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NEW CONSTITUTIVE EQUATIONS TO DESCRIBE INFINITESIMAL ELASTIC-PLASTIC DEFORMATIONS

B. Boecke, F. Link, G. Schneider
INTERATOM GmbH
D-5060 Bergisch Gladbach 1
Federal Republic of Germany

O. T. Bruhns
University of Kassel
D-3500 Kassel
Federal Republic of Germany

ABSTRACT

A set of constitutive equations is presented to describe infinitesimal elastic-plastic deformations of austenitic steel in the range up to 600 °C. This model can describe the hardening behaviour in the case of mechanical loading and hardening and softening behaviour in the case of thermal loading. The loading path can be either monotonic or cyclic. For this purpose, the well-known isotropic hardening model is continually transferred into the kinematic model according to Prager, whereby suitable internal variables are chosen. The occuring process-dependent material functions are to be determined by uniaxial experiments. The hardening function g and the translation function c are determined by means of a linearized stress-strain behaviour in the plastic range, whereby a coupling condition must be taken into account. As a linear hardening process is considered to be too unrealistic, nonlinearity is achieved by introducing a small function w, the determination procedure of which is given.

NOMENCLATURE

c	translation function
E	Young's modulus
E_t	tangent modulus
F	yield function
G	shear modulus
g	hardening function
LC	load condition
$\underset{\sim}{M}$	compliance tensor
t	time
w	disturbance of c
$\underset{\sim}{\alpha}$	thermal expansion tensor
$\underset{\sim}{\varepsilon}$	strain tensor
ϑ	temperature
κ	plastic work of reduced stresses
ν	Poisson's ratio
$\underset{\sim}{\xi}$	translation tensor
$\underset{\sim}{\sigma}$	stress tensor
$\dot{x}$	time derivative of x
$\underset{\sim}{x}'$	deviator of $\underset{\sim}{x}$

INTRODUCTION

New constitutive equations for the description of elastic-inelastic material behaviour in the case of cyclic mechanical or thermal loading on stainless austenitic steels at temperatures up to approximately 600 °C are being developed within the framework of a research programme. Such steel is used to construct the primary and secondary systems of fast sodium-cooled reactors, with operation temperatures around 550 °C.

The macroscopic model for the description of elastic-inelastic material behaviour under cyclic loading originally proposed by ORNL (1) and enlarged by several additional remarks seems to be not very suitable to express the fundamental properties as e.g.

- the interaction between rate-dependent and rate-independent material behaviour
- the rate-dependence of plastic deformations (viscoplasticity).

The linear kinematic model for the determination of the plastic deformations recommended by ORNL may be applied to general processes. However this is not valid for the creep-model describing the viscous influence. This model is more or less restricted to cases of radial loading. The aim of the research programme is to construct a model that includes the above mentioned material properties and that can be applied to general processes.

Theories are known, based only on one simple strain measure the total strain. Such procedures seem to be justified for a description of processes in the high temperature range. We therefore refer e. g. to the papers of Valanis (2,3), Krempl and co-workers (4,5) or Haupt (6). These theories are hardly proven for multiaxial homogeneous and inhomogeneous stress-strain problems. To get a realistic material description as quickly as possible the classical assumption is used of splitting the total infinitesimal strain into an elastic and an inelastic part.

This splitting of the total strain permits a separate formulation of the constitutive equations. This can be done in three steps.

Step 1: rate-independent material behaviour (elastic-plastic behaviour)
Step 2: rate-dependent permanent behaviour (viscoplastic behaviour)
Step 3: rate-dependent material behaviour below the yield point (viscoelastic behaviour).

This report only deals with step 1. In step 2 a modified overstress-model with internal variables will be developed, containing the plastic strain from step 1 as a limiting case. This leads to a "unified" model, in which will not be distinguished between plastic and creep strains. In step 3 a model of Norton type will be used. The respective strain will be added.

The assumptions for a rate-independent model are based on the concepts of classical elasticity and plasticity theories (7).

1. It is assumed that the theory is thermomechanically uncoupled, i. e. the temperature distribution is assumed to be independent of the deformation.

2. The elastic strains are determined by means of equations with in a general way temperature-dependent coefficients.

3. Even for high temperatures the existence of a yield surface is supposed. Following and generalizing an idea of Mróz (8) this yield surface can expand and change its position in "space" under mechanical loading. The proposed hardening model describes a continuous transfer from a purely isotropic to a purely kinematic hardening.

4. Suitably selected internal variables with the possibility of physical interpretations are introduced in order to

   (i) record the process history at least approximately, as is of particular importance for non-radial process paths, and
   (ii) permit a later description of interaction with rate-dependent deformations.

   As regards these internal variables, simple evolution equations are used in the form of ordinary differential equations.

5. In chapter 3 a quantity Δ is defined similar to a quantity "structure memory" in the sense of Krempl (4) or Dafalias-Popov (9). With such a quantity the nonlinear hardening under general loading conditions can be reproduced in a satisfactory manner.

6. The following points are assumed for the sake of simplicity:

   (i) isotropic material behaviour
   (ii) a convex yield surface only as a function of the second invariant of the reduced stress.
   (iii) incompressibility of the plastic part of the deformations.

FORMULATION OF THE CONSTITUTIVE EQUATIONS

The infinitesimal total strain $\underset{\sim}{\varepsilon}$ is assumed to be the sum of the elastic strains $\underset{\sim}{\varepsilon}^{el}$ and the plastic strains $\underset{\sim}{\varepsilon}^{pl}$

$$\underset{\sim}{\varepsilon} = \underset{\sim}{\varepsilon}^{el} + \underset{\sim}{\varepsilon}^{pl} \qquad (1)$$

Internal variables are introduced to be able to express the history of the material approximately through their variations. These internal variables are described by evolution equations, which have the form of ordinary differential equations (10). As the growth of the internal variables is required to be connected with the occurrence of plastic deformations it seems to be reasonable for the growth laws to have the following form

$$\begin{aligned} \dot{\underset{\sim}{\xi}} &= \underset{\sim}{C}(\underset{\sim}{\sigma},\vartheta,\underset{\sim}{\xi},\kappa)\cdot\dot{\underset{\sim}{\varepsilon}}^{pl} \\ \dot{\kappa} &= \underset{\sim}{D}(\underset{\sim}{\sigma},\vartheta,\underset{\sim}{\xi},\kappa)\cdot\dot{\underset{\sim}{\varepsilon}}^{pl} \end{aligned} \qquad (2)$$

where $\underset{\sim}{\sigma}$ is the stress, ϑ is the temperature and $\underset{\sim}{\xi}$ and κ are internal variables.

A function

$$F \equiv F(\underset{\sim}{\sigma},\vartheta,\underset{\sim}{\xi},\kappa)$$

furthermore is introduced separating the ranges of purely elastic and elastic-plastic deformations for F = 0 and fixed internal variables.

If now the internal variable $\underset{\sim}{\xi}$ is interpreted as a translation tensor describing the displacements of the centre of the "yield sphere" F = 0 in the stress space, the simpler form

$$F = F(\sigma-\xi,\vartheta,\kappa) = 0$$

can be taken. According to classical models of the plasticity theory

$$F \equiv f(\sigma-\xi) - g(\vartheta,\kappa) = 0 \qquad (3)$$

is assumed, where g is the "isotropic" hardening function.

Since under loading conditions only permanent deformations should occur the following equation

$$LC = \frac{\partial f}{\partial \sigma}\cdot\dot{\sigma} - \frac{\partial g}{\partial \vartheta}\dot{\vartheta} \qquad (4)$$

defines

$$\langle x\rangle = \begin{cases} x & \text{for } F = 0 \text{ and } LC > 0 \text{ what means "loading"} \\ 0 & \text{for } F = 0 \text{ and } LC < 0 \text{ what means "unloading" and "neutral loading"} \\ 0 & \text{for } F < 0 \end{cases}$$

The constitutive equation for the elastic part of "Eq. (1)" is assumed in the form

$$\varepsilon^{el} = M(\vartheta)\sigma + \alpha(\vartheta)\vartheta \qquad (5)$$

where $M(\vartheta)$ is the compliance tensor and $\alpha(\vartheta)$ is the thermal expansion tensor, which both here are functions of temperature. For constant material tensors M and α this relation is transformed into the linear Duhamel-Neumann law (7). It is assumed here that thermoelastic-plastic processes may be considered uncoupled, i.e. the strains are dependent on the temperature distribution, while the inverse effect is not valid.

Formal differentiation of (5) results in

$$\dot{\varepsilon}^{el} = M(\vartheta)\dot{\sigma} + \dot{M}(\vartheta)\sigma + \dot{\alpha}(\vartheta)\vartheta + \alpha(\vartheta)\dot{\vartheta} \qquad (6)$$

which still remains a rate-independent constitutive relation.

For the plastic part of the deformations a constitutive relation in the form of the "normality rule" is adopted

$$\dot{\varepsilon}^{pl} = \lambda \left\langle \frac{\partial f}{\partial \sigma} \right\rangle \qquad (7)$$

The herein still unknown parameter λ, which is dependent on the path of the process is determined by means of the condition of consistency $\dot{F} = 0$ to give

$$\lambda = \frac{LC}{\frac{\partial f}{\partial \sigma}\cdot C\langle\frac{\partial f}{\partial \sigma}\rangle + \frac{\partial g}{\partial \kappa}\, D\cdot\langle\frac{\partial f}{\partial \sigma}\rangle} \qquad (8)$$

If all the material parameters, some of which are functions of the path of the process, are known, any stress and strain conditions can be described for a given temperature distribution. What concerns inhomogeneous processes, the thermomechanical yield equations together with the given constitutive equations must be formulated and solved as initial boundary value problems.

SIMPLIFICATION OF THE MATERIAL LAWS

With reference to direct applications to problems in the field of engineering, it seems admissible to assume the material isotropic. As a further simplification, the fact that, in the case of above-mentioned materials, the plastic strains are almost incompressible, is used. We assume incompressibility by introducing

$$\mathrm{tr}\, \dot{\varepsilon}^{pl} = 0$$

As regards isotropic material behaviour, the function f in "Eq. (3)" only can be function of the invariants of the reduced stress $\sigma - \xi$. Incompressibility furthermore means that f is independent on the first invariant. The yield criterion therefore becomes

$$f(II, III) = g(\vartheta,\kappa)$$

where II and III represent the second and third invariants of $\sigma - \xi$. As consequence of the demand on convexity of the yield surface the third invariant III only has a limited influence on the form of the yield condition. We therefore neglect it entirely and write

$$f(II) = g(\vartheta,\kappa)$$

using

$$II = \mathrm{tr}(\sigma' - \xi')^2$$

where σ' and ξ' are the deviators of σ and ξ.

The simplest possible form is

$$f(II) = II = g(\vartheta,\kappa) \qquad (9)$$

what corresponds to the v. Mises Yield Criterion.

In order to keep the evolution laws (2) as simple as possible, we use

$$C = [c(\kappa,\vartheta) + w(\Delta,\vartheta)]\, 1$$
$$D = \frac{1}{2}\frac{\partial f}{\partial \sigma}$$

thus obtaining

$$\dot{\xi} = \dot{\xi}' = [c(\kappa,\vartheta) + w(\Delta,\vartheta)]\, \dot{\varepsilon}^{pl} \qquad (10a)$$
$$\dot{\kappa} = \frac{1}{2}\frac{\partial f}{\partial \sigma}\cdot\dot{\varepsilon}^{pl} \qquad (10b)$$

The expression $w(\Delta,\vartheta)$ herein depends on the value Δ, which is defined as follows

$$\Delta(\vartheta) = \int_{t_{Fl}}^{t} \sqrt{g_0/g}\; \dot{\kappa}\; d\tau \qquad (11)$$

t_{Fl} represents the time of the first onset of flow, g_0 is the value of the function g at $\kappa = 0$

$$g(\vartheta,\kappa=0) = g_0(\vartheta)$$

The parameter λ then becomes

$$\lambda = \frac{LC}{4II\left[c(\kappa,\vartheta) + w(\Delta,\vartheta) + \frac{1}{2}\frac{\partial g}{\partial \kappa}(\kappa,\vartheta)\right]}$$

The constitutive equations (6) and (7) now read

$$\dot{\varepsilon}^{el} = \frac{1}{2G(\vartheta)}\left[\dot{\sigma} - \frac{v(\vartheta)}{1+v(\vartheta)}\,\mathrm{tr}\,\dot{\sigma}\, 1\right] + \alpha(\vartheta)\,\dot{\vartheta}\, 1 + \frac{a(\vartheta)}{2}\,\sigma\,\dot{\vartheta} + \left[d(\vartheta)\,\vartheta - \frac{b(\vartheta)}{2G(\vartheta)}\,\mathrm{tr}\,\sigma - \frac{a(\vartheta)\, v(\vartheta)}{2(1+v(\vartheta))}\,\mathrm{tr}\,\sigma\right]\dot{\vartheta}\, 1 \qquad (12)$$

with

$$a(\vartheta) = \frac{d}{d\vartheta}\frac{1}{G(\vartheta)}\,;\quad b(\vartheta) = \frac{d}{d\vartheta}\frac{v(\vartheta)}{1+v(\vartheta)}\,;$$
$$d(\vartheta) = \frac{d}{d\vartheta}\alpha(\vartheta)$$

and

$$\dot{\varepsilon}^{pl} = \frac{LC\,\langle\sigma'-\xi'\rangle}{2\, II\left(c + w + \frac{1}{2}\frac{\partial g}{\partial \kappa}\right)} \qquad (13)$$

where as usual G is the shear modulus, v is the Poisson's ratio and α is the thermal expansion coefficient of an isotropic elastic material.

The following equations refer to the internal variables

$$\dot{\xi} = \frac{(c + w)\; LC\langle\sigma'-\xi'\rangle}{2\, II\,\left(c + w + \frac{1}{2}\frac{\partial g}{\partial \kappa}\right)} \qquad (14)$$

and

$$\dot{\kappa} = \frac{\langle LC\rangle}{2\,\left(c + w + \frac{1}{2}\frac{\partial g}{\partial \kappa}\right)} \qquad (15)$$

According to this, ξ is the translation tensor describing the displacements of the centre of the yield sphere, while κ is the plastic work of the reduced stresses. Both internal variables therefore can be interpreted as physical quantities.

Using "Eq. (1)" and the abbreviation

$$\delta(\kappa,\Delta,\vartheta) = \left(1 + \frac{1}{2\,G(\vartheta)}\left[c(\kappa,\vartheta) + w(\Delta,\vartheta) + \frac{1}{2}\frac{\partial g(\kappa,\vartheta)}{\partial \kappa}\right]\right)^{-1} \qquad (16)$$

"Eq. (12)" can be inverted

$$\dot{\sigma} = 2\,G(\vartheta)\Big[\dot{\varepsilon} + \frac{v(\vartheta)}{1-2v(\vartheta)}\,\mathrm{tr}\,\dot{\varepsilon}\, 1 - \dot{\vartheta}\Big(\frac{\alpha(\vartheta)(1+v(\vartheta))}{1-2v(\vartheta)}\, 1 + \frac{a(\vartheta)}{2}\,\sigma - \frac{b(\vartheta)(1+v(\vartheta))}{2G(\vartheta)(1-2v(\vartheta))}\,\mathrm{tr}\,\sigma\, 1 + \frac{d(\vartheta)\,(1+v(\vartheta))}{1-2v(\vartheta)}\, 1\Big)\Big] + G(\vartheta)\,\frac{\delta(\vartheta)}{II}\,\langle\sigma'-\xi'\rangle\Big[\frac{\partial g}{\partial \vartheta}\,\frac{\dot{\vartheta}}{2G(\vartheta)} - 2(\sigma'-\xi')\cdot\dot{\varepsilon} + a(\vartheta)\,\dot{\vartheta}\,[(\sigma'-\xi')\cdot\sigma]\Big] \qquad (17)$$

It is easy to determine the material functions $G(\vartheta)$, $v(\vartheta)$, and $\alpha(\vartheta)$ and thus also $a(\vartheta)$, $b(\vartheta)$ and $d(\vartheta)$ from experiments in the elastic range at various temperatures. The process-dependent functions $c(\kappa,\vartheta)$, $g(\kappa,\vartheta)$ and $w(\Delta,\vartheta)$ are only connected to each other by "Eq. (16)".

DETERMINATION OF THE FUNCTIONS $c(\kappa,\vartheta)$, $g(\kappa,\vartheta)$ AND $w(\Delta,\vartheta)$

The given functions should be determined by means of results of uniaxial tensile and compression tests. Therefore it should first be assumed that the process paths are isothermic ($\dot{\vartheta} = 0$). Using the expression for the tangent modulus

$$E_t(\Delta,\kappa) = \frac{E\,(1-\delta(\kappa,\Delta))}{1 - \frac{\delta(\kappa)}{3}\,(1-2v)}$$

the following linkage results from "Eq. (16)"

$$c(\kappa) + w(\Delta) + \frac{1}{2}\, g'(\kappa) = \frac{2}{3}\, E\, \frac{E_t(\Delta,\kappa)}{E - E_t(\Delta,\kappa)} \qquad (18)$$

We consider $w(\Delta)$ as a "disturbance" of the linear hardening case which for $w = 0$ will be achieved putting the right hand side of "Eq. (18)" constant

$$c(\kappa) + \frac{1}{2}\, g'(\kappa) = \frac{2}{3}\, E\, \frac{E_{t\infty}}{E - E_{t\infty}} = \text{const.} = c_0 \qquad (19)$$

with $E_{t\infty}$ the tangent modulus, which is asymptotically attained for large strains (Fig. 1).

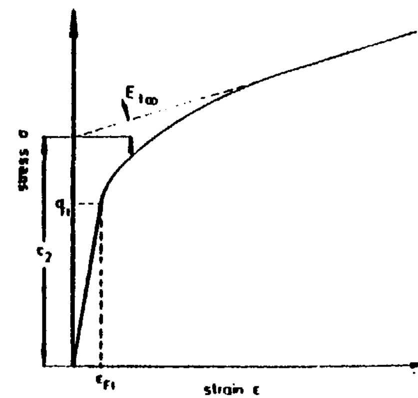

Figure 1: Stress-strain curve in uniaxial tensile test.

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The hardening model presented here should describe a gradually continuous transfer from a purely isotropic to a purely kinematic hardening behaviour. This is possible if the hardening function $g(\kappa)$ fulfils the requirements

$$g(\kappa=0) = g_o > 0$$
$$g'(\kappa) \geq 0 \quad \text{for all} \quad \kappa \in R^+$$
$$\lim_{\kappa\to\infty} g(\kappa) < \infty$$

and if the translation function $c(\kappa)$ fulfils the requirements

$$c(\kappa=0) = 0$$
$$c'(\kappa) \geq 0 \text{ for all } \kappa \in R^+$$
$$\lim_{\kappa\to\infty} c(\kappa) < \infty$$

(It is therefore impossible to describe softening due to mechanical loading). Some functions which fulfil these requirements and "Eq. (19)" are listed below

$c(\kappa)$	$g(\kappa)$	
$c_o\ (1-\exp(-c_1\kappa))$	$g_o + \frac{2c_o}{c_1}\ (1-\exp(-c_1\kappa))$	
$c_o\ (1-\frac{1}{(c_1\kappa+1)^2})$	$g_o + \frac{2c_o\kappa}{c_1\kappa+1}$	(20)
$c_o\ (1-\frac{1}{\cosh^2(c_1\kappa)})$	$g_o + \frac{2c_o}{c_1}\tanh\ (c_1\kappa)$	

hereby c_1, is a parameter which can be chosen freely.

A linear hardening behaviour is only realistic to a certain extent. The term $w(\Delta)$ has been introduced to permit the description of a weak non-linear behaviour. This term is therefore interpreted as a disturbance of the linear hardening behaviour. Using "Eq. (19)" from "Eq. (18)" results

$$w(\Delta) = \frac{2}{3}\ E\ \frac{E_t(\Delta)}{E - E_t(\Delta)} - c_o \qquad (21)$$

As c_o is known, $E_t(\Delta)$ still must be determined.

Let us assume that (Fig. 1)

$$\sigma(\varepsilon) = \begin{cases} E\ \varepsilon & \text{for } \varepsilon < \varepsilon_{Fl} \\ \dfrac{E_{t\infty}\ \varepsilon + c_2}{\varepsilon + \frac{c_2}{E} + (\frac{E_{t\infty}}{E} - 1)\varepsilon_{Fl}}\ \varepsilon & \text{for } \varepsilon \geq \varepsilon_{Fl} \end{cases} \qquad (22)$$

is valid whereby ε_{Fl} is the strain at the onset of flow and c_2 is a constant still to be determined.

For large values of $\varepsilon >> \varepsilon_{Fl}$ we find

$$\sigma(\varepsilon) = E_{t\infty}\ \varepsilon + c_2 = \sigma^*(\varepsilon) \qquad (23)$$

Using this, c_2 can be determined

$$c_2 = \sigma^*(\varepsilon = 0)$$

As $E_t(\varepsilon) = \frac{d\sigma}{d\varepsilon}$, the following holds

$$E_t(\varepsilon) = \begin{cases} E & \text{for } \varepsilon < \varepsilon_{Fl} \\ \dfrac{E_{t\infty}\varepsilon^2 + (2E_{t\infty}\varepsilon + c_2)((\frac{E_{t\infty}}{E} - 1)\varepsilon_{Fl} + \frac{c_2}{E})}{(\varepsilon + \frac{c_2}{E} + (\frac{E_{t\infty}}{E} - 1)\ \varepsilon_{Fl})^2} & \text{for } \varepsilon \geq \varepsilon_{Fl} \end{cases} \qquad (24)$$

The following may be calculated for the uniaxial tensile case

$$\dot{\kappa} = \sqrt{\frac{3}{2}\ g(\kappa)}\ \dot{\varepsilon}^{pl} \qquad (25)$$

Taking the requirement that the translation function should vanish at the beginning of the process, we obtain

$$g_o = \frac{2}{3}\ \sigma_{Fl}^2 \qquad (26)$$

where σ_{Fl} is the initial yield stress in the tensile test. Thus "Eq. (11)" can be written as

$$\Delta = \int_{t_{Fl}}^{t} \sigma_{Fl}\ \dot{\varepsilon}^{pl}\ d\tau = \sigma_{Fl}\ (\varepsilon - \frac{\sigma(\varepsilon)}{E}) \qquad (27)$$

This equation may be solved with respect to ε.

Thus $E_t(\varepsilon(\Delta)) = E_t(\Delta)$ is now known, and $w(\Delta)$ in "Eq. (21)" can be calculated. All functions mentioned above therefore can be determined by means of a uniaxial test using the functions $c(\kappa)$ and $g(\kappa)$ given above "Eq. (24)" and finally "Eq. (21)", whereby c_1 has been adapted to the test results. Multiaxial problems now can be treated promisingly using the functions determined.

The same method can be applied to non-isothermic processes, if suitable functions $g(\kappa,\vartheta)$ can be determined on the basis of test results.

FURTHER ACTIVITIES

The relations thus derived provide a tool which, used together with suitable FEM-codes, also permits the calculation of elastic-plastic processes for complicated structures. First check calculations for relevant experiments - which will be dealt with in the next step of the research programme - have already shown good consistency.

In the second step, constitutive relations to describe rate dependent material behaviour will be developed parallel to the verification of the relations just developed. This will be reported on at a later date.

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