WEAK MIXING ANGLE AND THE
SU(3)_C × SU(3) MODEL ON
M^4 × S^1/(Z_2 × Z'_2)

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and

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We show that the desirable weak mixing angle \( \sin^2 \theta_W = 0.2312 \) at \( m_Z \) scale can be generated naturally in the \( SU(3)_C \times SU(3) \) model on \( M^4 \times S^1/(Z_2 \times Z_2') \) where the gauge symmetry \( SU(3) \) is broken down to \( SU(2)_L \times U(1)_Y \) by orbifold projection. For a supersymmetric model with a TeV scale extra dimension, the \( SU(3) \) unification scale is about hundreds of TeVs at which the gauge couplings for \( SU(3)_C \) and \( SU(3) \) can also be equal in the mean time. For the non-supersymmetric model, \( SU(2)_L \times U(1)_Y \) are unified at order of 10 TeV. These models may serve as good candidates for physics beyond the SM or MSSM.
Grand Unification Theory (GUT) has long been an intriguing proposal for physics beyond the Standard Model (SM) or Minimal Supersymmetric Standard Model (MSSM)[1]. The impressive successes of supersymmetric (SUSY) GUTs include the successful prediction of the weak mixing angle at the electroweak (EW) scale, the unification of three running gauge couplings of the $G_{SM} = SU(3)_C \times SU(2)_L \times U(1)_Y$ group at GUT scale $M_{GUT} \approx 2 \times 10^{16}$ GeV. Besides the proton decay problem, an unsatisfactory feature of this idea is that there exists a SUSY desert for new physics from TeV scale to the GUT scale. Several years ago, gauge coupling unification has been reconsidered in the large extra dimension scenarios. The point of this proposal is that beyond the compactification scale, the KK modes arising from the bulk fields would give radiative corrections to the gauge couplings which would behave as polynomial functions of the energy scale[2]. Gauge coupling unification can be achieved around $10^6$ GeV if the compactification scale is $10^5$ GeV.

On the other hand, the discrete symmetry on the extra space manifold gives another interesting proposal for GUT breaking or gauge symmetry breaking[3, 4, 5]. The doublet-triplet splitting problem and proton decay problem can be solved neatly in this kind of scenario. In addition, the discrete symmetry may not act freely on the extra space manifold. When the discrete symmetry does not act freely on the extra space manifold, there exists a brane at each fixed point, line or hypersurface, where only part of the gauge symmetry and SUSY might be preserved and the SM fermions can be located at[5]. However, the SUSY desert still remains since the compactification scale is close to the 4-dimensional (4D) GUT scale.

In a recent paper[6], Dimopoulos and Kaplan pointed out that the weak mixing angle arising from the breaking of a $SU(3)$ group is around 0.25 while leaving $SU(3)_C$ untouched. New physics scale is expected to be around 3.75 TeV using the SM gauge couplings and beta functions. The idea is quite interesting but the model they proposed is too complicated. Their model involved the $SU(3) \times SU(2) \times U(1)$ gauge group altogether to get the SM $SU(2)_L \times U(1)_Y$ gauge group by Higgs mechanism. One problem is that, as discussed in their paper, to achieve a good value of $\sin^2 \theta_W$ around 0.25, the gauge couplings of the extra $SU(2) \times U(1)$ have to be much larger than that of $SU(3)$ group, which determines the gauge couplings of the $SU(2)_L \times U(1)_Y$ group. Consequently the $U(1)$ gauge group may have Landau pole problem not far beyond the 3.75 TeV.

How to combine above ideas to construct a minimal simple model is an interesting question. As we know, the gauge symmetry breaking by the discrete symmetry on extra space manifold will not reduce the rank of the bulk gauge group if the discrete symmetry is abelian. So, only the $SU(5)$, $SU(4)_C \times SU(2)_L$ and $SU(3)_C \times SU(3)$ gauge symmetries can be broken directly down to $G_{SM}$ without extra $U(1)$. The $SU(5)$ model has been discussed extensively and the $SU(4)_C \times SU(2)_L$ have relatively higher unification scale comparing to TeV. In this letter, we would like to discuss the $SU(3)_C \times SU(3)$ model on the space-time $M^4 \times S^1/(Z_2 \times Z_2')$ in which the correct weak mixing angle $\sin^2 \theta_W = 0.2312$ at the $m_Z$ scale can be obtained naturally with
a TeV scale extra dimension.

We will concentrate on the SUSY model because the discussion of non-SUSY model is similar. Here is our convention. We consider the 5-dimensional space-time as the product of the 4D Minkowski space-time $M^4$, and the circle $S^1$ with radius $R$. The corresponding coordinates for the space-time are $x^\mu$, $(\mu = 0, 1, 2, 3, y \equiv x^5)$. In addition, the orbifold $S^1/(Z_2 \times Z_2')$ is obtained from $S^1$ modded by two equivalent classes: $y \sim -y$ and $y' \sim -y'$ where $y' = y + \pi R/2$. $y = 0, \pi R/2$ are two fixed points. We assume that the bulk gauge symmetry is $G = SU(3)_C \times SU(3)$, and two Higgs hypermultiplets $\Psi_u$ and $\Psi_d$ transforming as $(1, 3)$ and $(1, 3)$ are in the bulk.

The $N = 1$ SUSY theory with gauge group $G$ in 5-dimension have 8 real supercharges, corresponding to $N = 2$ SUSY in 4-dimension. The vector multiplet physically contains a vector boson $A_M$ where $M = 0, 1, 2, 3, 5$, two Weyl gauginos $\lambda_{1,2}$, and a real scalar $\sigma$. In the language of 4D $N = 1$ SUSY, it contains a vector multiplet $V(A_{\mu}, \lambda_1)$ and a chiral multiplet $\Sigma((\sigma + iA_2)/\sqrt{2}, \lambda_2)$ in the adjoint representation. And the 5-dimensional hypermultiplet physically has two complex scalars $\phi$ and $\phi^c$, two Weyl fermions $\psi$ and $\psi^c$, and can be decomposed into two chiral multiplets $\Phi(\phi, \psi)$ and $\Phi^c(\phi^c, \psi^c)$, which transform as conjugate representations of each other under $G$.

We define the $P$ and $P'$ as corresponding operators for the $Z_2$ symmetry transformations $y \rightarrow -y$ and $y' \rightarrow -y'$, respectively. With general action for the 5-dimensional $N = 1$ SUSY gauge theory and its couplings to the bulk hypermultiplets in [8], we obtain the symmetry transformations for the gauge fields, and $\Phi_u$, $\Phi^c_u$ under $P$

\begin{align*}
V(x^\mu, y) &\rightarrow V(x^\mu, -y) = PV(x^\mu, y)P^{-1}, \\
\Sigma(x^\mu, y) &\rightarrow \Sigma(x^\mu, -y) = -P\Sigma(x^\mu, y)P^{-1}, \\
\Phi_u(x^\mu, y) &\rightarrow \Phi_u(x^\mu, -y) = P\Phi_u(x^\mu, y), \\
\Phi^c_u(x^\mu, y) &\rightarrow \Phi^c_u(x^\mu, -y) = -P^*\Phi^c_u(x^\mu, y),
\end{align*}

where $P^*$ is the complex conjugate of $P$. And under $P$, the transformation properties of $\Phi_d$ and $\Phi^c_d$ are similar to those of $\Phi_u$ and $\Phi^c_u$. Similar results hold for $P'$. Because of the $Z_2$ symmetry, the eigenvalues of $P$ and $P'$ must be $\pm 1$. Denoting the general bulk field $\phi$ with parities $(\pm, \pm)$ under $(P, P')$ as $\phi_{\pm\pm}$, the solutions under Fourier-expansion can be found in [7]. Reducing to 4D fields, $\phi_{++}$ and $\phi_{--}$ correspond respectively to the KK modes with masses $2n/R$ and $(2n+2)/R$, $\phi_{+-}$ and $\phi_{-+}$ are the KK modes with masses $(2n+1)/R$ (n is non negative integer). We emphasize that the zero modes are contained only in $\phi_{+++}$ fields. $\phi_{--}$ and $\phi_{-+}$ would vanish at $y = 0$ and $\phi_{++}$ and $\phi_{+-}$ would vanish at $y = \pi R/2$.

We choose the matrix representations for $P$ and $P'$ as $P = \text{diag}(1, 1, 1) \otimes \text{diag}(1, 1, -1)$ and $P' = \text{diag}(1, 1, 1) \otimes \text{diag}(1, 1, 1)$, which are the group elements of $SU(3)_C \times SU(3)$ up to a $Z_2$ phase. With general discussions and results for the gauge symmetry breaking by the discrete symmetry on extra space manifold in [5], we obtain that under $P$, the $SU(3)_C \times SU(3)$ gauge symmetry is broken down to $G_{SM}$ on the 3-brane at $y = 0$ for all the KK modes and in the bulk
for zero modes. And under $P'$, the 4D $N = 2$ SUSY is broken down to $N = 1$ for zero modes. The 3-branes at $y = 0$ and $y = \pi R/2$ preserve only 4D $N = 1$ SUSY due to the $Z_2$ projections. These results can also be obtained by noticing the parities of $V$ and $\Sigma$ for the $SU(3)$ group, and of $\Phi_u$ and $\Phi_u^c$ (similar for $\Phi_d$ and $\Phi_d^c$) under $(P, P')$:

\[ V: \begin{pmatrix} (+,+) & (+,+) & (-,+), \\ (+,-) & (-,+) & (+,+), \end{pmatrix} \]

\[ \Sigma: \begin{pmatrix} (-,+) & (-,-) & (+,+), \\ (-,-) & (-,+) & (+,+) \end{pmatrix} \]

\[ \Phi_u: \begin{pmatrix} (+,+) \\ (+,+) \\ (-,-) \end{pmatrix}, \quad \Phi_u^c: \begin{pmatrix} (-,-) \\ (+,-) \end{pmatrix} \]

For $SU(3)_C$ group, $V$ and $\Sigma$ have parities $(+,+)$ and $(-,-)$, respectively. Then, the MSSM quarks and leptons can be put on the observable 3-brane at $y = 0$, and the Higgs doublets $H_u$ and $H_d$ are obtained from the zero modes of $\Phi_u$ and $\Phi_d$. The Yukawa terms are similar to those in the MSSM for on the observable 3-brane, we only have Higgs doublets $H_u$ and $H_d$ for all the KK modes.

$SU(2)_L$ group is generated by $T^a \approx \sigma^a/2$ ($a = 1, 2, 3$) of the corresponding $SU(3)$ group, where $\sigma^a$ is the Pauli matrix. $U(1)_Y$ is generated by $\sqrt{3} T^8$, so, the hypercharges of the remaining two Higgs doublets $H_u$ and $H_d$ are 1/2 and $-1/2$, respectively. From the $SU(3)$ relation we would get $g_1 = g_2/\sqrt{3}$. On the observable 3-brane at $y = 0$, we can then put the left handed quark and lepton fields in the fundamental representation of the $SU(2)_L$ group while the right handed quarks and leptons are singlets. Knowing that the total $U(1)_Y$ charges for each Yukawa term must be zero, together with four anomaly-free conditions: $(3C)^2 Y, (2L)^2 Y, Y^3$ and $(grav)^2 Y$, we can determine the correct hypercharges for $Q, U, D, L$ and $E$ fields. Since the hypercharges of two Higgs doublets are quantized as they are from triplets of $SU(3)$ group, an understanding of the charge quantization is achieved as a result of the consistency of our setup.

There exist two energy scales in our models: the compactification scale $M_C = 1/R$, and the $SU(3)$ unification scale which is considered as the cutoff scale $\Lambda$ in the theory. Because there might exist additional $SU(2)_L$ and $U(1)_Y$ kinetic terms localized on the observable 3-brane which may violate the $SU(3)$ gauge coupling relation, the cutoff scale might be much larger than the compactification scale if those localized kinetic terms were not small. We will neglect those extra kinetic terms for two reasons: (1) It is natural to set those localized kinetic terms to zero at tree level in the fundamental theory, so, they can only be generated at loop level as counter terms and are very small if the theory is weakly coupled at cutoff scale; (2) If the gauge interaction is strong coupled at $\Lambda$ and $M_C \sim 0.01 \Lambda$, the effects of those kinetic terms are very small [9].

First, let us study the scenario with $M_C \sim \Lambda$. Assuming the sparticle mass scale, $M_S$, in the range $200 - 1000$ GeV, using the MSSM beta function $(-11, -1, 3)$ above this scale and the SM
beta function \((-41/6, 19/6, 7)\) down to the \(m_Z\) scale, in the limit that \(\sin^2 \theta_W = 0.25\) is realised at \(M_C\) or \(\Lambda\), we find out that \(M_C \sim 77 - 14\) TeV.

Second, we discuss the scenario with \(M_C \sim 0.01\Lambda\). For \(M_C < \mu < \Lambda\), we should include the massive KK modes in counting the radiative corrections to gauge couplings. There are two kinds of beta functions, \(b_o\) and \(b_e\), from the KK modes of bulk fields with masses \(2n/R\) and \((2n + 1)/R\), respectively. The radiative corrected gauge couplings above the \(M_C\) scale is

\[
\alpha_i^{-1}(\mu) = \alpha_i^{-1}(M_C) + \frac{b_i}{2\pi} \ln \frac{\mu}{M_C} + \sum_n \frac{1}{2\pi} \left( b_o \ln \frac{\mu}{M_{2n-1}} + b_e \ln \frac{\mu}{M_{2n}} \right),
\]

where \(\alpha_i = g_i^2/4\pi\), \(b_e = (-2, 2, 6)\) and \(b_o = (14, 2, 0)\). One obtains that the compactification scale is several TeVs in this kind of scenario.

Third, because we include the \(SU(3)_C\) gauge symmetry in the bulk, it is interesting to study whether there exists the scenario in which the \(SU(3)\) unification scale or \(\Lambda\) is the scale at which the gauge couplings for \(SU(3)_C\) and \(SU(3)\) are also equal in the mean time. And if the 5D \(SU(3)\) theory is strong coupled at cutoff scale, it is reasonable to assume that the gauge coupling for \(SU(3)\) is similar to that of \(SU(3)_C\) at \(\Lambda\). To have an estimate, we extract the leading \(SU(3)\) symmetric contributions given by the KK modes, that is \(\sum k \frac{b_k}{4\pi} (\ln \frac{\mu}{M_k} + \ln \frac{\mu}{M_k^*})\). Similar to the case in [2], for \(\mu \gg M_C\), we may resum this leading contribution to be a power law running, that is \(\frac{b_k}{4\pi} (\frac{\mu}{M_C} - 1)\). Using the beta function listed above, we get a rough estimate of the cutoff scale \(\Lambda\):

\[
\frac{1}{2\pi} \left( \frac{\Lambda}{M_C} - 1 \right) \approx \alpha_2^{-1} - \alpha_3^{-1},
\]

where \(\alpha_{2,3}\) are the running gauge couplings for \(SU(2)_L\) and \(SU(3)_C\) at scale \(M_C\). The difference in the power law running is given by the KK tower from the Higgs fields which we choose living in the bulk. \(\frac{\Lambda}{M_C}\) is about 100 – 130 for \(M_S \sim 200 – 1000\) GeV and \(M_C < 10\) TeV. At first sight this large gap between the cutoff scale and \(M_C\) requires less than TeV \(M_C\) to meet the purpose of unifying simultaneously \(SU(2)_L\) and \(U(1)_Y\), because the \(SU(3)\) asymmetric relative running given by the zero mode is \(\frac{b_2}{2\pi} \ln \frac{\Lambda}{M_C}\) which is about \(-2.03\) for \(\Lambda/M_C = 120\) \((\alpha_i^{-1}/3 - \alpha_2^{-1} \approx 3.21\) at \(m_Z\) scale). However, there are also \(SU(3)\) asymmetric radiative corrections given by the massive KK modes. It is \(\sum k \frac{b_k}{4\pi} \ln \frac{M_{2k}}{M_{2k-1}} = \frac{b_k}{4\pi} \ln ^{\frac{M_{2k}}{1.3}} \cdot \cdots\). This relative running between \(3\alpha_1\) and \(\alpha_2\) is completely given by the \(U(1)_Y\) part and is about 1.11 for \(\Lambda/M_C = 120\). Detailed calculation using logarithmic running including step by step massive KK states shows that in the whole range \(M_S \sim 200 – 1000\) GeV, \(3\alpha_1\), \(\alpha_2\) and \(\alpha_3\) can meet simultaneously for \(M_C \sim 9.3 - 1.5\) TeV with \(\alpha^{-1} \sim 60 - 66\) and \(\Lambda/M_C\) around 110. And the requirement that \(3\alpha_1\), \(\alpha_2\) and \(\alpha_3\) meet simultaneously further constrains the model, i.e., \(M_C\) and \(\Lambda\) will be correlated if the SUSY threshold is known. In Fig. 1, we give a plot for \(M_S = 400\) GeV showing the \(\alpha^{-1}\) versus the energy scale. We may see clearly the running behaviours of the three gauge couplings. \(3\alpha_1\),
\( \alpha_2 \) and \( \alpha_3 \) meet with a value of about 1/63 at \( \Lambda \approx 494 \) TeV. In this case \( M_C \approx 4.5 \) TeV and \( \Lambda/M_C \approx 110 \). We may see that for \( \mu \gg 10 \) TeV, the difference between \( 3\alpha_1 \) and \( \alpha_2 \) is quite small (the discrepancy is less than 0.2% for \( \mu > 100 \) TeV).

The SUSY breaking can be done by the Scherk-Schwarz mechanism or by gauge mediated SUSY breaking. For the first approach, one can solve the \( \mu \) problem, and get the gaugino masses, the \( \mu B \) terms and Higgs masses [4]. The radiative electroweak (EW) symmetry breaking may also be realized. For the second approach, the 3-brane at the fixed point \( y = \pi R/2 \) can be considered as hidden sector for SUSY breaking. The SUSY breaking effects may be communicated to the observable sector via gauge-mediated mechanism. This may be realized if there is a SUSY breaking gauge singlet field \( S \) living in the bulk. Interaction like \( Tr(SW^aW_a) \) gives gaugino masses via F-term. \( \mu \) term, \( \mu B \) and Higgs mass terms can also be generated via the interaction of \( \Phi \) and \( S \)[10]. Squarks and sleptons can then obtain their masses through the radiative corrections. EW symmetry breaking may be achieved in the usual way via loop corrections at the EW scale because of the large top quark Yukawa coupling[11].

Similarly, we can construct a non-SUSY model on \( M^4 \times S^1/(Z_2 \times Z_2') \). The triplet scalar Higgs field can live in the bulk, too. The SM fermions are confined on the observable 3-brane. Because the relative running between \( 3\alpha_1 \) and \( \alpha_2 \) are much faster than that in the SUSY case for \( \mu < M_C \), and the relative power law running between \( 3\alpha_1 \) and \( \alpha_3 \) or between \( \alpha_2 \) and \( \alpha_3 \) is 1/12 times slower for \( \mu > M_C \), the gauge couplings for \( SU(3)_C \) and \( SU(3) \) can not be equal in the mean time at the \( SU(3) \) unification scale. Since \( \sin^2 \theta_W = 0.25 \) is expected to be reached using the SM beta functions at about 3.75 TeV, we should have \( M_C < 3.5 \) TeV for this case. Cutoff scale is of order 10 TeV. Another problem of this case is that the masses of Higgs zero modes might be a problem. Studies on EW symmetry breaking, if it's accounted for solely by the radiative corrections (for which 1% fine tuning is clearly needed), would give us the strong

Figure 1: \( \alpha^{-1} \) versus energy scale. From below three lines are for \( \alpha_3^{-1} \), \( \alpha_2^{-1} \) and \( \alpha_1^{-1}/3 \).
constraints on this setup.

Furthermore, we can discuss the $SU(3)_C \times SU(3)$ model on $M^4 \times S^1/Z_2$ with gauge-Higgs unification. However, the weak mixing angle $\sin^2 \theta_W$ is expected about 0.75 at the cutoff scale.

The constraints on $M_C$ from the current high energy experiment data are model dependent, and one need to study it in detail. It is possible that the present knowledges might give the strong constraints on the non-SUSY model for $M_C < 3.5$ TeV. Moreover, the extra $SU(2)_L$ doublets in the off-diagonal $\Sigma$ ($A_5$ for non-SUSY case), and the third components of the $\Phi_u$ and $\Phi_d$, may be produced at LHC via s-channel $W^\pm, Z^0$ and $\gamma$ exchange for $M_C$ is several TeVs, and the decay modes are model dependent. For the $SU(2)_L$ doublets in the off-diagonal $V$ ($A_\mu$ for non-SUSY case) and the third components of the $\Phi_u$ and $\Phi_d$, there might exist the derivative interactions with the leptons, so, they might also be produced at LHC and decay into leptons. Of course, the decay channels of the KK modes of bulk fields are dependent on the loop corrections to the masses of KK modes, which should be considered in the collider test of the models.

Conclusion. We have presented the $SU(3)_C \times SU(3)$ model on the space-time $M^4 \times S^1/Z_2 \times Z_2'$ with natural explanation for the weak mixing angle which is a consequence of the $SU(3)$ symmetry breaking in our setup. For the SUSY model with a TeV scale extra dimension, the $SU(3)$ unification scale is about hundreds of TeVs at which the gauge couplings for $SU(3)_C$ and $SU(3)$ can be equal in the mean time. For the non-SUSY model, the $SU(3)$ unification of $SU(2)_L \times U(1)_Y$ may be realized at tens of TeVs at which the gauge couplings for $SU(3)_C$ and $SU(3)$ can not be equal simultaneously. Therefore, the supersymmetric desert does not appear in our models, which is an unsatisfactory feature in the usual GUTs. We stress that both models presented in this letter, supersymmetric or not, are of strong predictive power because of their clean setup. Studying the non-supersymmetric model, one may soon reveal whether it's realistic or not. For the supersymmetric model, we can break the SUSY by Scherk-Schwarz mechanism or gauge mediated SUSY breaking, and might test it at LHC. We also comment on the charge quantization, radiative EW symmetry breaking, and the models with gauge-Higgs unification. We conclude that this kind of interesting models may be good candidates for the minimal extensions of the SM or MSSM, and their phenomenology deserves further study.

After sending our letter, we noticed the papers [12, 13] which discuss similar issue.

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References


