_v^p \in W_v^p$, there exists k such that $f_i \in f_j + V_v^p$ for all $i, j > k$. Then $\|(f_i - f_j)v\|_p \leq 1$ i.e.$\|f_i v - f_j v\|_p \leq 1$ for all $i, j \geq k$. Since $f_i v \in M_o(X)$ for each $v \in V$, then $f_i v$ is a Cauchy net in $(M_o(X), \|.\|_p)$ and hence by Lemma 4.1, there exists $f_v \in M_o(X)$ such that $f_i v \to f_v(\|\cdot\|_p)$. By assumption, if $x \in X$, there exists $u \in U^p$ such that $u(x) > 0$ a.e. Then since $U^p \leq V^p$ there is a $v \in V^p$ such that $v(x) > 0$ a.e. and so $fv = f_v$ a.e. for every $v \in V^p$. Since $fv \in M_o(X)$ for every $v \in V$, $f \in MV_o^p(X)$. Also since $f_i v \to fv(\|.\|_p)$ for every

$v \in V$, there is k such that for all $i > k$, say, $\|f_i v - fv\|_p \leq 1$ and so $f_i - f \in V_v^p$ thus $f_i \to f(W_v^p)$.

COROLLARY 4.3: *Let V^p be an N_p family on X such that $M_o^+(X) \leq V^p$. Then $MV_o^p(X)$ is complete.*

PROOF: Setting $L^p(\mu) = MU_o^p(X)$ with $U^p = M_o^+(X)$, (see Example 1), since $L^p(\mu)$ is complete, then by Theorem 4.2, $MV_o^p(X)$ is complete.

We will now consider the inductive limits of weighted spaces. We shall examine the projective description of weighted inductive limits as an analogue of the weighted locally convex spaces studied in [1].

Let $\{V_n^p,\ n \in N\}$ be a sequence of N_p families on X such that $V_{n+1}^p \leq V_n^p$ for each $n \in N$. We shall denote *ind.* $M(V_n^p)_o(X)$ by $V^pM(X)$. We want to describe the weighted inductive limit $V^pM(X)$, analogous to the case of weighted spaces of continuous functions, in terms of an associated N_p family on X.

Let $v_n \in V_n^p$ and $\alpha_n > 0$ for each n; if we put $\overline{v}(x) = inf\{\alpha_n v_n(x), n \in N, x \in X\}$, then $\overline{v}(x)$ is clearly a weight on X. Scalar multiples of all those weights on X form an N_p family on X which we will denote $\overline{V}^p$. Clearly $\overline{V}^p$ contains every N_p family V^p on X that satisfies $V^p \leq V_n^p$ for each $n \in N$.

We first state the following results:

LEMMA 4.4: *Let V^p be an N_p family on a σ-compact space X and μ a probability measure, then $M\overline{V}_o^p(X)$ and $V^pM(X)$ induce the same topology on $M_o(X)$.*

PROOF: We follow the proof of the analogous result in the weighted spaces of continuous functions(See [4, p114,Lemma 4]) with some modifications. Since the canonical injection of $V^pM(X)$ into $M\overline{V}_o^p(X)$ is continuous, we can fix an arbitrary neighbourhood U of zero in $V^pM(X)$ and then have to prove that the intersection of $M_o(X)$ with some zero neighbourhood in $M\overline{V}_O^p(X)$ is contained in U. By the description of a basis of zero neighbourhood in an inductive limit, we may assume without loss of generality that U is an absolutely convex hull of the form $\Gamma(\bigcup_n B_n)$, where

$$B_n = \{f \in M(V_n^p)_o(X) : \|fv_n\|_p \leq \rho_n, v_n \in V_n\}$$

and ρ_n is positive for each $n \in N$. Put $\overline{v} = \inf\limits_{n \in N} \frac{2^n}{\rho_n} v_n \in \overline{V}^p$. It remains to show that $\{f \in M_o(X) : \|f\overline{v}\|_p < 1 \subset U$. Fix $f \in M_o(X)$ with $\|f\overline{v}\|_p < 1$. For each n, let F_n denote the measurable subset $\{x \in X : \frac{2^n}{\rho_n} v_n(x)|f(x)| \geq 1\}$ of X. We observe that $\bigcap F_n$ is empty because for any $x \in \bigcap F_n$, $\frac{2^n}{\rho_n} v_n(x)|f(x)| \geq 1$ holds for each n, whereby $\|f\overline{v}\|_p \geq 1$ contradicting $\|f\overline{v}\|_p < 1$. For each F_n, we can find a closed set C_n such that $C_n \subset F_n$[10, Theorem 2.17a].

$\bigcap C_n$ is empty. Hence putting $U_n = X \backslash C_n$ for each n, each U_n is measurable and $(U_n, n \in N)$ is an open covering of X. Let $(\psi_n)_n \subset C_c(X)$ be a continuous partition of unity on *supp* f which is surbordinate to $(U_n)_n$. We then take $g_n = 2^n \psi_n f \in M_o(X) \subset M(V_n^p)_o(X)$ for each n and estimate $\|g_n v_n\|_p = \|\psi_n 2^n v_n f\|_p = \|\rho_n \psi_n \frac{2^n}{\rho_n} v_n f\|_p \leq \rho_n$. Thus each $g_n \in B_n$, and hence f $= \sum \psi_n f$ is an element of $\Gamma(\bigcup_n B_n) = U$ and the proof is complete .

The following result will also be needed.

LEMMA 4.5: *Given a semiconvex space (E_1, ϵ_1), let E_2 denote a linear subspace and ϵ_2 a semiconvex topology on E_2 which in finer than the topology induced by ϵ_1. If ϵ_1 and ϵ_2 induce the same topology on some dense linear subspace D of (E_2, ϵ_2), then $\epsilon_2 = \epsilon_1/E_2$.*

PROOF: The proof is essentilly the same as the one for locally convex space in [1, Lemma 1.2].

We now have the following result which is an analogue of [1, Theorem 1.3].

THEOREM 4.6: *Let X be a σ-compact space and μ a probability measure.*

(1) If $\{V_n^p, n \in N\}$ is a sequence of N_p families on X such that $V_{n+1}^p \leq V_n^p$ for each $n \in N$, then the canonical injection from $\mathbb{V}^p M(X)$ into $M\overline{V}_o^p(X)$ is a topological isomorphism.

(2) Suppose $M_o^+(X) \leq V_n^p$ for each $n \in N$, then $M\overline{V}_o^p(X)$ is the completion of $V^p M(X)$.

PROOF: (1) If $(E_1, \epsilon_1) = M\overline{V}^p(X)$, $(E_2, \epsilon_2) = V^p M(X)$ and D=$M_o(X)$ in Lemma 4.5, then the proof follows clearly from Lemma 4.4.

(2) Since $M\overline{V}_o^p(X)$ is complete by Corollary 4.3, and the fact that $M_o(X)$ is dense in $V^p M(X)$, an application of (1) completes the proof.

Linear subspaces and Quotient spaces of weighted spaces

We now answer the question as to when a linear subspace of a weighted semiconvex space is of the same sort.

THEOREM 5.1: *Let U^p be a N_p family on X such that $M_o^+(X) \leq U^p$, and $L \neq 0$ a subspace of $MU_o^p(X)$ such that $L \subseteq M_b(X)$. If there is an N_p family V^p on X such that $U^p \leq V^p$ and $\alpha_L.v \in U^p$ for each $v \in V^p$, where $\alpha_L = \sup_{f \in L} |f|$, then $L = MV_o^p(X)$ and $W_u^p / MV_o^p(X) \leq W_v^p$.*

PROOF: Let $f \in L$, then $f.\alpha_L.v \in M_o^+(X)$ for all $v \in V^p$. Therefore $f \in MV_o^p(X)$ and thus $L \subseteq MV_o^p(X)$. Conversely, let $f_1 \in MV_o^p(X) \subseteq MU_o^p(X)$ such that $f_1 \notin L$. We can choose $f_1 > \alpha_L$. $f_1 \in MU_o^p(X)$ implies $f_1.\alpha_L.v \in M_o^+(X)$ for all $v \in V^p$. By the construction of α_L, $v \in M_o^+(X)$ i.e. $V^p \leq M_o^+(X)$. Hence by our assumption $M_o^+(X) \leq U^p \leq V^p \leq M_o^+(X)$ and so $U^p \sim V^p$. Thus $MV_o^p(X) = MU_o^p(X)$. This contradicts our assumption. Therefore $f_1 \in L$ and so $MV_o^p(X) \subseteq L$. Thus $L = MV_o^p(X)$. The second part of the Theorem follows immeditely from Theorem 2.6.

Let L be a linear subspace of a weighted space $MV_o^p(X)$. Consider the canonical map $k : MV_o^p(X) \to MV_o^p(X)/L$. The sets $k(V_v^p)$, where $\{V_v^p, v \in V^p\}$, is a base of neighbourhood for $MV_o^p(X)$ with the topology W_v^p, form a base of neighbourhoods for the quotient topology on $MV_o^p(X)/L$[7].

We wish to show when $MV_o^p(X)/L$ with the quotient topology is a weighted semiconvex space. First we show when it is a weighted space algebraically.

<u>PROPOSITION 5.2</u>: *Let V^p be an N_p family on X and L a linear subspace of $MV_o^p(X)$. If there is an N_p family U^p on X such that $U^p \leq V^p$ and for each $v \in V^p$, there is a measurable set B in X, with measure zero, such that $v(x) \leq u(x)\ \forall\ x \in X/B$ for some $u \in U$, then the quotient space $MV_o^p(X)/L = MU_o^p(X)$(algebraically).*
<u>PROOF</u>: $L \subseteq MV_o^p(X) \subseteq MU_o^p(X)$. By partitioning $MU_o^p(X)$ into equivalence classes $\{f + L\}$, $f \in MU_o^p(X)$; it is easy to see that $MV_o^p(X)/L = MU_o^p(X)$.

We now consider the question of weighted semiconvex quotient topology i.e. when is $MV_o^p(X)/L = MU_o^p(X)$ topologically.

We adopt Warner's method[14] in proving the following Lemma.

<u>LEMMA 5.3</u>: *Let $(V_n^p, n \in N)$ be a sequence of N_p families on X and U^p an N_p family on X such that $U^p \leq V_n^p$ for each n, and $\bigcup_n M(V_n)_o^p(X)$ spans $MU_o^p(X)$. Let i_n be an inclusion map from $M(V_n)_o^p(X) \to MU_o^p(X)$, then the semiconvex(sc.) inductive limit topology on $MU_o^p(X)$ coincides with the weighted sc.inductive limit topology.*
<u>PROOF</u>: Let V be a sc. neighbourhood of zero in $MU_o^p(X)$ for the sc. inductive limit topology. Then $V \cap M(V_n)_o^p(X)$ is a neighbourhood of 0 in $M(V_n)_o^p(X)$ for each n. Hence there is a $v_n \in V_n^p$ such that $V_{v_n}^p \subseteq V \cap M(V_n)_o^p(X)$. Let $W = \bigcup_n V_{v_n}^p$, then $W \subset MU_o^p(X)$. For let $f \in W$, then $f \in V_{v_n}^p$ for some n by the construction of W. Hence $fv \in M_o(X)$ for all v in some N_p family V_n^p. Since $U^p \leq V_n^p$ for each n, then $fv \in M_o(X)$ for all $v \in U^p$ *i.e.* $W \subseteq MU_o^p(X)$. Let $\overline{W}$ be the semiconvex, balanced envelope(or hull) of W, then $\overline{W}$ is a neighbourhood of zero for the sc.inductive limit topology on $MU_o^p(X)$[7]. Since $W \subseteq MU_o^p(X), \overline{W} \subseteq MU_o^p(X)$. Since V is semiconvex and balanced and $W \subseteq V$, then $\overline{W} \subseteq V$. Since W is absorbing so is also $\overline{W}$. Finally, it is easy to see that $V_{v_n}^p \subseteq W \cap M(V_n)_o^p(X) \subseteq \overline{W} \cap M(V_n)_o^p(X)$ for each n. Thus $\overline{W}$ is a neighbourhood of zero for the weighted sc.inductive limit topology on $MU_o^p(X)$. Hence the weighted sc.inductive limit is finer than the sc.inductive limit topology and since the sc.inductive limit topology is the finest sc.topology on $MU_o(X)$ under which each inclusion map i_n is continuous[9], then two inductive limit topologies coincide.

Since the quotient topology is a special case of inductive limit topology[9, chap.5], then in view of Proposition 5.2 and Lemma 5.3, we have

COROLLARY 5.4: *Let V^p be an N_p family on X and L be a linear subspace of $MV_o^p(X)$ with the assumptions in Proposition 5.2, then $MV_o^p(X)/L$ with the quotient topology is a weighted semiconvex space.*

Let $MV_o^p(X)$ be a Hausdorff weighted semiconvex space and Γ a base of balanced semiconvex neighbourhoods. If $U^p \in \Gamma$, choose a sequence $(U_n^p)_{n=1}^{\infty}$ from Γ such that $U_1^p + U_1^p \subseteq U^p$ and $U_{n+1}^p + U_{n+1}^p \subseteq U_n^p \ \forall\, n$. Set
$N(u) = \bigcap_{n=1}^{\infty} U_n^p$ *and let* $k_{N(u)} : MV_o^p(X) \to MV_o^p(X)/N(u)$ be the quotient map. $MV_o^p(X)/N(u)$ with the quotient topology is locally bounded ([5, p114]). If the conditions of Proposition 5.2 are assumed, and setting L = N(u) in Corollary 5.4, $MV_o^p(X)/N(u)$ with the quotient topology is a weighted locally bounded space. The map $k_{N(u)} : MV_o^p(X) \to MV_o^p(X)/N(u)$ is a topological isomorphism from $MV_o^p(X)$ into the product $\underset{U^p \in \Gamma}{\times} MV_o^p(X)/N(u)$[7] and hence we have the following result.

THEOREM 5.5: *Let V^p be an N_p family on X. If there is an N_p family U^p on X such that $U^p \leq V^p$, and for each $v \in V^p$, there is a measurable set B in X, with measure zero such that $v(x) \leq u(x) \ \forall\, x \in X/B$ for some $u \in U$, then $MV_o^p(X)$ is topologically isomorphic to a subspace of a product of weighted locally bounded spaces.*

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