IC/2003/14

XA0303048

the
abdus salam
international
centre
for theoretical
physics

united nations educational scientific and cultural organization

international atomic energy agency

preprint

PARTIALLY COHERENT IMAGING AND SPATIAL COHERENCE WAVELETS

Román Castañeda

Available at. http://www.ictp.trieste.it/~pub_off IC/2003/14

United Nations Educational Scientific and Cultural Organization
and
International Atomic Energy Agency

THE ABDUS SALAM INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

PARTIALLY COHERENT IMAGING AND SPATIAL COHERENCE WAVELETS

Román Castañeda[1]
Physics Department, Universidad Nacional de Colombia Sede Medellín,
A.A. 3840 Medellín, Colombia
and
The Abdus Salam International Centre for Theoretical Physics, Trieste, Italy.

Abstract

A description of spatially partially coherent imaging based on the propagation of second order spatial coherence wavelets and marginal power spectra (Wigner distribution functions) is presented. In this dynamics, the spatial coherence wavelets will be affected by the system through its elementary transfer function.

The consistency of the model with the both extreme cases of full coherent and incoherent imaging was proved. In the last case we obtained the classical concept of optical transfer function as a simple integral of the elementary transfer function. Furthermore, the elementary incoherent response function was introduced as the Fourier transform of the elementary transfer function. It describes the propagation of spatial coherence wavelets form each object point to each image point through a specific point on the pupil planes. The point spread function of the system was obtained by a simple integral of the elementary incoherent response function.

MIRAMARE – TRIESTE

March 2003

[1] Regular Associate of the Abdus Salam ICTP. rcastane@perseus unalmed.edu co

1. FUNDAMENTALS

The term Wigner Optics denotes the set of optical phenomena, which involve waves whose amplitudes are described by Wigner distribution functions (WDF) [1,2]. So, Wigner Optics provides a phase-space description of these optical phenomena.

Interference, diffraction and imaging with spatially partially coherent optical fields can be included in the Wigner Optics. These phenomena are usually described with basis on the propagation of the cross-spectral density between specific planes, a wave-like quantity that represents the correlation of any of the frequency components of the optical field at each plane [3].

It was shown that the cross-spectral density can be expressed in terms of the superposition of spatial coherence wavelets [4,5], which are the elementary vehicles for transporting both the correlation and the power of the optical field. For propagation in the Fresnel-Fraunhofer domain and by introducing the centre and difference co-ordinates as defined in Fig. 1 [4], the cross-spectral density of frequency ω at the observation plane, $W\left(\mathbf{r}_A+\frac{\mathbf{r}_D}{2},\mathbf{r}_A-\frac{\mathbf{r}_D}{2};\omega\right)$, takes the mathematical form

$$W\left(\mathbf{r}_A+\frac{\mathbf{r}_D}{2},\mathbf{r}_A-\frac{\mathbf{r}_D}{2};\omega\right)=\left(\frac{1}{\lambda z}\right)^2 e^{i\frac{k}{z}\mathbf{r}_A\cdot\mathbf{r}_D}\int_{\mathcal{A}}\mathcal{W}\left(\mathbf{r}_A+\frac{\mathbf{r}_D}{2},\mathbf{r}_A-\frac{\mathbf{r}_D}{2},\mathbf{r}'_A;\omega\right)d^2r'_A, \quad (1)$$

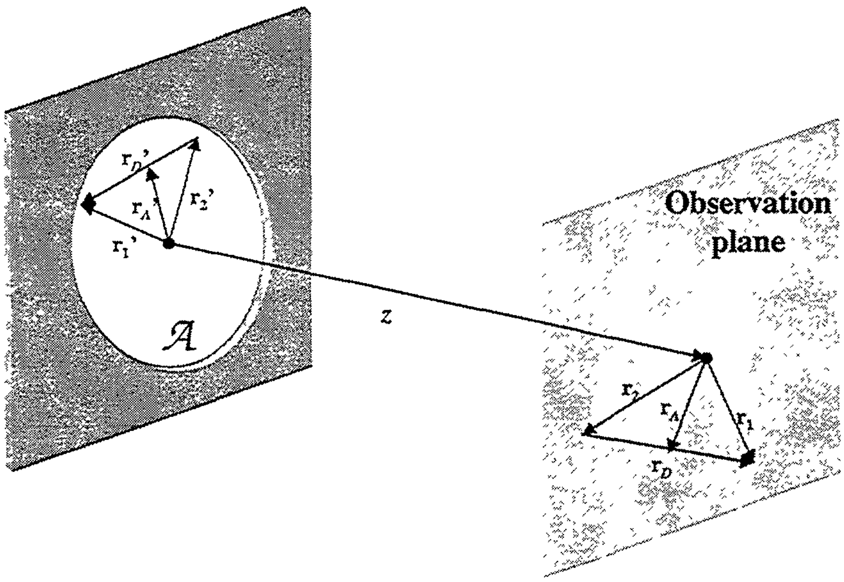

Fig. 1: Illustrating the centre and difference co-ordinates

where λ is the mean wavelength of the field, z is the distance between the planes, $k=\frac{2\pi}{\lambda}$ and

$$\mathcal{W}\left(\mathbf{r}_A+\frac{\mathbf{r}_D}{2},\mathbf{r}_A-\frac{\mathbf{r}_D}{2},\mathbf{r}'_A;\omega\right)=\mathcal{S}(\mathbf{r}_A,\mathbf{r}'_A;\omega)\,e^{-i\frac{k}{z}\mathbf{r}_D\cdot\mathbf{r}'_A} \quad (2)$$

represents the second-order spatial coherence wavelets. The amplitude of each wavelet is called the marginal power spectrum [4,5] and takes the form of a WDF, i.e. it is defined as

$$\mathcal{S}(\mathbf{r}_A,\mathbf{r}'_A;\omega)=\int_{\mathcal{A}}\mu\left(\mathbf{r}'_A+\frac{\mathbf{r}'_D}{2},\mathbf{r}'_A-\frac{\mathbf{r}'_D}{2};\omega\right)\sqrt{S\left(\mathbf{r}'_A+\frac{\mathbf{r}'_D}{2};\omega\right)}\,t\left(\mathbf{r}'_A+\frac{\mathbf{r}'_D}{2}\right)\,, \qquad (3)$$
$$\sqrt{S\left(\mathbf{r}'_A-\frac{\mathbf{r}'_D}{2},\omega\right)}\,t^*\left(\mathbf{r}'_A-\frac{\mathbf{r}'_D}{2}\right)e^{\imath\frac{k}{z}(\mathbf{r}'_A-\mathbf{r}_A)\mathbf{r}'_D}\,d^2r'_D$$

with $\mu\left(\mathbf{r}'_A+\frac{\mathbf{r}'_D}{2},\mathbf{r}'_A-\frac{\mathbf{r}'_D}{2};\omega\right)=\left|\mu\left(\mathbf{r}'_A+\frac{\mathbf{r}'_D}{2},\mathbf{r}'_A-\frac{\mathbf{r}'_D}{2};\omega\right)\right|e^{\imath\alpha\left(\mathbf{r}'_A+\frac{\mathbf{r}'_D}{2},\mathbf{r}'_A-\frac{\mathbf{r}'_D}{2},\omega\right)}$ the complex degree of spatial coherence of the light within the aperture, $S\left(\mathbf{r}'_A\pm\frac{\mathbf{r}'_D}{2};\omega\right)$ its power spectra at $\mathbf{r}'_A\pm\frac{\mathbf{r}'_D}{2}$, and $t\left(\mathbf{r}'_A\pm\frac{\mathbf{r}'_D}{2}\right)=\left|t\left(\mathbf{r}'_A\pm\frac{\mathbf{r}'_D}{2}\right)\right|e^{\imath\phi\left(\mathbf{r}'_A\pm\frac{\mathbf{r}'_D}{2}\right)}$ the aperture transmission at $\mathbf{r}'_A\pm\frac{\mathbf{r}'_D}{2}$.

The phase and space co-ordinates of this WDF can be identified in the arguments of the integrand in equation (3). Accordingly, the vector $\mathbf{r}'_A$, which describe the position of individual centres of secondary disturbance within the aperture, will specify the space co-ordinate and $\frac{k}{z}\mathbf{r}_A$ specifies the phase co-ordinate. It is essentially an angular coordinate. Indeed, in paraxial approach (Fresnel-Fraunhofer domain) it takes the form $\frac{k}{z}|\mathbf{r}_A|\approx\frac{2\pi}{\lambda}\sin\theta$, where θ is the inclination angle of the axis of the marginal power spectrum (i.e. the straight line from $\mathbf{r}'_A$ to $\mathbf{r}_A$) with respect to the optical axis, which is orthogonal to both the aperture and the observation planes.

Equations (1) and (2) lead to

$$S(\mathbf{r}_A;\omega)=\left(\frac{1}{\lambda z}\right)^2\int_{\mathcal{A}}\mathcal{S}(\mathbf{r}_A,\mathbf{r}'_A;\omega)\,d^2r'_A, \qquad (4)$$

for the power spectrum at each point $\mathbf{r}_A$ on the observation plane. Equations (1) and (4) confirm the above affirmation that the spatial coherence wavelets are the elementary vehicles for transporting both the correlation and the power of the optical field from the diffracting aperture to the observation plane

The above ideas can be used for describing the spatially partially coherent imaging, by assuming it as the result from the cascade of two diffraction phenomena of spatial coherence wavelets in Fresnel-Fraunhofer domain. The first one describes the propagation of the optical field from the object plane or entrance window (*ew*) to the entrance pupil (*ep*) of the imaging system, which are at a distance z' to each other. The second one is corresponding to the propagation from the exit pupil (*EP*) to the image plane or exit window (*EW*), which are at a distance z to each other. Furthermore, we will assume the lateral magnification of the system equal to one for the sake of simplicity and without lack of generality.

According to equations (2) and (3), the spatial coherence wavelets that propagate between the *ew* and the *ep* planes, i.e. the object wavelets, will be

$$\mathcal{W}_{ep}^{ew}\left(\xi_A+\frac{\xi_D}{2},\xi_A-\frac{\xi_D}{2},\mathbf{r}'_A;\omega\right)=e^{-\imath\frac{k}{z'}\xi_D\cdot\mathbf{r}'_A}\int_{ew}\mu_{ew}\left(\mathbf{r}'_A+\frac{\mathbf{r}'_D}{2},\mathbf{r}'_A-\frac{\mathbf{r}'_D}{2};\omega\right)\sqrt{S_{ew}\left(\mathbf{r}'_A+\frac{\mathbf{r}'_D}{2};\omega\right)},$$

$$t\left(\mathbf{r}'_A+\frac{\mathbf{r}'_D}{2}\right)\sqrt{S_{ew}\left(\mathbf{r}'_A-\frac{\mathbf{r}'_D}{2},\omega\right)}\;t^*\left(\mathbf{r}'_A-\frac{\mathbf{r}'_D}{2}\right)e^{\imath\frac{k}{z'}(\mathbf{r}'_A-\xi_A)\mathbf{r}'_D}\,d^2r'_D \tag{5}$$

where $(\mathbf{r}'_A,\mathbf{r}'_D)$ and (ξ_A,ξ_D) denote the centre and difference co-ordinates at the *ew* and the *ep* planes respectively. According to equation (3), their marginal power spectra are given by the integral in equation (5).

The transmission of the imaging system is given by the function $P(\xi)=|P(\xi)|\,e^{\imath\Phi(\xi)}\,e^{-\imath\frac{k}{f}|\xi|^2}$, with $|P(\xi)|$ the amplitude of the pupil function, $\frac{\Phi(\xi)}{k}$ its wave-aberration function and *f* its focal length [6]. Therefore, the spatial coherence wavelets that propagate between the *EP* and the *EW* planes, i.e. the image wavelets, will be

$$\mathcal{W}_{EW}^{EP}\left(\mathbf{r}_A+\frac{\mathbf{r}_D}{2},\mathbf{r}_A-\frac{\mathbf{r}_D}{2},\xi_A;\omega\right)=e^{-\imath\frac{k}{z}\mathbf{r}_D\,\xi_A}\int_{\substack{EP\\ \xi_D\neq 0}}\mu_{EP}\left(\xi_A+\frac{\xi_D}{2},\xi_A-\frac{\xi_D}{2};\omega\right)\sqrt{S_{EP}\left(\xi_A+\frac{\xi_D}{2};\omega\right)}$$

$$P\left(\xi_A+\frac{\xi_D}{2}\right)\sqrt{S_{EP}\left(\xi_A-\frac{\xi_D}{2};\omega\right)}\;P^*\left(\xi_A-\frac{\xi_D}{2}\right)\,e^{\imath k\left(\frac{1}{z}-\frac{1}{f}\right)\xi_A\,\xi_D}\,e^{-\imath\frac{k}{z}\mathbf{r}_A\xi_D}\,d^2\xi_D \tag{6}$$

whose marginal power spectra are given by the integral of this expression. Taking into account that

$$W_{ep}\left(\xi_A+\frac{\xi_D}{2},\xi_A-\frac{\xi_D}{2};\omega\right)=\sqrt{S_{EP}\left(\xi_A+\frac{\xi_D}{2};\omega\right)}\sqrt{S_{EP}\left(\xi_A-\frac{\xi_D}{2};\omega\right)}\,\mu_{EP}\left(\xi_A+\frac{\xi_D}{2},\xi_A-\frac{\xi_D}{2};\omega\right)$$

$$=\left(\frac{1}{\lambda z'}\right)^2 e^{\imath\frac{k}{z'}\xi_A\xi_D}\int_{ew}\mathcal{W}_{ep}^{ew}\left(\xi_A+\frac{\xi_D}{2},\xi_A-\frac{\xi_D}{2},\mathbf{r}'_A;\omega\right)d^2r'_A$$

equation (6) becomes

$$\mathcal{W}_{EW}^{EP}\left(\mathbf{r}_A+\frac{\mathbf{r}_D}{2},\mathbf{r}_A-\frac{\mathbf{r}_D}{2},\xi_A;\omega\right)=\left(\frac{1}{\lambda z'}\right)^2 e^{-\imath\frac{k}{z}\mathbf{r}_D\xi_A}\int_{ew}\int_{EP}\mathcal{W}_{ep}^{ew}\left(\xi_A+\frac{\xi_D}{2},\xi_A-\frac{\xi_D}{2},\mathbf{r}'_A;\omega\right)$$

$$P\left(\xi_A+\frac{\xi_D}{2}\right)P^*\left(\xi_A-\frac{\xi_D}{2}\right)e^{\imath\frac{k}{F}\xi_A\xi_D}\,e^{-\imath\frac{k}{z}\mathbf{r}_A\xi_D}\,d^2\xi_D\,d^2r'_A \quad . \qquad (7)$$

with $\frac{1}{F}=\frac{1}{z'}+\frac{1}{z}-\frac{1}{f}$ the defocus parameter (i.e. $\frac{1}{F}=0$ for perfectly focused imaging and $\frac{1}{F}\neq 0$ for defocused imaging) [7,8]. Equations (5) and (7) provide an elementary description of the spatially partially imaging in terms of the transport of both power and correlation of the optical field from the *ew* to the *EW* planes via the second order spatial coherence wavelets. Specifically, equation (7) yields the following transformation for the marginal power spectrum, i.e. the transformation of the object WDF into the image WDF:

$$\mathcal{S}_{EW}^{EP}(\mathbf{r}_A,\xi_A;\omega)=\left(\frac{1}{\lambda z'}\right)^2\int_{ew}\int_{EP}\mathcal{S}_{ep}^{ew}(\xi_A,\mathbf{r}'_A;\omega)P\left(\xi_A+\frac{\xi_D}{2}\right)P^*\left(\xi_A-\frac{\xi_D}{2}\right)e^{\imath\frac{k}{F}\xi_A\xi_D}$$

$$e^{-\imath\frac{k}{z'}\left(\mathbf{r}'_A+\frac{z'}{z}\mathbf{r}_A\right)\xi_D}\,d^2\xi_D\,d^2r'_A \quad , \qquad (8)$$

with $\mathcal{S}_{EW}^{EP}(\mathbf{r}_A,\xi_A;\omega)$ the image marginal power spectrum and

$$\mathcal{S}_{ep}^{ew}(\xi_A,\mathbf{r}'_A;\omega)=\int_{ew}\mu_{ew}\left(\mathbf{r}'_A+\frac{\mathbf{r}'_D}{2},\mathbf{r}'_A-\frac{\mathbf{r}'_D}{2};\omega\right)\sqrt{S\left(\mathbf{r}'_A+\frac{\mathbf{r}'_D}{2};\omega\right)}\,t\left(\mathbf{r}'_A+\frac{\mathbf{r}'_D}{2}\right)$$

$$\sqrt{S\left(\mathbf{r}'_A-\frac{\mathbf{r}'_D}{2},\omega\right)}\,t^*\left(\mathbf{r}'_A-\frac{\mathbf{r}'_D}{2}\right)e^{\imath\frac{k}{z}(\mathbf{r}'_A-\xi_A)\mathbf{r}'_D}\,d^2r'_D \qquad (9)$$

the object marginal power spectrum.

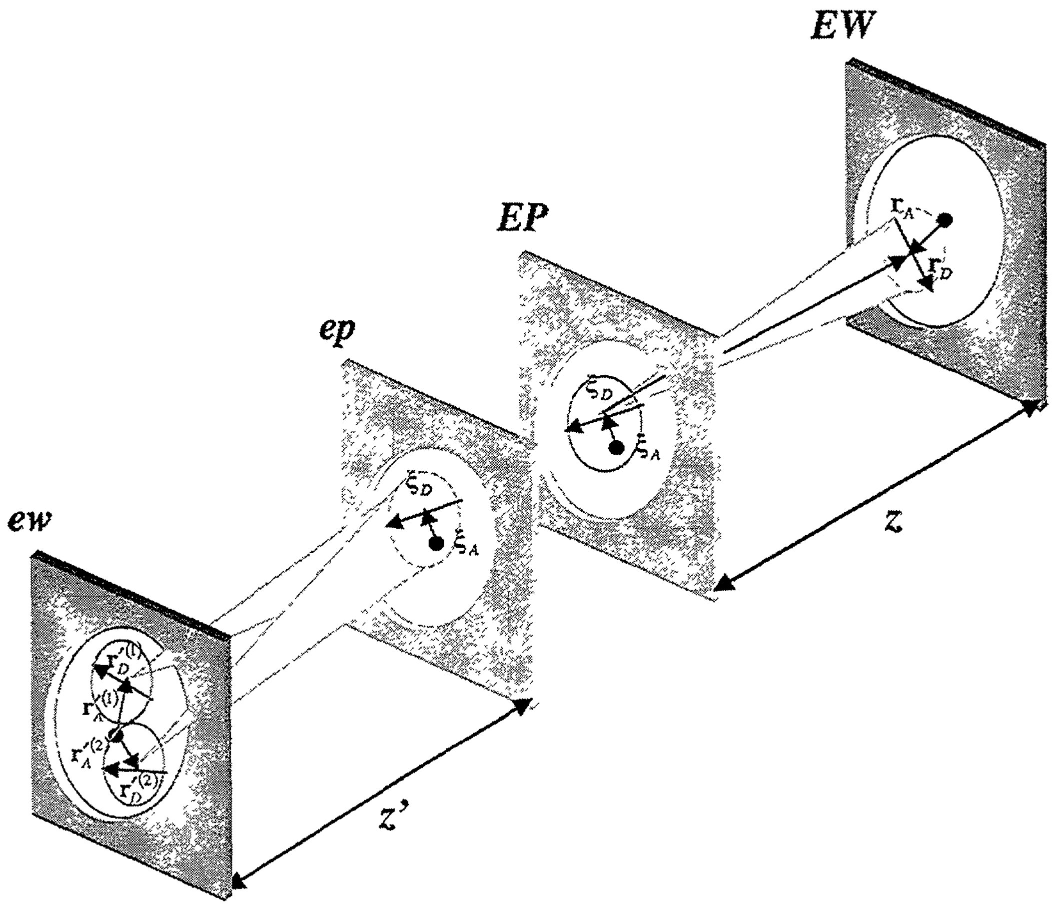

Fig. 2: Imaging description based on spatial coherence wavelets.

The kern of the transformation in both equations (7) and (8) is the same function

$$\mathcal{P}\left(\xi_A,\xi_D;\frac{1}{F}\right)=P\left(\xi_A+\frac{\xi_D}{2}\right)P^*\left(\xi_A-\frac{\xi_D}{2}\right)e^{i\frac{k}{F}\xi_A\xi_D}. \quad (10)$$

It only depends on system parameters at the *EP* plane, so that it behaves as a transfer function. Indeed, $\int_{EP} S_{ep}^{ew}(\xi_A,\mathbf{r}'_A;\omega)\mathcal{P}\left(\xi_A,\xi_D;\frac{1}{F}\right)e^{-i\frac{k}{z'}\left(\mathbf{r}'_A+\frac{z'}{z}\mathbf{r}_A\right)\xi_D}\,d^2\xi_D$ from equation (8) represents the fraction of marginal power spectrum that propagates along the axis $\mathbf{r}'_A \to \xi_A \to \mathbf{r}_A$. So, this fraction could be modified at the *EP* plane depending on the values of $\mathcal{P}\left(\xi_A,\xi_D;\frac{1}{F}\right)$ in a vicinity of ξ_A determined by the separation vector ξ_D, i.e. that fraction will be provided not only by the individual centre of secondary disturbance at ξ_A but also by the pairs of centres with separation vector ξ_D around ξ_A.

A similar analysis of the propagation of spatial coherence wavelets can be performed on eq.(7) but taking into account the corresponding phase factors. Consequently, we call $\mathcal{P}\left(\xi_A, \xi_D ; \frac{1}{F}\right)$ the *(defocused) elementary transfer function.* For illustrating purposes, we depict the transference of spatial coherence wavelets and marginal power spectra between the *ew* and the *EW* planes through cones in Fig.2. The circles centred at their vertices denote the correlation area, that is, the set of pairs of centres that affect the contribution transferred along the cone axis.

It could be interesting to analyse the transport of correlation and power between the *ew* and the *EW* planes in more detail. By introducing the dimensionless function $\lambda z' \delta(\mathbf{r}'_D) + [1 - \lambda z' \delta(\mathbf{r}'_D)]$ in the integrand of equation (5) and following the same procedure as in Ref.(4), equation (5) becomes

$$\mathcal{W}_{ep}^{ew}\left(\xi_A + \frac{\xi_D}{2}, \xi_A - \frac{\xi_D}{2}, \mathbf{r}'_A; \omega\right) = S_{ew}(\mathbf{r}'_A;\omega)|t(\mathbf{r}'_A)|^2 e^{-i\frac{k}{z'}\xi_D \mathbf{r}'_A}$$

$$+2e^{-i\frac{k}{z'}\xi_D \mathbf{r}'_A} \int\limits_{\substack{ew \\ \mathbf{r}'_D \neq 0}} \left|\mu_{ew}\left(\mathbf{r}'_A + \frac{\mathbf{r}'_D}{2}, \mathbf{r}'_A - \frac{\mathbf{r}'_D}{2}; \omega\right)\right| \sqrt{S_{ew}\left(\mathbf{r}'_A + \frac{\mathbf{r}'_D}{2};\omega\right)} \left|t\left(\mathbf{r}'_A + \frac{\mathbf{r}'_D}{2}\right)\right| \sqrt{S_{ew}\left(\mathbf{r}'_A - \frac{\mathbf{r}'_D}{2}, \omega\right)}$$

$$\left|t\left(\mathbf{r}'_A - \frac{\mathbf{r}'_D}{2}\right)\right| \cos\left(\frac{k}{z'}\xi_A \cdot \mathbf{r}'_D - \frac{k}{z'}\mathbf{r}'_A \cdot \mathbf{r}'_D - \phi_{12} - \alpha\right) d^2 r'_D \tag{11}$$

with $\phi_{12} = \phi\left(\mathbf{r}'_A + \frac{\mathbf{r}'_D}{2}\right) - \phi\left(\mathbf{r}'_A - \frac{\mathbf{r}'_D}{2}\right)$. So, the object spatial coherence wavelet will result from the superposition of:

- a spatially incoherent contribution (i.e. the first term of equation (11)) provided by the centre of secondary disturbance at $\mathbf{r}'_A$, and
- a partially coherent contribution (the second term of equation (11)), which is a modulation on the incoherent contribution due to the correlation between pairs of centres with separation vectors $\mathbf{r}'_D$ around the position $\mathbf{r}'_A$.

In a similar way, equations (7) and (11) yield

$$\mathcal{W}_{EW}^{EP}\left(\mathbf{r}_A+\frac{\mathbf{r}_D}{2},\mathbf{r}_A-\frac{\mathbf{r}_D}{2},\xi_A;\omega\right)=\left(\frac{1}{\lambda z'}\right)^2 e^{-\imath\frac{k}{z}\mathbf{r}_D\xi_A}\left\{\left|P(\xi_A)\right|^2\int_{ew}S_{ew}(\mathbf{r}'_A;\omega)\left|t(\mathbf{r}'_A)\right|^2 d^2r'_A\right.$$

$$+2\left|P(\xi_A)\right|^2\int_{ew}\int_{\substack{ew\\ \mathbf{r}'_D\neq 0}}\left|\mu_{ew}\left(\mathbf{r}'_A+\frac{\mathbf{r}'_D}{2},\mathbf{r}'_A-\frac{\mathbf{r}'_D}{2};\omega\right)\right|\sqrt{S_{ew}\left(\mathbf{r}'_A+\frac{\mathbf{r}'_D}{2};\omega\right)}\left|t\left(\mathbf{r}'_A+\frac{\mathbf{r}'_D}{2}\right)\right|\sqrt{S_{ew}\left(\mathbf{r}'_A-\frac{\mathbf{r}'_D}{2},\omega\right)}$$

$$\left|t\left(\mathbf{r}'_A-\frac{\mathbf{r}'_D}{2}\right)\right|\cos\left(\frac{k}{z'}\xi_A\cdot\mathbf{r}'_D-\frac{k}{z'}\mathbf{r}'_A\cdot\mathbf{r}'_D-\phi_{12}-\alpha\right)d^2r'_A\,d^2r'_D$$

$$+2\int_{ew}\int_{\substack{EP\\ \xi_D\neq 0}}S_{ew}(\mathbf{r}'_A;\omega)\left|t(\mathbf{r}'_A)\right|^2\left|P\left(\xi_A+\frac{\xi_D}{2}\right)\right|\left|P\left(\xi_A-\frac{\xi_D}{2}\right)\right|\cos\left[\frac{k}{z'}\left(\frac{z'}{z}\mathbf{r}_A+\mathbf{r}'_A\right)\cdot\xi_D-\frac{k}{F}\xi_A\cdot\xi_D-\Phi_{12}\right]d^2\xi_D\,d^2r'_A$$

$$+2\int_{ew}\int_{\substack{ew\\ \mathbf{r}'_D\neq 0}}\int_{\substack{EP\\ \xi_D\neq 0}}\left|\mu_{ew}\left(\mathbf{r}'_A+\frac{\mathbf{r}'_D}{2},\mathbf{r}'_A-\frac{\mathbf{r}'_D}{2};\omega\right)\right|\sqrt{S_{ew}\left(\mathbf{r}'_A+\frac{\mathbf{r}'_D}{2};\omega\right)}\left|t\left(\mathbf{r}'_A+\frac{\mathbf{r}'_D}{2}\right)\right|\sqrt{S_{ew}\left(\mathbf{r}'_A-\frac{\mathbf{r}'_D}{2},\omega\right)}\left|t\left(\mathbf{r}'_A-\frac{\mathbf{r}'_D}{2}\right)\right|$$

$$\left.\left|P\left(\xi_A+\frac{\xi_D}{2}\right)\right|\left|P\left(\xi_A-\frac{\xi_D}{2}\right)\right|\cos\left(\frac{k}{z'}\left[\left(\frac{z'}{z}\mathbf{r}_A+\mathbf{r}'_A\right)\cdot\xi_D+\xi_A\cdot\mathbf{r}'_D\right]-\frac{k}{z'}\mathbf{r}'_A\cdot\mathbf{r}'_D-\frac{k}{F}\xi_A\cdot\xi_D-\phi_{12}-\alpha-\Phi_{12}\right)d^2r'_D\,d^2\xi_D\,d^2r'_A\right\} \tag{12}$$

for the image spatial coherence wavelets, with $\Phi_{12}=\Phi\left(\xi_A+\frac{\xi_D}{2}\right)-\Phi\left(\xi_A-\frac{\xi_D}{2}\right)$. The first two terms in equation (12) have a similar meaning as the terms in equation (11), with the difference that now they are integrated over all individual centres of secondary disturbance in the *ew* plane.

The third and fourth terms denote modulations on the spatially incoherent and spatially coherent contributions from the *ew* plane respectively due to the correlation between pairs of centres with separation vectors ξ_D around the position ξ_A.

Note that the integration region of the second and the fourth terms of equation (12) is the minor in size between the aperture at the *ew* plane and the support of the complex degree of spatial coherence there. So, for spatially incoherent illumination, i.e. $\mu_{ew}\left(\mathbf{r}'_A+\frac{\mathbf{r}'_D}{2},\mathbf{r}'_A-\frac{\mathbf{r}'_D}{2};\omega\right)=\delta(\mathbf{r}'_D)$ these terms nullify. For full spatially coherent illumination, i.e. $\left|\mu_{ew}\left(\mathbf{r}'_A+\frac{\mathbf{r}'_D}{2},\mathbf{r}'_A-\frac{\mathbf{r}'_D}{2};\omega\right)\right|=1$ and arbitrary constant phase $\alpha\left(\mathbf{r}'_A+\frac{\mathbf{r}'_D}{2},\mathbf{r}'_A-\frac{\mathbf{r}'_D}{2};\omega\right)$, they are integrated over the whole aperture. But, for partially coherent illumination, they are integrals running over the support of the complex degree of spatial coherence, i.e. the region within it takes the more significant values. In other words, pairs of centres of secondary disturbance will be correlated only if both are located inside the support.

On the other hand, imaging can change the spatial coherence properties of the optical field. From equations (1), (5), (7) and (10) we have for the image cross-spectral density the expression

$$W_{EW}\left(\mathbf{r}_A+\frac{\mathbf{r}_D}{2},\mathbf{r}_A-\frac{\mathbf{r}_D}{2};\omega\right)=\left(\frac{1}{\lambda z}\right)^2\left(\frac{1}{\lambda z'}\right)^2 e^{\imath\frac{k}{z}\mathbf{r}_A\mathbf{r}_D}\iiiint_{ewewEPEP}\mu_{ew}\left(\mathbf{r}'_A+\frac{\mathbf{r}'_D}{2},\mathbf{r}'_A-\frac{\mathbf{r}'_D}{2};\omega\right)$$

$$\sqrt{S_{ew}\left(\mathbf{r}'_A+\frac{\mathbf{r}'_D}{2};\omega\right)}\; t\left(\mathbf{r}'_A+\frac{\mathbf{r}'_D}{2}\right)\sqrt{S_{ew}\left(\mathbf{r}'_A-\frac{\mathbf{r}'_D}{2},\omega\right)}\; t^*\left(\mathbf{r}'_A-\frac{\mathbf{r}'_D}{2}\right)\mathcal{P}\left(\xi_A,\xi_D;\frac{1}{F}\right)\cdot$$

$$e^{\imath\frac{k}{z'}\mathbf{r}'_A\mathbf{r}'_D}\, e^{-\imath\frac{k}{z'}(\xi_A\mathbf{r}'_D+\xi_D\cdot\mathbf{r}'_A)}\, e^{-\imath\frac{k}{z}(\mathbf{r}_A\cdot\xi_D+\mathbf{r}_D\cdot\xi_A)}\, d^2\xi_A\, d^2\xi_D\, d^2r'_A\, d^2r'_D$$

The complex degree of spatial coherence at the *EW* plane, i.e. $\mu_{EW}\left(\mathbf{r}_A+\frac{\mathbf{r}_D}{2},\mathbf{r}_A-\frac{\mathbf{r}_D}{2};\omega\right)$, will be obtained by normalizing the cross-spectral density with respect to the image power spectrum, which results after the evaluation of the cross spectral density for $\mathbf{r}_D=0$. $\mu_{EW}\left(\mathbf{r}_A+\frac{\mathbf{r}_D}{2},\mathbf{r}_A-\frac{\mathbf{r}_D}{2};\omega\right)$ clearly differs from $\mu_{ew}\left(\mathbf{r}'_A+\frac{\mathbf{r}'_D}{2},\mathbf{r}'_A-\frac{\mathbf{r}'_D}{2};\omega\right)$ except for the ideal imaging as we discuss below.

2. CORRESPONDENCE TO THE CLASSICAL DESCRIPTION OF IMAGING

The ideal imaging

It is known that ideal imaging will be performed if the following conditions are fulfilled [7,8]:

- Diffraction-limited system $(\Phi_{12}=0)$ with uniform pupil $\left(\left|P\left(\xi_A\pm\frac{\xi_D}{2}\right)\right|=1\right)$.
- Enough great pupils so that the Fourier transform of the pupil function approaches to a Dirac's delta function.
- The *EW* plane must coincide with the best focus plane $\left(\frac{1}{F}=0\right)$.
- Diffraction in Fraunhofer domain, i.e. $e^{\imath\frac{k}{z'}\mathbf{r}'_A\cdot\mathbf{r}'_D}\approx 1$ and $e^{\imath\frac{k}{z}\mathbf{r}_A\mathbf{r}_D}\approx 1$.

Under such conditions $\mathcal{P}\left(\xi_A,\xi_D;\frac{1}{F}\right)\equiv 1$ and equation (7) reduces to

$$W_{EW}^{EP}\left(\mathbf{r}_A+\frac{\mathbf{r}_D}{2},\mathbf{r}_A-\frac{\mathbf{r}_D}{2},\xi_A;\omega\right)=\left(\frac{1}{\lambda z'}\right)^2 e^{-\imath\frac{k}{z}\mathbf{r}_D\cdot\xi_A}\iint_{ewEP} W_{ep}^{ew}\left(\xi_A+\frac{\xi_D}{2},\xi_A-\frac{\xi_D}{2},\mathbf{r}'_A;\omega\right)e^{-\imath\frac{k}{z}\mathbf{r}_A\,\xi_D}\,d^2\xi_D\,d^2r'_A$$

$$=\left(\frac{1}{\lambda z'}\right)^2 S_{ep}^{ew}\left(\xi_A,-\frac{z'}{z}\mathbf{r}_A;\omega\right)e^{-\imath\frac{k}{z}\mathbf{r}_D\,\xi_A} \qquad (13)$$

According to equation (8), it means that

$$S_{EW}^{EP}(\xi_A,\mathbf{r}_A;\omega)=\left(\frac{1}{\lambda z'}\right)^2 S_{ep}^{ew}\left(\xi_A,-\frac{z'}{z}\mathbf{r}_A;\omega\right). \qquad (14)$$

Thus, the WDF (marginal power spectrum) that describes the ideal imaging propagates along the axis $\mathbf{r}'_A=-\frac{z'}{z}\mathbf{r}_A\rightarrow\xi_A\rightarrow\mathbf{r}_A$, without change. The scaling factor is due to the lateral magnification of the system. As a consequence, we obtain the well-known results [6,7]

$$W_{EW}\left(\mathbf{r}_A+\frac{\mathbf{r}_D}{2},\mathbf{r}_A-\frac{\mathbf{r}_D}{2};\omega\right)=\left(\frac{1}{\lambda z'}\right)^2\left(\frac{1}{\lambda z}\right)^2 W_{ew}\left(-\frac{z'}{z}\left(\mathbf{r}'_A+\frac{\mathbf{r}'_D}{2}\right),-\frac{z'}{z}\left(\mathbf{r}'_A-\frac{\mathbf{r}'_D}{2}\right);\omega\right) \qquad (15)$$

and

$$S_{EW}(\mathbf{r}_A;\omega)=\left(\frac{1}{\lambda z'}\right)^2\left(\frac{1}{\lambda z}\right)^2 S_{ew}\left(-\frac{z'}{z}\mathbf{r}_A;\omega\right). \qquad (16)$$

Equation (15) shows that the ideal image cross-spectral density is an inverted and scaled version of the object cross-spectral density. The same holds for the ideal image power spectrum as shown by equation (16). It also means that the same spatial coherence properties hold for the optical field at both the *ew* and the *EW* planes. Indeed, equations (15) and (16) yield $\mu_{EW}\left(\mathbf{r}_A+\frac{\mathbf{r}_D}{2},\mathbf{r}_A-\frac{\mathbf{r}_D}{2};\omega\right)=\mu_{ew}\left(-\frac{z'}{z}\left(\mathbf{r}'_A+\frac{\mathbf{r}'_D}{2}\right),-\frac{z'}{z}\left(\mathbf{r}'_A-\frac{\mathbf{r}'_D}{2}\right);\omega\right)$.

Imaging under spatially incoherent illumination

Under spatially incoherent illumination, i.e. $\mu_{ew}\left(\mathbf{r}'_A+\frac{\mathbf{r}'_D}{2},\mathbf{r}'_A-\frac{\mathbf{r}'_D}{2};\omega\right)=\delta(\mathbf{r}'_D)$, equation (5) yields $W_{ep}^{ew}\left(\xi_A+\frac{\xi_D}{2},\xi_A-\frac{\xi_D}{2},\mathbf{r}'_A;\omega\right)=S_{ew}(\mathbf{r}'_A;\omega)|t(\mathbf{r}'_A)|^2 e^{-\imath\frac{k}{z}\xi_D\cdot\mathbf{r}'_A}$, so that the object power spectrum will be $S_{ep}^{ew}(\xi_A,\mathbf{r}'_A;\omega)=S_{ew}(\mathbf{r}'_A,\omega)|t(\mathbf{r}'_A)|^2$. It means that there are no contributions from pairs of centres of secondary disturbance within the *ew* plane to these quantities.

According to equations (7) and (10), the image spatial coherence wavelets will take the form

$$\mathcal{W}_{EW}^{EP}\left(\mathbf{r}_A+\frac{\mathbf{r}_D}{2},\mathbf{r}_A-\frac{\mathbf{r}_D}{2},\xi_A;\omega\right)=\left(\frac{1}{\lambda z'}\right)^2 e^{-\imath\frac{k}{z}\mathbf{r}_D\xi_A}\int_{ew} S_{ew}(\mathbf{r}'_A;\omega)|t(\mathbf{r}'_A)|^2\, p\left(\frac{z'}{z}\mathbf{r}_A+\mathbf{r}'_A\,;\xi_A;\frac{1}{F}\right)d^2r'_A \tag{17}$$

with

$$p\left(\frac{z'}{z}\mathbf{r}_A+\mathbf{r}'_A\,,\xi_A\,;\frac{1}{F}\right)=\int_{EP}\mathcal{P}\left(\xi_A,\xi_D\,;\frac{1}{F}\right)e^{-\imath\frac{k}{z'}\left(\frac{z'}{z}\mathbf{r}_A+\mathbf{r}'_A\right)\xi_D}\,d^2\xi_D\,. \tag{18}$$

According to equations (2) and (8), equation (17) means that the image marginal power spectra will results from the convolution between the object power spectra and the function $p\left(\frac{z'}{z}\mathbf{r}_A+\mathbf{r}'_A\,;\xi_A;\frac{1}{F}\right)$, which is the Fourier transform of the elementary transfer function. In other words, this function behaves as a response function for the propagation of both the spatial coherence wavelets and the marginal power spectra through the system in spatially incoherent imaging. For this reason we call it the *(defocused) elementary incoherent response function.* It describes a linear transport along the axis $\mathbf{r}'_A=-\frac{z'}{z}\mathbf{r}_A\rightarrow\xi_A\rightarrow\mathbf{r}_A$. So, by spatially incoherent imaging, the WDF transforms (or propagates) following a linear convolution.

From equations (2), (4) and (17), the image power spectrum will take the form

$$S_{EW}(\mathbf{r}_A;\omega)=\left(\frac{1}{\lambda z}\right)^2\left(\frac{1}{\lambda z'}\right)^2\int_{ew} S_{ew}(\mathbf{r}'_A;\omega)|t(\mathbf{r}'_A)|^2\, p\left(\frac{z'}{z}\mathbf{r}_A+\mathbf{r}'_A\,;\frac{1}{F}\right)d^2r'_A\,, \tag{19}$$

with

$$p\left(\frac{z'}{z}\mathbf{r}_A+\mathbf{r}'_A\,;\frac{1}{F}\right)=\int_{EP} p\left(\frac{z'}{z}\mathbf{r}_A+\mathbf{r}'_A\,;\xi_A;\frac{1}{F}\right)d^2\xi_A\,. \tag{20}$$

Equations (10), (18) and (20) allow us to express $p\left(\frac{z'}{z}\mathbf{r}_A+\mathbf{r}'_A\,;\frac{1}{F}\right)$ as the Fourier transform of

$$\int_{EP} P\left(\xi_A+\frac{\xi_D}{2}\right)P^*\left(\xi_A-\frac{\xi_D}{2}\right)e^{\imath\frac{k}{F}\xi_A\xi_D}\,d^2\xi_A=\int_{EP}\mathcal{P}\left(\xi_A,\xi_D\,;\frac{1}{F}\right)d^2\xi_A\,, \tag{21}$$

which is the defocused optical transfer function (OTF). It reduces to the well-known Duffieux integral for imaging onto the best focus plane [6]. It means that $p\left(\frac{z'}{z}\mathbf{r}_A+\mathbf{r}'_A\,;\frac{1}{F}\right)$ is the defocused point spread function (PSF) of the system. So, equation (19) is a well known result for spatially incoherent imaging.

Now, because of convolution with the elementary incoherent response function the imaging will be not ideal. Diffraction effects of the spatial coherence wavelets at the *ew* plane will be unavoidable in general. They produce some blur not only on the power spectrum (as expected) but also on the cross-spectral density, which is revealed by an increase in spatial coherence of the optical field. Indeed, the cross-spectral density at the *EW* plane will be

$$W_{EW}\left(\mathbf{r}_A+\frac{\mathbf{r}_D}{2},\mathbf{r}_A-\frac{\mathbf{r}_D}{2};\omega\right)=\left(\frac{1}{\lambda z}\right)^2\left(\frac{1}{\lambda z'}\right)^2 e^{i\frac{k}{z}\mathbf{r}_A\cdot\mathbf{r}_D}\int_{ew} S_{ew}(\mathbf{r}'_A;\omega)\left|t(\mathbf{r}'_A)\right|^2 \quad , \quad (22)$$

$$\int_{EP} p\left(\frac{z'}{z}\mathbf{r}_A+\mathbf{r}'_A;\xi_A;\frac{1}{F}\right)e^{-i\frac{k}{z}\mathbf{r}_D\xi_A}\,d\xi_A\,d^2r'_A$$

so that the complex degree of spatial coherence at this plane takes the form

$$\mu_{EW}\left(\mathbf{r}_A+\frac{\mathbf{r}_D}{2},\mathbf{r}_A-\frac{\mathbf{r}_D}{2};\omega\right)=e^{i\frac{k}{z}\mathbf{r}_A\mathbf{r}_D}\frac{\int_{ew} S_{ew}(\mathbf{r}'_A;\omega)\left|t(\mathbf{r}'_A)\right|^2\int_{EP} p\left(\frac{z'}{z}\mathbf{r}_A+\mathbf{r}'_A;\xi_A;\frac{1}{F}\right)e^{-i\frac{k}{z}\mathbf{r}_D\cdot\xi_A}\,d\xi_A\,d^2r'_A}{\int_{ew} S_{ew}(\mathbf{r}'_A;\omega)\left|t(\mathbf{r}'_A)\right|^2\int_{EP} p\left(\frac{z'}{z}\mathbf{r}_A+\mathbf{r}'_A;\xi_A;\frac{1}{F}\right)d^2\xi_A\,d^2r'_A}$$

(23)

which differs in general from the complex degree of spatial coherence at the *ew* plane, i.e. $\mu_{ew}\left(\mathbf{r}'_A+\frac{\mathbf{r}'_D}{2},\mathbf{r}'_A-\frac{\mathbf{r}'_D}{2};\omega\right)=\delta(\mathbf{r}'_D)$.

Imaging under spatially coherent illumination

Under spatially incoherent illumination, i.e. $\left|\mu_{ew}\left(\mathbf{r}'_A+\frac{\mathbf{r}'_D}{2},\mathbf{r}'_A-\frac{\mathbf{r}'_D}{2};\omega\right)\right|=1$ and arbitrary phase $\alpha\left(\mathbf{r}'_A+\frac{\mathbf{r}'_D}{2},\mathbf{r}'_A-\frac{\mathbf{r}'_D}{2};\omega\right)$ (for mathematical simplicity and without lack of generality we assume the phase equals to null), equations (1), (5) and (7) yield

$$W_{EW}\left(\mathbf{r}_A+\frac{\mathbf{r}_D}{2},\mathbf{r}_A-\frac{\mathbf{r}_D}{2};\omega\right)=\left(\frac{1}{\lambda z}\right)^2\left(\frac{1}{\lambda z'}\right)^2 e^{\imath\frac{k}{z}\mathbf{r}_A\mathbf{r}_D}\int_{ew}\int_{ew}\int_{EP}\int_{EP}\sqrt{S_{ew}\left(\mathbf{r}'_A+\frac{\mathbf{r}'_D}{2};\omega\right)}\; t\left(\mathbf{r}'_A+\frac{\mathbf{r}'_D}{2}\right)$$

$$\sqrt{S_{ew}\left(\mathbf{r}'_A-\frac{\mathbf{r}'_D}{2},\omega\right)}\; t^*\left(\mathbf{r}'_A-\frac{\mathbf{r}'_D}{2}\right)P\left(\xi_A+\frac{\xi_D}{2}\right)P^*\left(\xi_A-\frac{\xi_D}{2}\right)$$

$$e^{\imath\frac{k}{z'}\mathbf{r}'_A\cdot\mathbf{r}'_D}\, e^{\imath\frac{k}{F}\xi_A\xi_D}\, e^{-\imath\frac{k}{z'}(\xi_A\cdot\mathbf{r}'_D+\xi_D\mathbf{r}'_A)}\, e^{-\imath\frac{k}{z}(\mathbf{r}_A\xi_D+\mathbf{r}_D\xi_A)}\, d^2\xi_A\, d^2\xi_D\, d^2r'_A\, d^2r'_D \tag{24}$$

for the image cross-spectral density. The integral in equation (24) can be separate by introducing the canonical co-ordinate transformations $\mathbf{x}_1=\mathbf{x}_A+\frac{\mathbf{x}_D}{2}$ and $\mathbf{x}_2=\mathbf{x}_A-\frac{\mathbf{x}_D}{2}$ for $\mathbf{x}=\mathbf{r}',\xi$, so that $d^2x_A\, d^2x_D=d^2x_1\, d^2x_2$. Indeed,

$$W_{EW}(\mathbf{r}_1,\mathbf{r}_2;\omega)=\left(\frac{1}{\lambda z}\right)^2\left(\frac{1}{\lambda z'}\right)^2 e^{\imath\frac{k}{2z}|\mathbf{r}_1|^2}\int_{ew}\int_{EP}\sqrt{S_{ew}(\mathbf{r}'_1;\omega)}\, t(\mathbf{r}'_1)$$

$$P(\xi_1)\, e^{\imath\frac{k}{2z'}|\mathbf{r}'_1|^2}\, e^{\imath\frac{k}{2F}|\xi_1|^2}\, e^{-\imath\frac{k}{z'}\xi_1\mathbf{r}'_1}\, e^{-\imath\frac{k}{z}\mathbf{r}_1\xi_1}\, d^2\xi_1\, d^2r'_1 \tag{25}$$

$$e^{-\imath\frac{k}{2z}|\mathbf{r}_2|^2}\int_{ew}\int_{EP}\sqrt{S_{ew}(\mathbf{r}'_2,\omega)}\, t^*(\mathbf{r}'_2)\, P^*(\xi_2)\, e^{-\imath\frac{k}{2z'}|\mathbf{r}'_2|^2}\, e^{-\imath\frac{k}{2F}|\xi_2|^2}\, e^{\imath\frac{k}{z'}\xi_2\mathbf{r}'_2}\, e^{\imath\frac{k}{z}\mathbf{r}_2\cdot\xi_2}\, d^2\xi_2\, d^2r'_2.$$

So, the image power spectrum will be obtained by evaluating equation (25) for $\mathbf{r}_1=\mathbf{r}_2=\mathbf{r}$, i.e.

$$S_{EW}(\mathbf{r};\omega)=\left|\left(\frac{1}{\lambda z}\right)\left(\frac{1}{\lambda z'}\right)\int_{ew}\sqrt{S_{ew}(\mathbf{r}';\omega)}\, t(\mathbf{r}')\, e^{\imath\frac{k}{2z'}|\mathbf{r}'|^2}\int_{EP}P(\xi)\, e^{\imath\frac{k}{2F}|\xi|^2}\, e^{-\imath\frac{k}{z'}\left(\frac{z'}{z}\mathbf{r}+\mathbf{r}'\right)\xi}\, d^2\xi\, d^2r'\right|^2. \tag{26}$$

But $h\left(\mathbf{r}'+\frac{z'}{z}\mathbf{r};\frac{1}{F}\right)=\int_{EP}P(\xi)\, e^{\imath\frac{k}{2F}|\xi|^2}\, e^{-\imath\frac{k}{z'}\left(\mathbf{r}'+\frac{z'}{z}\mathbf{r}\right)\xi}\, d^2\xi$ is the *(defocused) impulse response* of the imaging system. Thus, according to the well-known result for full coherent imaging, the image power spectrum will be proportional to the square modulus of the convolution between the amplitude that emerges from the *ew* plane and the (defocused) impulse response of the system, i.e.

$$S_{EW}(\mathbf{r};\omega)=\left|\left(\frac{1}{\lambda z'}\right)\left(\frac{1}{\lambda z}\right)\int_{ew}\sqrt{S_{ew}(\mathbf{r}';\omega)}\, t(\mathbf{r}')\, e^{\imath\frac{k}{z'}|\mathbf{r}'|^2}\, h\left(\mathbf{r}'+\frac{z'}{z}\mathbf{r};\frac{1}{F}\right)d^2r'\right|^2.$$

SUMMARY

We have shown a description of spatially partially coherent imaging based on the use of Wigner distribution functions (WDF). The cross spectral density was expressed as the superposition of second order spatial coherence wavelets, whose amplitude is the marginal power spectra. Such spectra take the mathematical form of WDFs.

Therefore, spatial coherence wavelets are the vehicles for transport of both correlation and power through the imaging system. In this dynamics, the spatial coherence wavelets will be affected by the system through its elementary transfer function. We depicted their propagation along the axis of cones with vertices on each point of both the object plane and the pupil plane. However, the wavelets were composed by two types of contributions: the incoherent one from the individual centre of secondary disturbance located at the cone vertex, and the partially coherent one due the correlation between pairs of centers, around the vertex, which modulates the first contribution. The contributing pairs are those located within the support of the complex degree of spatial coherence.

The consistency of the model with the both extreme cases of full coherent and incoherent imaging was proved. In the last case we obtained the classical concept of optical transfer function as a simple integral of the elementary transfer function. Furthermore, the elementary incoherent response function was introduced as the Fourier transform of the elementary transfer function. It describes the propagation of spatial coherence wavelets form each object point to each image point through a specific point on the pupil planes. The point spread function of the system was obtained by a simple integral of the elementary incoherent response function.

Acknowledgments
This work was finished during a visit of the author to the Abdus Salam International Centre for Theoretical Physics (AS-ICTP), Trieste, Italy, in February 2003. The author thanks the DIMED (Universidad Nacional de Colombia Sede Medellin Grant 2010100604), Colciencias (Grant 1118-05-11518),) and the AS-ICTP for the financial support.

REFERENCES

1. Simon, R. and N. Mukunda. *Optical phase space, Wigner representation and invariant quality parameters*. J. Opt. Soc. Am. A. **17** (2000) 2440-2463.
2. Dragoman, D. *The Wigner distribution function in optics and optoelectronics.* Progress in Optics 37, Emil Wolf ed., Elsevier, Amsterdam, 1997. 1-56.
3. Mandel, L. and E. Wolf. *Optical coherence and quantum optics.* Cambridge University Press, Cambridge, 1995.
4. Castañeda, R. and J. García. *Spatial coherence wavelets: mathematical properties and physical features.* Accepted for publication in *Journal of Modern Optics*, 2003.
5. Castañeda, R. and J. García. *Spatial coherence wavelets.* In press in *Journal of Modern Optics*, 2002.
6. Born, M and E. Wolf. *Principles of Optics.* 5. ed. Pergamon Press, Oxford, 1975.
7. García, J. and R. Castañeda. *Spatial partially coherent imaging.* J. Mod. Opt. **49** (2002) 2093-2104.
8. Castañeda, R and F. Medina. *Partially coherent imaging with Schell-model beams.* Opt. Laser Tech. **29** (1997), 165-170.