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LAVRENTIEV REGULARIZATION METHOD FOR NONLINEAR ILL-POSED PROBLEMS

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Abstract

In this paper we shall be concerned with Lavrentiev regularization method to reconstruct solutions x_0 of nonlinear ill-posed problems $F(x) = y_0$, where instead of y_0 noisy data $y_\delta \in X$ with $\|y_\delta - y_0\| \leq \delta$ are given and $F : X \to X$ is an accretive nonlinear operator from a real reflexive Banach space X into itself. In this regularization method solutions x_α^δ are obtained by solving the singularly perturbed nonlinear operator equation $F(x) + \alpha(x - x^*) = y_\delta$ with some initial guess x^*. Assuming certain conditions concerning the operator F and the smoothness of the element $x^* - x_0$ we derive stability estimates which show that the accuracy of the regularized solutions is order optimal provided that the regularization parameter α has been chosen properly.

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1 Introduction

In this paper we study the stable solution of nonlinear ill-posed problems

$$F(x) = y_0 \tag{1}$$

where $F : X \to X$ is a nonlinear operator from a real reflexive Banach space X into itself. Throughout this paper we assume that X is a real reflexive Banach space with a dual X^* that is strictly convex and F is an accretive operator, that is, there holds

$$(F(x_2) - F(x_1), U(x_2 - x_1)) \geq 0 \quad \text{for all } x_1, x_2 \in X \tag{2}$$

where the notation (z, f) is the value of the linear functional $f \in X^*$ on the element $z \in X$. We further assume throughout that (1) has a unique solution $x_0 \in X$ and that $y_\delta \in X$ are the available noisy data with

$$\|y_\delta - y_0\| \leq \delta \tag{3}$$

known noise level $\delta > 0$, $\delta \to 0$ and that F possesses a locally uniformly bounded Fréchet-derivative $F'(.)$ in a ball $B_r(x_0)$ of radius r around $x_0 \in X$.

The problem of solving (1) is, in general, ill-posed (see [35]). By this we mean that the solutions do not depend continuously on the data. The numerical treatment of nonlinear ill-posed problems requires the application of special regularization methods. A well known and effective technique is Tikhonov regularization. In this method a regularized approximation x_α^δ is obtained by minimizing Tikhonov's functional

$$J_{\alpha,\delta}(x) = \|F(x) - y_\delta\|^2 + \alpha\|x - x^*\|^2$$

with some guess $x^* \in X$ (see [6]) and some properly chosen regularization parameter $\alpha > 0$. If F is Fréchet differentiable, then the regularized approximation x_α^δ satisfies the necessary condition for minimum (cf. e.g., [17]), i.e.

$$F'(x)^*[F(x) - y_\delta] + \alpha(x - x^*) = 0$$

where $F'(x)^*$ is the adjoint of the Fréchet-derivative $F'(x)$. Tikhonov regularization theory for nonlinear problems has been studied by some authors (cf, e.g., [6, 22, 23, 24, 30, 31, 32, 33, 13, 27, 28, 11, 12, 15, 29]). While for linear ill-posed problems the regularization theory is rather complete (cf, e.g., [4, 10, 7, 16, 3, 21]).

In the case of monotone operators F the least squares minimization can be avoided and one can use the simpler regularized equation

$$F(x) + \alpha(x - x^*) = y_\delta. \tag{4}$$

This method, in which the regularized approximation x_α^δ is obtained by solving the singularly perturbed operator equation (4), is called the Lavrentiev regularization method (see [18]), or sometimes, the singular perturbation method (see [19]). In addition that F is a linear operator, the regularized approximation x_α^δ of equation (4) is given by

$$x_\alpha^\delta = (F + \alpha I)^{-1}(y_\delta + \alpha x^*).$$

Hence, in this case (4) reduces to the well Lavrentiev regularization method for linear ill-posed problems $F(x) = y$, $F \in L(X, X)$, which has been extensively studied by many authors, specially for solving Volterra integral equations of the first kind (cf. e.g. [8, 9, 26]). Here the advantage is that Lavrentiev regularization preserves the natural evolutionary structure of Volterra equations and therefore, leads to quick and simple numerical procedures.

While for nonlinear ill-posed problems, there are still many open problems for the regularization theory; recently, some authors have studied Lavrentiev regularization method solving (1) containing monotone operators in Hilbert spaces (cf. e.g., [19, 14, 34]) and earlier by Al'ber [2].

The plan of this paper is as follows. In section 2 we derive error bounds for $\|x_\alpha^\delta - x_0\|$ and show that under certain conditions concerning the nonlinear operator F and concerning the smoothness of the element $x^* - x_0$, the accuracy of Lavrentiev regularization method is order optimal provided that α is chosen a priori. In section 3 we discuss a posteriori rule of choosing the regularization parameter α which is the solution of the nonlinear scalar equation

$$\|\alpha(F'(x_\alpha^\delta) + \alpha I)^{-1}[F(x_\alpha^\delta) - y_\delta\| = C\delta$$

with constant $C > 1$. We prove that under certain assumptions this equation has a unique solution $\alpha = \alpha(\delta)$ which guarantees order optimal error bounds for $\|x_\alpha^\delta - x_0\|$. In section 4 we briefly discuss an example to illutrate the assumptions required in the foregoing sections.

2 Lavrentiev regularization with priori parameter choice

Throughout this and the next section we shall use the notations

$$A = F'(x_0), \quad A_\alpha^\delta = F'(x_\alpha^\delta)$$

$$M_\alpha = \alpha(A + \alpha I)^{-1}, \quad M_\alpha^\delta = \alpha(A_\alpha^\delta + \alpha I)^{-1}.$$

Since F is an accretive operator, it follows that the operators A, A_α^δ, M_α, M_α^δ, respectively, are also accretive. It is well known that the dual mapping $U : X \to X^*$ holds the following properties

$$\|U(x)\| = \|x\|, \ \|\left(x, U(x)\right)\| = \|U(x)\|\|x\| = \|x\|^2, \text{ for all } x \in X.$$

Moreover, by the stated assumption of X, X^*, it follows that the dual mapping is single-valued, monotone, hemicontinuous, uniformly continuous on bounded subsets of X(see [36]).

First, we discuss the existence and uniqueness of the regularized approximation x_α^δ of problem (4).

Theorem 2.1 *Under the stated assumption, the regularized problem (4) has a unique solution* x_α^δ *in the ball* $B_r(x_0)$ *of radius* r *around* x_0 *in* X *with* $r = \|x^* - x_0\| + \delta/\alpha$.

Proof. By the stated assumption on X, X^*, F, it follows that equation (4) has a solution (see [5, 36]). Moreover, equation (4) has a unique solution. Indeed, suppose that x_1, x_2 are solutions of (4), then

$$F(x_1) + \alpha(x_1 - x^*) = y_\delta \tag{5}$$

$$F(x_2) + \alpha(x_2 - x^*) = y_\delta \tag{6}$$

Subtracting equations (5) and (6) yields $F(x_2) - F(x_1) + \alpha(x_2 - x_1) = 0$ and scalar multiplication by $U(x_2 - x_1)$, gives

$$(F(x_2) - F(x_1), U(x_2 - x_1)) + \alpha\|x_2 - x_1\|^2 = 0.$$

Due to the assumption (2) the first summand on the left-hand side is non-negative, hence $x_2 = x_1$. Let x_α^δ denote a solution of (4), we have

$$F(x_\alpha^\delta) + \alpha(x_\alpha^\delta - x^*) = y_\delta.$$

Consequently, $F(x_\alpha^\delta) - F(x_o) + \alpha(x_\alpha^\delta - x^*) = y_\delta - y_0$ and scalar multiplication by $U(x_\alpha^\delta - x_o)$, gives

$$\left(F(x_\alpha^\delta) - F(x_0), U(x_\alpha^\delta - x_0)\right) + \alpha\left(x_\alpha^\delta - x^*, U(x^\delta - x_0)\right) = \left(y_\delta - y_0, U(x^\delta - x_0)\right).$$

Using the accretive property (2) and (3) we obtain

$$\|x_\alpha^\delta - x_0\| \le \|x_0 - x^*\| + \delta/\alpha.$$

Hence, $x_\alpha^\delta \in B_r(x_0)$.
The proof is complete.

Theorem 2.2 *Assume the accretive property (2). Let* x_α *be the (unique) solution of the singularly perturbed operator equation*

$$F(x) + \alpha(x - x^*) = y_0 \tag{7}$$

and x_α^δ *the (unique) solution of the singularly perturbed operator equation (4). Then,*

$$\|x_\alpha^\delta - x_\alpha\| \le \delta/\alpha. \tag{8}$$

Proof. Since x_α^δ, x_α are respectively solutions of (4), (7), it follows that

$$F(x_\alpha^\delta) - F(x_\alpha) + \alpha(x_\alpha^\delta - x_\alpha) = y_\delta - y_0.$$

Scalar multiplication by $U(x_\alpha^\delta - x_\alpha)$, gives

$$\left(F(x_\alpha^\delta) - F(x_\alpha), U(x_\alpha^\delta - x_\alpha)\right) + \alpha\left(x_\alpha^\delta - x_\alpha, U(x^\delta - x_\alpha)\right) = \left(y_\delta - y_0, U(x_\alpha^\delta - x_\alpha)\right).$$

Using the accretive property (2) and (3) we obtain

$$\|x_\alpha^\delta - x_\alpha\| \le \delta/\alpha.$$

Theorem 2.3 *Assume the accretive property (2). Let* x_α *be the (unique) solution of equation (7). Then,*

$$x_\alpha \to x_0 \quad \text{as } \alpha \to 0.$$

Proof. Since x_α is a solution of equation (7) we have

$$F(x_\alpha) - F(x_0) + \alpha(x_\alpha - x^*) = 0.$$

Consequently,

$$(F(x_\alpha) - F(x_0), U(x_\alpha - x_0)) + \alpha(x_\alpha - x^*, U(x_\alpha - x_0)) = 0.$$

Due to the assumption (2) the first summand on the left-hand side is non-negative. By neglecting this summand we obtain

$$(x_\alpha - x_0, U(x_\alpha - x_0)) + (x_0 - x^*, U(x_\alpha - x_0)) \leq 0.$$

Consequently,

$$\|x_\alpha - x_0\| \leq \|x^* - x_0\|. \tag{9}$$

Hence, the sequence (x_α) is bounded in the real reflexive Banach X; therefore, there exists a subsequence (x_β) that weakly converges to $\bar{x} \in X$ as $\beta \to 0$. We shall prove that $F(\bar{x}) = y_0$, i.e., that $\bar{x} = x_0$. Let x be any fixed element of X, then

$$\begin{aligned}(F(x) - y_0, U(x - x_\beta)) &= (F(x) - F(x_\beta), U(x - x_\beta)) - \beta\,(x_\beta - x^*, U(x - x_\beta)) \\ &\geq -\beta\,(x_\beta - x^*, U(x - x_\beta)). \end{aligned} \tag{10}$$

Let $\beta \to 0$. Since the sequences $\|x_\beta\|$ and $\|U(x - x_\beta)\| = \|x - x_\beta\|$ are bounded, when we take the limit in (10) we obtain

$$(F(x) - y_0, U(x - \bar{x})) \geq 0, \text{ for all } x \in X.$$

Consequently,

$$(F(x) - F(\bar{x}), U(x - \bar{x})) + (F(\bar{x}) - y_0, U(x - \bar{x})) \geq 0, \text{ for all } x \in X.$$

Due to the assumption (2) the first summand on the left-hand side is non-negative. By neglecting this summand we obtain

$$(F(\bar{x}) - y_0, U(x - \bar{x})) \geq 0, \text{ for all } x \in X.$$

Hence $F(\bar{x}) = y_0$. Since the solution of equation (1) is unique, we obtain $\bar{x} = x_0$. On the other hand, since x_β is a solution of equation (7) we have

$$F(x_\beta) - F(x_0) + \beta(x_\beta - x^*) = 0.$$

Scalar multiplication by $U(x_\beta - x_0)$, gives

$$(F(x_\beta) - F(x_0), U(x_\beta - x_0)) + \beta\,(x_\beta - x^*, U(x_\beta - x_0)) = 0.$$

Due to the assumption (2) the first summand on the left-hand side is non-negative. By neglecting this summand we obtain $(x_\beta - x^*, U(x_\beta - x_0)) \le 0$. Consequently,

$$\|x_\beta - x_0\|^2 \le (x^* - x_0, U(x_\beta - x_0))\,. \tag{11}$$

Since the dual mapping is sequentially weakly continuous in X and $U(0) = 0$ (see [36]), from (11) it follows that

$$\|x_\beta - x_0\| \to 0, \quad \beta \to 0.$$

The uniqueness of limit implies that $x_\alpha \to x_0,\ \alpha \to 0$. The proof is complete.

By Theorem 2.2 and Theorem 2.3, we obtain the following result

Theorem 2.4 *Under the stated assumption, $x_\alpha^\delta \to x_0$, as $\alpha \to 0, \delta/\alpha \to 0$.*

The estimate of (9) shows that $x_\alpha \in B_r(x_0)$ with $r = \|x^* - x_0\|$. In order to derive error bounds in terms of α, certain source condition and certain nonlinearlity condition are needed. This we shall discuss below.

Theorem 2.5 *Assume the following conditions hold:*

(a) There exists $w \in X$ satisfying $x^* - x_0 = F'(x_0)w$.

(b) The Fréchet-derivative $F'(.)$ is locally Lipschitz in a ball $B_r(x_0)$ of radius r around $x_0 \in X$, that is, there exists a Lipschitz constant $L \ge 0$ such that

$$\|F'(x) - F'(x_0)\| \le L\|x - x_0\| \quad \textit{for all} \quad x \in B_r(x_0) \textit{ with } r = \|x^* - x_0\|.$$

Let x_α be the (unique) solution of the singularly perturbed equation (7). Then, for all α,

$$\|x_\alpha - x_0\| \le \left(\|w\| + \frac{L}{2}\|w\|^2\right)\alpha. \tag{12}$$

Proof. We introduce the notations

$$z_\alpha = x_\alpha - x_0 \text{ and } B = AM_\alpha = M_\alpha A$$

with $M_\alpha = \alpha(A + \alpha I)^{-1}$. Since x_α is a solution of equation (7) and (a), it follows that

$$\begin{aligned} F(x_\alpha) - F(x_0 + Bw) + \alpha(z_\alpha - Bw) &= F(x_0) - F(x_0 + Bw) + \alpha Aw - \alpha Bw \\ &= F(x_0) + ABw - F(x_0 + Bw). \end{aligned}$$

Scalar multiplication by $U(z_\alpha - Bw)$ we obtain

$$(F(x_\alpha) - F(x_0 + Bw), U(z_\alpha - Bw)) + \alpha\,(z_\alpha - Bw, U(z_\alpha - Bw))$$
$$= (F(x_0) + ABw + F(x_0 + Bw), U(z_\alpha - Bw))\,.$$

By the assumption (2) the first summand on the left-hand side is non-negative and

$$\|U(z_\alpha - Bw)\| = \|z_\alpha - Bw\| \text{ and } (z_\alpha - Bw, U(z_\alpha - Bw)) = \|z_\alpha - Bw\|^2,$$

we obtain

$$\alpha\|z_\alpha - Bw\|^2 \le \|F(x_0) + ABw - F(x_0 + Bw)\|\|z_\alpha - Bw\|.$$

Due to the assumption (b), we have

$$\|F(x) - F(x_0) - F'(x_0)(x - x_0)\| \le \frac{L}{2}\|x - x_0\|^2.$$

By applying this property with $x = x_0 + Bw$ and $\|B\| \le \alpha$ (see [20]), we obtain

$$\|z_\alpha - Bw\| \le \frac{L}{2}\alpha\|w\|^2.$$

Since $\|z_\alpha\| \le \|z_\alpha - Bw\| + \|Bw\|$, it follows that

$$\|x_\alpha - x_o\| \le \left(\|w\| + \frac{L}{2}\|w\|^2\right)\alpha.$$

Theorem 2.6 *Assume the following conditions hold:*

(a') There exists $w \in X$ satisfying $x^* - x_0 = F'(x_0)^p w,\ p \in (0, 1]$.

(b') There exists a constant $k_0 \ge 0$ such that for all $x \in B_r(x_0)$ with $r = \|x^* - x_0\|$ and $z \in X$ there exists some element $k(x, x_0, z) \in X$ with property

$$\left[F'(x) - F'(x_0)\right](z) = F'(x_0)k(x, x_0, z) \text{ and } \|k(x, x_0, z)\| \le k_0\|z\|.$$

Let x_α be the (unique) solution of the singularly perturbed equation (7). Then, for all α,

$$\|x_\alpha - x_0\| \le (1 + k_0)\|w\|\alpha^p. \tag{13}$$

Proof. We introduce the operator $N_\alpha = \int_0^1 F'(x_0 + t(x_\alpha - x_0))\,dt$. From the mean value theorem in integral, we have $F(x_\alpha) - F(x_0) = N_\alpha(x_\alpha - x_0)$. Since x_α is a solution of equation (7) we have

$$F(x_\alpha) - F(x_0) + \alpha(x_\alpha - x^*) = 0.$$

Consequently,

$$N_\alpha(x_\alpha - x_0) + \alpha(x_\alpha - x_0) = \alpha(x^* - x_0).$$

Due to (a') we have $x^* - x_0 = A^p w$. Furthermore, from the assumpion (2), it follows that the inverse operators $(N_\alpha + \alpha I)^{-1}$ and $(A + \alpha I)^{-1}$, respectively, exist. Consequently,

$$\begin{aligned} x_\alpha - x_0 &= \alpha(N_\alpha + \alpha I)^{-1}(x^* - x_0) \\ &= \alpha(A + \alpha I)^{-1}A^p w + \alpha\left[(N_\alpha + \alpha I)^{-1} - (A + \alpha I)^{-1}\right]A^p w \\ &= M_\alpha A^p w + (N_\alpha + \alpha I)^{-1}(A - N_\alpha)M_\alpha A^p w. \end{aligned}$$

By the assumption (b'), it follows that

$$\|x_\alpha - x_0\| \le \|M_\alpha A^p w\| + k_0\|(N_\alpha + \alpha I)^{-1}N_\alpha\|\|M_\alpha A^p w\|.$$

Using the estimate $\|M_\alpha A^p\| \le \alpha^p$ (see [20, 25]), we obtain (13).

Remark. Using the estimate of (8) and the results of Theorem 2.5 and Theorem 2.6, we obtain that for the a priori parameter choice $\alpha = k(\delta/\|w\|)^{1/p+1}$ there holds the order optimal error estimate

$$\|x_\alpha^\delta - x_0\| \le c\|w\|^{1/p+1}\delta^{p/p+1}$$

where $c = c(k)$ is a constant which does not depend on δ, α and $\|w\|$.

We also look at the sample of the given error estimate in [34]. In this paper the assumption (2) has been replaced by the stronger assumption that the operator F is monotone in a Hilbert space.

Under the assumptions (a) and (b), the error bounds $\|x_\alpha - x_0\|$ in [3, 1] have the form

$$\|x_\alpha - x_0\| \le \frac{2\|w\|\alpha}{2 - L\|w\|}$$

and in addition requires the smallness condition $L\|w\| < 2$. The error bound in [19] has the form $\|x_\alpha - x_0\| \le c\sqrt{\alpha}$ with some constant c and requires also the smallness condition $L\|w\| \le 2$. This smallness condition is not necessary for our estimate of (13).

Some estimate for $\|x_\alpha^\delta - x_0\|$ has recently been given in [14]. In this paper $F'(x_0)$ is accretive, i.e. $Re\,(F'(x_0)x, x) \ge 0$ for all x in a Hilbert space X. The error estimate has (for the a priori parameter choice $\alpha = k\sqrt{\delta}$) the form

$$\|x_\alpha^\delta - x_0\| \le \frac{k}{L}\left(1 - \sqrt{1 - \frac{2L}{k^2}(1 + k^2\|w\|)}\right)\sqrt{\delta}$$

and requires the smallness condition $L\|w\| < \frac{1}{2}$ and some additional restriction for the constant k of the form $k > \sqrt{2L}/\sqrt{1 - 2L\|w\|}$.

Let us compare our order optimal error bounds with corresponding bounds Tikhonov regularization. For Tikhonov regularization instead of (a) the source condition

$$x^* - x_0 = \left[F'(x_0)^* F'(x_0)\right]^{p/2} w, \quad w \in X,\ p > 0$$

has been exploited. Then it can be shown that in the case of a priori parameter choice $\alpha \sim \delta^{2/p+1}$ order optimal error bounds $\|x_\alpha^\delta - x_0\| \sim \delta^{p/p+1}$ hold:

(i) for $p \in [1,2]$ provided (b) and the smallness condition $L\|w\| < 1$ are satisfied (see [7, 6]);

(ii) for $p \in (0,1]$ provided some additional conditions concerning the non-linear operator F similar to condition (b') are satisfied (see [11]).

Hence for Tikhonov regularization order optimal error bounds can be guaranteed for the larger range $p \in (0,1]$. Lavrentiev regularization is more simple and does not require the smallness conditions $L\|w\| < 1$.

3 Lavrentiev regularization with posteriori parameter choice

Throughout this section we use the notations which have been introduced in section 2 and we shall briefly state the given assumptions (a), (b), (a'), (b') in section 2. We shall study a posteriori rule for choosing the regularization parameter α in Lavrentiev regularization method (4) as follows

$$\rho(\alpha) := \|\alpha(F'(x_\alpha^\delta) + \alpha I)^{-1} \left[F(x_\alpha^\delta) - y_\delta\right] \| = C\delta \tag{14}$$

where x_α^δ is the (unique) solution of the singularly perturbed operator equation (4). This posteriori rule has been studied in [34]. In this paper the assumption of the operator F is monotone in a Hilbert space.

Let us start our study with the justification of rule (14).

Lemma 3.1 *Assume the accretive property (2) and $C > 1$. If the initial guess x^* satisfies $\|F(x^*) - y_\delta\| > C\delta$, then there exists a solution $\alpha = \alpha(\delta)$ of equation (14) with*

$$\alpha \geq \alpha_0 := \frac{C-1}{\|x^* - x_0\|}\delta. \tag{15}$$

Proof. First, we show that $\rho(\alpha)$ is continuous for $\alpha \in (0, +\infty)$. By Theorem 2.1, we have x_α^δ in $B_r(x_0)$ with $r = \|x^* - x_0\| + \delta/\alpha$. We consider (4) with positive regularization parameters α and β, respectively, then we obtain

$$F(x_\alpha^\delta) - y_\delta = \alpha(x^* - x_\alpha^\delta),$$

$$F(x_\beta^\delta) - F(x_\alpha^\delta) = \beta(x^* - x_\beta^\delta) - \alpha(x^* - x_\alpha^\delta).$$

Consequently,

$$\begin{aligned}
\beta(x_\alpha^\delta - x_\beta^\delta) &= (\alpha - \beta)(x^* - x_\alpha^\delta) + \beta(x^* - x_\beta^\delta) - \alpha(x^* - x_\alpha^\delta) \\
&= \frac{\alpha - \beta}{\alpha}\left[F(x_\alpha^\delta) - y_\delta\right] + F(x_\beta^\delta) - F(x_\alpha^\delta) \\
&= \frac{\alpha - \beta}{\alpha}\left[F(x_\alpha^\delta) - y_\delta\right] - R_\beta^\delta(x_\alpha^\delta - x_\beta^\delta)
\end{aligned}$$

with $R^\delta_\beta = \int_0^1 F'(x^\delta_\beta + t(x^\delta_\alpha - x^\delta_\beta))dt$.

Hence,

$$x^\delta_\alpha - x^\delta_\beta = \frac{\alpha-\beta}{\alpha}(R^\delta_\beta + \beta I)^{-1}\left[F(x^\delta_\alpha) - y_\delta\right].$$

Since $\|(R^\delta_\beta + \beta I)^{-1}\| \le 1/\beta$ (see [20]), it follows that

$$\|x^\delta_\alpha - x^\delta_\beta\| \le \left|\frac{\alpha-\beta}{\alpha\beta}\right| \left(\|F(x^\delta_\alpha) - F(x_\alpha)\| + \|F(x_\alpha) - y_0\| + \delta\right). \tag{16}$$

On the other hand, we have

$$F(x^\delta_\alpha) - F(x_\alpha) + \alpha(x^\delta_\alpha - x_\alpha) = y_\delta - y_0.$$

Consequently,

$$\|F(x^\delta_\alpha) - F(x_\alpha)\| \le \|y_\delta - y_0\| + \alpha\|x^\delta_\alpha - x_\alpha\|.$$

Using (8) and (3) we obtain

$$\|F(x^\delta_\alpha) - F(x_\alpha)\| \le 2\delta \quad \text{for all } \alpha > 0. \tag{17}$$

By the definition of the element x_α we have

$$F(x_\alpha) - F(x_0) + \alpha(x_\alpha - x_0) = \alpha(x^* - x_0).$$

Consequently,

$$\|F(x_\alpha) - F(x_0)\| \le \alpha\left(\|x_\alpha - x_0\| + \|x^* - x_0\|\right).$$

Using (9) we obtain

$$\|F(x_\alpha) - F(x_0)\| \le 2\alpha\|x^* - x_0\| \quad \text{for all } \alpha > 0. \tag{18}$$

From (16), (17), (18) we conclude that $\|x^\delta_\beta - x^\delta_\alpha\| \to 0$, as $\beta \to \alpha$. From this property, the Fréchet-differentiability of F in $B_r(x_0)$ and (14), it follows that $\rho(\alpha)$ is continuous for $\alpha > 0$.

Next, we show that

$$\rho(\alpha_0) \le C\delta \text{ and } \lim_{\alpha\to\infty} \rho(\alpha) = \|F(x^*) - y_\delta\|. \tag{19}$$

From (4),(7), $\|M^\delta_{\alpha_0}\| \le 1$, (8), (15), (14) we obtain

$$\begin{aligned}
\rho(\alpha_0) &= \|M^\delta_{\alpha_0}\left[\alpha_0(x^\delta_{\alpha_0} - x_{\alpha_0}) + \alpha_0(x_{\alpha_0} - x^*)\right]\| \\
&\le \alpha_0\|x^\delta_{\alpha_0} - x_{\alpha_0}\| + \|\alpha_0 M^\delta_{\alpha_0}(x_{\alpha_0} - x^*)\| \\
&\le \delta + \alpha_0\|x^* - x_0\| \\
&= C\delta.
\end{aligned}$$

From the definition of x^δ_α, it follows that $x^\delta_\alpha \to x^*$ as $\alpha \to \infty$. Since in addition $M^\delta_\alpha \to I$ as $\alpha \to \infty$ we obtain (19). Now Lemma 3.1 is followed from the continuity of $\rho(\alpha)$, (19) and the mean value theorem.

In order to prove order optimal error bounds for $\|x^\delta_\alpha - x_0\|$ with α chosen from rule (14) three preparatory lemmas are required. The proofs of these lemmas are similar to [34].

Lemma 3.2 *Let assumptions (2) and (b') with $k_0 < 1$, $r = \|x^* - x_0\|$ be satisfied. Then, for all $0 < \beta \leq \alpha$,*

$$\|x_\alpha - x_0\| \leq \frac{\|\alpha M_\alpha (x^* - x_\alpha)\|}{(1-k_0)\beta} + \|x_\beta - x_0\|.$$

Lemma 3.3 *Let assumptions (2), (b') with $r = \|x^* - x_0\| + \delta/\alpha$ be satisfied and let $\alpha = \alpha(\delta)$ be chosen by rule (14). Then,*

$$\|\alpha M_\alpha (x^* - x_\alpha)\| \leq (C+1)(1+k_0)\delta.$$

Lemma 3.4 *Let assumptions (2), (a') with $p \in (0,1]$, (b') with $r = \|x^* - x_0\| + \delta/\alpha$ be satisfied and let $\alpha = \alpha(\delta)$ be chosen by rule (14) with $C > 1$. Then,*

$$\alpha \geq \left[\frac{C-1}{(1+k_0)^3\|w\|}\right]^{\frac{1}{p+1}} \delta^{\frac{1}{p+1}}.$$

Now we are in a position to prove the main result which consists in providing order optimal error bounds for $\|x_\alpha^\delta - x_0\|$ provided α is chosen from rule (14).

Theorem 3.1 *Assume (2), (a') with fixed $p \in (0,1]$, (b') with $k_0 < 1$ and $r = \|x^* - x_0\| + \delta/\alpha$. Let $\alpha = \alpha(\delta)$ be chosen by rule (14) with $C > 1$. Then, for all $\delta > 0$,*

$$\|x_\alpha^\delta - x_0\| \leq c_p \|w\|^{\frac{1}{p+1}} \delta^{\frac{p}{p+1}} \tag{20}$$

with a constant c_p independent of δ and $\|w\|$ of the form

$$c_p = 2(1+k_0)\left[\frac{C+1}{1-k_0}\right]^{\frac{p}{p+1}} + \left[\frac{(1+k_0)^3}{C-1}\right]^{\frac{1}{p+1}}. \tag{21}$$

Proof. We consider a fixed regularization parameter $\alpha = \beta$ of the form

$$\beta = c_0 \left(\frac{\delta}{\|w\|}\right)^{\frac{1}{p+1}} \text{ with } c_0 = \left[\frac{C+1}{1-k_0}\right]^{\frac{1}{p+1}} \tag{22}$$

and we distinguish two cases. In the first case we assume that the solution $\alpha = \alpha(\delta)$ rule (14) satisfies $\alpha \leq \beta$. In this case we obtain from Theorem 2.6 with $\alpha = \alpha(\delta)$, estimate (8), Lemma 3.4 and (22) that

$$\begin{aligned} \|x_\alpha^\delta - x_0\| &\leq \|x_\alpha^\delta - x_\alpha\| + \|x_\alpha - x_0\| \\ &\leq \left[\frac{(1+k_0)^3\|w\|}{C-1}\right]^{\frac{1}{p+1}} \delta^{\frac{p}{p+1}} + (1+k_0)\|w\|\beta^p \\ &= \left((1+k_0)c_0^p + \left[\frac{(1+k_0)^3}{C-1}\right]^{\frac{1}{p+1}}\right) \|w\|^{\frac{1}{p+1}} \delta^{\frac{p}{p+1}}. \end{aligned} \tag{23}$$

In the second case we assume that the solution $\alpha = \alpha(\delta)$ of rule (14) satisfies $\alpha \geq \beta$. In this case we obtain from Lemma 3.2, estimate (8), Theorem 2.6 with $\alpha = \beta$, Lemma 3.3, Lemma 3.4

and (22) that

$$\begin{aligned}
\|x_\alpha^\delta - x_0\| &\leq \|x_\alpha^\delta - x_\alpha\| + \|x_\alpha - x_0\| \\
&\leq \delta/\alpha + \frac{\|\alpha M_\alpha(x^* - x_\alpha)\|}{(1-k_0)\beta} + \|x_\beta - x_0\| \\
&\leq \left[\frac{(1+k_0)^3}{C-1}\right]^{\frac{1}{p+1}} \|w\|^{\frac{1}{p+1}} \delta^{\frac{p}{p+1}} + \frac{(C+1)(1+k_0)\delta}{(1-k_0)\beta} + (1+k_0)\|w\|\beta^p \\
&= \left((1+k_0)c_0^p + \frac{(C+1)(1+k_0)}{(1-k_0)c_0} + \left[\frac{(1+k_0)^3}{C-1}\right]^{\frac{1}{p+1}}\right) \|w\|^{\frac{1}{p+1}} \delta^{\frac{p}{p+1}}. \qquad (24)
\end{aligned}$$

Since the error bound (23) of the first case is smaller than the error bound (24) of the second case we conclude that the error bound for $\|x_\alpha^\delta - x_0\|$ with α chosen rule from (14) is given by (24). By substituting the special value c_0 from (22) we obtain (20) and (21).

Now we also assume that the element $x^* - x_0$ is sufficiently smooth, i.e., $x^* - x_0 = F'(x_0)w$. In this case order optimal error bound for $\|x_\alpha^\delta - x_0\|$ with α chosen from rule (14) can be guaranteed under (b) instead of (b').

Theorem 3.2 *Let (2), (a) and (b) with radius $r = \|x^* - x_0\| + \delta/\alpha$ be satisfied and let $\alpha = \alpha(\delta)$ be chosen by rule (14). Suppose that the constant*

$$C > 1 + L\|w\| \left(1 + \frac{L\|w\|}{2} + \frac{L^2\|w\|^2}{2} + \frac{L^3\|w\|^2}{8}\right)$$

and that the Lipschitz constant L of the assumption (b) is sufficiently small such that

$$2L\|w\| \left(1 + \frac{L\|w\|}{2} + \frac{L^2\|w\|^2}{4}\right)^2 < 1.$$

Then, there exists a constant C_0 independent of δ such that

$$\|x_\alpha^\delta - x_0\| \leq C_0 \delta^{\frac{1}{2}}.$$

Proof. The proof is similar to [34].

4 Application

In this section we will introduce an example to illustrate some of the assumptions required in the foregoing sections.

We consider the Volterra equation of the first kind in the real Hilbert space $X = L_2(0,T)$, with $T < \infty$

$$F(x)(t) = \int_0^t x(t-\tau)x(\tau)\,d\tau = y_0(t), \quad t \in (0,T). \qquad (25)$$

This equation occurs in many engineering problems. Let us provide $L_2(0,T)$ with the weighted inner products

$$(x,y)_\sigma = \int_0^T e^{-2\sigma t} x(t)y(t)\,dt$$

and the corresponding weighted norms $\|x\|_\sigma = \left(\int\limits_0^T e^{-2\sigma t}x^2(t)\,dt\right)^{\frac{1}{2}}$, where the parameter σ is non-negative. These specially chosen inner products and norms are necessary to get better sufficient conditions for the Fréchet derivative F' and the source condition in the sequel. Note that the scale $\|.\|_\sigma$ satisfies the relations

$$e^{-\sigma T}\|.\|_0 \leq \|.\|_\sigma \leq \|.\|_0, \quad \sigma \geq 0.$$

Suppose that instead of the exact right-hand side y_0 we only know some approximation $y_\delta \in L_2(0,T)$ such that $\|y_\delta - y_0\|_0 \leq \delta$. It was shown in [11] that the problem (25) is ill-posed in $L_2(0,T)$, i.e., the convergence $y_\delta \to y_0$ does not necessarily imply the convergence of corresponding solutions of (25). It is well known that Lavrentiev regularization approximation of (25) is a solution of the equation of the second kind

$$\int\limits_0^t x(t-\tau)x(\tau)\,d\tau + \alpha(x(t) - x_*(t)) = y_\delta(t), \qquad (26)$$

where $\alpha > 0$ is a regularization parameter and $x_*(t)$ is some guess in $L_2(0,T)$.

Equation (26) has a unique solution x for every $\alpha > 0$ and every right-hand side from X (see [37]). Moreover, the operator F is accretive with respect to the inner products $(.,.)_\sigma$, $\sigma \geq 0$ and satisfies the assumptions (a), (b), (b') (see [14]).

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