IC/IR/2003/2
INTERNAL REPORT
(Limited Distribution)

United Nations Educational Scientific and Cultural Organization
and
International Atomic Energy Agency
THE ABDUS SALAM INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

ELECTRODYNAMICS AS A THEORY OF INTERACTING COMPLEX CHARGES

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Abstract

In this paper, we formulate a general theory of electrodynamics which incorporates both electric and magnetic charges. The mathematical origin of a second vector potential and magnetic charge is established. Electrodynamics is then reformulated in complex form as a theory of complex charges moving in a complex force field. This provides the framework for complex charged particle interactions as a generalization of Schwinger's theory of dyon-dyon interactions. The concept of duality transformation relating electric and magnetic charge spaces is developed within the general framework of electrodynamics in complex form.

MIRAMARE – TRIESTE
April 2003

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1 Introduction

A lot of effort has been made towards achieving a firm theoretical basis and experimental confirmation of the existence of magnetic charge (or magnetic monopoles) in nature, particularly after Dirac used them in a formulation of electrodynamics to account for the observed quantization of electric charge (Dirac, 1931; 1948). A comprehensive review of the current theoretical and experimental status, based mainly on Schwinger's model (Schwinger, 1966; 1968; 1969; 1975; Schwinger et al, 1976), has been presented by Gamberg et al and Milton et al (Gamberg et al, 2000; Milton et al, 2002). An important theoretical direction in this effort has involved the introduction of a second four-vector potential associated with the hypothesized magnetic current density four-vector (Cabbibo and Ferrari, 1962; Salam, 1966; Barker and Graziani, 1978; Singleton, 1995, 1996a, b; Ziino, 2000).

The main observation to make is that in all the good work cited above, magnetic charge has been introduced by hand as a logical modification of Maxwell's equations to achieve the expected symmetry between all the electric and magnetic components of the electrodynamic system. It is therefore to be expected that all the work to date has concentrated on investigating the compatibility of magnetic charge with the mathematical and physical principles of classical and quantum electrodynamics, as well as the possibility of experimental confirmation of their existence in nature. The current theoretical position is that it is possible to formulate a consistent classical and quantum theory of electrodynamics with magnetic charge and one or two four-vector field potential(s), the case of two four-vector potentials being the more preferable. On the experimental side, magnetic monopoles of masses lower than about 300GeV have been ruled out (Milton et al, 2002).

Despite the fact that the inclusion of magnetic charge in the equations of electrodynamics has had the important consequence of adequately accounting for the observed quantization of electric charge up to the quark level (Schwinger, 1969, 1970), there has not been any formal theory of electrodynamics incorporating magnetic charge as a basic component of the electrodynamic medium. This is the issue to be addressed in the present paper. There are two goals to achieve: (i) establishing a mathematical origin of magnetic charge following the introduction of a second field potential four-vector in the basic field equations of electrodynamics and (ii) formulating a formal theory of electrodynamics incorporating magnetic charge, alongside electric charge, as basic components of the electrodynamic medium.

The formulation of electrodynamics in terms of vector potentials together with the electric and magnetic charge sources is presented in section 2. The resulting field equations are a set of modified Maxwell's equations governing the coupling of electric and magnetic field intensities on the one hand, and the two field potential four-vectors on the other hand, which pave the way for a reformulation of electrodynamics in complex form in section 3. Electrodynamics is then modelled as a theory of complex charges moving in a complex force field. The important concept of duality transformation is then developed in section 4, followed by the covariant form of the field equations in section 5. The classical theory of complex charged particle (or dyon-dyon) interaction is presented in section 6. The conclusions are given in section 7.

2 Maxwell's Equations, Vector Potentials and Magnetic Charge

2.1 The First Vector Potential

Maxwell's equations of standard electrodynamics are expressed in the following form using gaussian units:

$$\nabla \cdot \mathbf{E} = 4\pi\rho \tag{1a}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{1b}$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \tag{1c}$$

$$\nabla \times \mathbf{B} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c}\mathbf{J} \tag{1d}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0 \ , \tag{1e}$$

where the last equation expressing the conservation of electric charge has been included to complete the set of equations. As usual, we have denoted the electric field intensity $\mathbf{E}$, the magnetic field intensity $\mathbf{B}$, electric charge density ρ, electric current density $\mathbf{J}$ and the speed of light c. In the coordinate frame specified by the set of coordinates $(x_0 = ct \ , \ \mathbf{r})$, the electric current density four-vector is defined by,

$$J = (J_0 \ , \ \mathbf{J}) = (\rho c \ , \ \mathbf{J}) \tag{1f}$$

The standard vector calculus results for any vector $\mathbf{V}$ and scalar χ,

$$\nabla \cdot (\nabla \times \mathbf{V}) = 0 \tag{2a}$$

$$\nabla \times (\nabla \chi) = 0 \ , \tag{2b}$$

are applied to Maxwell's eqs. $(1b)$ and $(1c)$ to introduce a four-vector electromagnetic field potential denoted as usual by $A = (A_0 \ , \ \mathbf{A})$ to define the electric and magnetic field intensities in the form,

$$\mathbf{E} = -\nabla A_0 - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} \tag{3a}$$

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{3b}$$

These are then used in Maxwell's eqs. $(1a)$ and $(1d)$, applying the standard vector calculus results,

$$\nabla \times (\nabla \times \mathbf{V}) = \nabla(\nabla \cdot \mathbf{V}) - \nabla^2 \mathbf{V} \tag{4a}$$

$$\nabla \cdot (\nabla \chi) = \nabla^2 \chi \tag{4b}$$

to obtain wave propagation equations for A_0 and $\mathbf{A}$ in the form,

$$\frac{1}{c^2}\frac{\partial^2 A_0}{\partial t^2} - \nabla^2 A_0 = 4\pi\rho \tag{5a}$$

$$\frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla^2 \mathbf{A} = \frac{4\pi}{c}\mathbf{J} \ , \tag{5b}$$

under the Lorentz gauge condition,

$$\frac{1}{c}\frac{\partial A_0}{\partial t} + \nabla \cdot \mathbf{A} = 0 \tag{5c}$$

The single field potential four-vector $A = (A_0 \ , \ \mathbf{A})$ is normally considered to specify the electrodynamic field adequately. We see below that Maxwell's equations would admit a second field potential four-vector.

2.2 The Second Vector Potential

We now take up the issue of whether or not a single four-vector potential is mathematically adequate for describing electrodynamics within the framework of Maxwell's eqs. $(1a)$-$(1e)$. The starting point is the definition of the electric field intensity $\mathbf{E}$ given in eq. $(3a)$ following the introduction of the four-vector potential $A = (A_0 \ , \ \mathbf{A})$ through the standard vector calculus results in eqs. $(2a)$-$(2b)$ applied to Maxwell's eqs. $(1b)$ and $(1c)$.

In particular, we consider the definition of $\mathbf{E}$ in relation to its substitution in Maxwell's eq. $(1a)$ to derive the wave equation for the scalar potential A_0 presented in eq. $(5a)$. We notice that according to eq. $(2a)$, we can extend the definition of $\mathbf{E}$ in eq. $(3a)$ by including the curl of any arbitrary vector without changing the final result in eq. $(5a)$ derived through substituting the so modified $\mathbf{E}$ in Maxwell's eq. $(1a)$. In this respect, we introduce a second vector potential $\mathbf{W}$ to extend the definition of $\mathbf{E}$ given in eq. $(3a)$ in the form,

$$\mathbf{E} = -\nabla A_0 - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - \nabla \times \mathbf{W} \tag{6}$$

Substituting for $\mathbf{E}$ from eq. (6) into Maxwell's eq. $(1a)$ and applying eq. $(2a)$, we obtain

$$\nabla \cdot \mathbf{E} = \nabla \cdot (-\nabla A_0 - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}) \ ,$$

as before. The modified $\mathbf{E}$ therefore leads to the same wave equation for A_0 as given in eq. $(5a)$ under the Lorentz gauge condition given in eq. $(5c)$.

From the above, we conclude that the definition of the electric field intensity given by eq. $(3a)$ is mathematically inadequate, since Maxwell's eq. $(1a)$ admits an additional term in the form of the curl of a second vector without altering the result obtained through application of the same Lorentz gauge condition for the first four-vector potential A. We observe that there is no compelling physical objection against the introduction of a second vector potential as desired mathematically. We therefore adopt eq. (6) as the mathematically adequate definition of the electric field intensity to be used in Maxwell's equations.

Let us then proceed by substituting for $\mathbf{E}$ from eq. (6) into Maxwell's eq. (1d), applying the Lorentz gauge condition from eq. (5c) and the vector calculus result from eq. (4a) to obtain,

$$\frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla^2 \mathbf{A} = -\nabla \times (\mathbf{B} + \frac{1}{c}\frac{\partial \mathbf{W}}{\partial t} - \nabla \times \mathbf{A}) + \frac{4\pi}{c}\mathbf{J} \tag{7a}$$

Since the mathematical extension of the definition of $\mathbf{E}$ does not change the form of the wave equation for the temporal component A_0 of the original four-vector potential $A = (A_0\ ,\ \mathbf{A})$, it must also not change the form of the wave equation for the spatial component $\mathbf{A}$ given in eq. (5b), if the final wave equation for the four-vector A is to remain the same. This condition is fulfilled in eq. (7a) if we set,

$$\nabla \times (\mathbf{B} + \frac{1}{c}\frac{\partial \mathbf{W}}{\partial t} - \nabla \times \mathbf{A}) = 0 \tag{7b}$$

We then apply the vector calculus result from eq. (2b) to introduce a scalar potential conveniently denoted by W_0 to solve eq. (7b) in the form,

$$\mathbf{B} + \frac{1}{c}\frac{\partial \mathbf{W}}{\partial t} - \nabla \times \mathbf{A} = -\nabla W_0\ ,$$

from which we obtain the modified definition of the magnetic field intensity in the form,

$$\mathbf{B} = -\nabla W_0 - \frac{1}{c}\frac{\partial \mathbf{W}}{\partial t} + \nabla \times \mathbf{A} \tag{8}$$

This provides the desired mathematically adequate definition of the magnetic field intensity $\mathbf{B}$. The introduction of a second vector potential $\mathbf{W}$ and a second scalar potential W_0 to complete the mathematical definitions of the electric and magnetic field intensities has not interfered with the wave propagation equation for the original four-vector potential derived through application of Maxwell's eqs. (1a) and (1d) together with the Lorentz gauge condition in eq. (5c).

2.3 Magnetic Charge

We now want to consider the physical implication of the mathematical generalization of the definitions of $\mathbf{E}$ and $\mathbf{B}$ in relation to Maxwell's eqs. (1c) and (1b), which made the starting point of the whole process.

Taking the divergence of $\mathbf{B}$ in eq. (8), using eq. (2a) and applying a condition,

$$\frac{1}{c}\frac{\partial W_0}{\partial t} + \nabla \cdot \mathbf{W} = 0\ , \tag{9}$$

we obtain,

$$\frac{1}{c^2}\frac{\partial^2 W_0}{\partial t^2} - \nabla^2 W_0 = \nabla \cdot \mathbf{B}\ , \tag{10a}$$

which is a wave equation for W_0 driven by a scalar quantity defined by $\nabla \cdot \mathbf{B}$. We introduce a scalar function Φ_0 defined by,

$$\nabla \cdot \mathbf{B} = \Phi_0\ , \tag{10b}$$

which we use to write eq. (10a) in the form,

$$\frac{1}{c^2}\frac{\partial^2 W_0}{\partial t^2} - \nabla^2 W_0 = \Phi_0 \tag{10c}$$

We observe that eq. (9) is just the Lorentz condition for a four-vector

$$W = (W_0 \ , \ \mathbf{W})$$

Next, we take the curl of $\mathbf{E}$ in eq. (6) and then apply eqs. (2b), (4a), (8) and (9) to obtain the wave equation for $\mathbf{W}$ in the form,

$$\frac{1}{c^2}\frac{\partial^2 \mathbf{W}}{\partial t^2} - \nabla^2 \mathbf{W} = -(\nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}) \tag{11a}$$

We introduce a vector $\mathbf{\Phi}$ defined by,

$$\mathbf{\Phi} = -(\nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}) \ , \tag{11b}$$

which we use to write the wave equation in eq. (11a) in the form,

$$\frac{1}{c^2}\frac{\partial^2 \mathbf{W}}{\partial t^2} - \nabla^2 \mathbf{W} = \mathbf{\Phi} \tag{11c}$$

We express eq. (11b) in the familiar form,

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} - \mathbf{\Phi} \ , \tag{11d}$$

Finally, we take the divergence of eq. (11d) and then apply eqs. (2a) and (10b) to obtain,

$$\frac{1}{c}\frac{\partial \Phi_0}{\partial t} + \nabla \cdot \mathbf{\Phi} = 0 \ , \tag{12a}$$

which suggests that we can define Φ_0 as the temporal component of a four-vector Φ whose spatial component is vector $\mathbf{\Phi}$. Thus we have the four-vector,

$$\Phi = (\Phi_0 \ , \ \mathbf{\Phi}) \tag{12b}$$

Comparision of eqs. (10b) and (11d) with Maxwell's equations leads to the conclusion that $\Phi = (\Phi_0 \ , \ \mathbf{\Phi})$ should be interpreted as a magnetic current density four-vector. We therefore introduce appropriately defined magnetic charge density ϱ and magnetic current density $\mathbf{K}$ forming a four-vector $K = (\varrho c \ , \ \mathbf{K})$ so that in gaussian units we have,

$$\Phi = \frac{4\pi}{c}K = \frac{4\pi}{c}(\varrho c \ , \ \mathbf{K}) \ , \tag{13a}$$

according to which we define,

$$\Phi_0 = 4\pi\varrho \ ; \qquad \mathbf{\Phi} = \frac{4\pi}{c}\mathbf{K} \tag{13b}$$

Substituting for Φ_0 and $\mathbf{\Phi}$ from eq. (13b) in eq. (12a), we obtain the continuity equation for magnetic charge in the form,

$$\frac{\partial \varrho}{\partial t} + \nabla \cdot \mathbf{K} = 0 \ , \tag{13c}$$

We have thus established the mathematical origin of a second field potential and the associated magnetic current density four-vectors in electrodynamics.

2.4 Modified Field Equations

We can now collect together the set of modified field equations expressed in gaussian units as below.

Modified Maxwell's Equations

$$\nabla \cdot \mathbf{E} = 4\pi\rho \tag{14a}$$

$$\nabla \cdot \mathbf{B} = 4\pi\varrho \tag{14b}$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} - \frac{4\pi}{c}\mathbf{K} \tag{14c}$$

$$\nabla \times \mathbf{B} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c}\mathbf{J} \tag{14d}$$

$$J = (\rho c\ ,\ \mathbf{J})\ ; \qquad \frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0 \tag{14e}$$

$$K = (\varrho c\ ,\ \mathbf{K})\ ; \qquad \frac{\partial \varrho}{\partial t} + \nabla \cdot \mathbf{K} = 0 \tag{14f}$$

Four-vector Potential Equations

$$\frac{1}{c}\frac{\partial A_0}{\partial t} + \nabla \cdot \mathbf{A} = 0 \tag{15a}$$

$$\frac{1}{c}\frac{\partial W_0}{\partial t} + \nabla \cdot \mathbf{W} = 0 \tag{15b}$$

$$\mathbf{E} = -\nabla A_0 - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - \nabla \times \mathbf{W} \tag{15c}$$

$$\mathbf{B} = -\nabla W_0 - \frac{1}{c}\frac{\partial \mathbf{W}}{\partial t} + \nabla \times \mathbf{A} \tag{15d}$$

$$\frac{1}{c^2}\frac{\partial^2 A_0}{\partial t^2} - \nabla^2 A_0 = 4\pi\rho\ ; \qquad \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla^2 \mathbf{A} = \frac{4\pi}{c}\mathbf{J} \tag{15e}$$

$$\frac{1}{c^2}\frac{\partial^2 W_0}{\partial t^2} - \nabla^2 W_0 = 4\pi\varrho\ ; \qquad \frac{1}{c^2}\frac{\partial^2 \mathbf{W}}{\partial t^2} - \nabla^2 \mathbf{W} = \frac{4\pi}{c}\mathbf{K} \tag{15f}$$

We observe that eqs. (15*c*)-(15*d*) can be rearranged to take the form of eqs. (14*c*)-(14*d*) of Maxwell equations. The two electrodynamic four-vectors $A = (A_0\ ,\ \mathbf{A})$ and $W = (W_0\ ,\ \mathbf{W})$ are thus coupled through Maxwell type equations. But, as we have seen during the derivation, they can be effectively decoupled through application of Maxwell's equations and the Lorentz gauge condition to propagate as independent waves in accordance with eqs. (15*e*)-(15*f*). On the other hand, the electric and magnetic current density four-vectors $J = (\rho c\ ,\ \mathbf{J})$ and $K = (\varrho c\ ,\ \mathbf{K})$ are coupled indirectly through eqs. (15*c*) and (15*d*) taken together with the solutions of the wave equations (15*e*)-(15*f*) for A and W. As it is already clear, the electric field intensity $\mathbf{E}$ and the magnetic field intensity $\mathbf{B}$ are coupled through Maxwell's equations. We have therefore obtained a complete set of field equations coupling four-vector current densities, four-vector potentials and the field intensities, which characterize the electrodynamic field.

The derivation presented here has established the mathematical and physical foundation of the introduction of a second field potential four-vector and a magnetic current density four-vector which acts as a source in the basic equations of electrodynamics. As we pointed out in the Introduction, the case of electrodynamics with two four-vector field potentials within the context of electric and magnetic charges has been presented and discussed before in the literature, normally starting with the modified Maxwell's equations as in eqs. $(14a)$-$(14f)$ on the assumption that magnetic charge exists to complete the expected symmetry between the electric and magnetic components of the electromagnetic field. The important outcome of those works is that the introduction of a second four-vector potential in this respect does not violate the physical principles governing electrodynamics, both classical and quantum. The role of magnetic charge in the observed quantization of electric charge has been clearly specified, except in a recent work (Ziino, 2000) where it has been observed that the anticommutativity between electric and magnetic charge operators forbids the expected interaction between electric and magnetic charges which would lead to the quantization condition.

The occurence of two four-vector potentials, one generated by electric charge sources and the other generated by magnetic charge sources is nolonger in dispute. The issue is why magnetic charge and effects associated with the second four-vector field potential have not been observed experimentally. The popular theoretical explanation is that magnetic monopoles (particles carrying magnetic charge) are very heavy, with mass lower than about 300GeV already ruled out in a recent experiment (Milton et al, 2002). The Higg's mechanism has been applied (Singleton, 1995) to establish that the second four-vector potential W may represent a massive photon whose effects can be observed only when the approapriate mass scale is achieved. In contrast, it has also been suggested (Ziino, 2000) that electric and magnetic charges are complementary, owing to the anticommutativity of their charge operators. In this respect, both four-vector potentials represent massless photons, but magnetic monopoles and their associated photons cannot be observed in experiments where electric charges and their associated photons play a prominent role. We shall give a related idea based on duality transformation in section 6.

3 Electrodynamics in Complex Form

The basic field equations presented in eqs. $(14a)$-$(15f)$ describe the coupling of three pairs of related physical quantities. The form the equations makes it possible to introduce complex quantities which combine each pair as real and imaginary parts. The end result will be a simple reformulation of electrodynamics in terms of complex current density four-vector, field potential four-vector and field intensity vector.

3.1 Complex Field Intensity and Current Density Four-vector

Let us start with the modified Maxwell's equations. We find it convenient to multiply eq. $(14a)$ by $\frac{1}{\sqrt{2}}$ and eq. $(14b)$ by $\frac{i}{\sqrt{2}}$ (the imaginary number $i = \sqrt{-1}$) and then add the results. Likewise, we multiply eq. $(14c)$ by $\frac{1}{\sqrt{2}}$ and eq. $(14d)$ by $\frac{i}{\sqrt{2}}$ and then add the results. On introducing a complex field intensity denoted by $\mathbf{Q}$, a complex charge density denoted by Λ and a complex current density denoted by **Z**, which we define by,

$$\mathbf{Q} = \frac{1}{\sqrt{2}}(\mathbf{E} + i\mathbf{B}) \tag{16a}$$

$$\Lambda = \frac{1}{\sqrt{2}}(\rho + i\varrho) \; ; \qquad \mathbf{Z} = \frac{1}{\sqrt{2}}(\mathbf{J} + i\mathbf{K}) \; , \tag{16b}$$

with complex current density four-vector Z defined by,

$$Z = \frac{1}{\sqrt{2}}(J + iK) = (\Lambda c \; , \; \mathbf{Z}) \; , \tag{16c}$$

we finally express the Maxwell equations in terms of the complex quantities in the form,

$$\nabla \cdot \mathbf{Q} = 4\pi\Lambda \tag{17a}$$

$$\nabla \times \mathbf{Q} = \frac{i}{c}\frac{\partial \mathbf{Q}}{\partial t} + \frac{4\pi i}{c}\mathbf{Z} \tag{17b}$$

Taking the divergence of eq. (17*b*) and using eq. (17*a*) leads to,

$$\frac{\partial \Lambda}{\partial t} + \nabla \cdot \mathbf{Z} = 0 \; , \tag{17c}$$

which is the continuity equation for the complex current density four-vector $Z = (\Lambda c \; , \; \mathbf{Z})$. Taking the complex conjugation of eqs. (17*a*)-(17*c*), we obtain the Maxwell equations for the complex conjugates,

$$\nabla \cdot \mathbf{Q}^* = 4\pi\Lambda^* \tag{18a}$$

$$\nabla \times \mathbf{Q}^* = -\frac{i}{c}\frac{\partial \mathbf{Q}^*}{\partial t} - \frac{4\pi i}{c}\mathbf{Z}^* \tag{18b}$$

$$\frac{\partial \Lambda^*}{\partial t} + \nabla \cdot \mathbf{Z}^* = 0 \; , \tag{18c}$$

where from eqs. (16*a*)-(16*c*) we have,

$$\mathbf{Q}^* = \frac{1}{\sqrt{2}}(\mathbf{E} - i\mathbf{B}) \tag{19a}$$

$$\Lambda^* = \frac{1}{\sqrt{2}}(\rho - i\varrho) \; ; \qquad \mathbf{Z}^* = \frac{1}{\sqrt{2}}(\mathbf{J} - i\mathbf{K}) \tag{19b}$$

$$Z^* = \frac{1}{\sqrt{2}}(J - iK) = (\Lambda^* c \; , \; \mathbf{Z}^*) \tag{19c}$$

We use eqs. (16*a*) and (19*a*) to obtain,

$$\mathbf{Q}^* \cdot \mathbf{Q} = \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2) \tag{20a}$$

$$\mathbf{Q}^2 = \mathbf{Q} \cdot \mathbf{Q} = \frac{1}{2}(\mathbf{E}^2 - \mathbf{B}^2) + i\mathbf{E} \cdot \mathbf{B} \; ; \qquad \mathbf{Q}^{*2} = \frac{1}{2}(\mathbf{E}^2 - \mathbf{B}^2) - i\mathbf{E} \cdot \mathbf{B} \tag{20b}$$

$$\mathbf{Q} \times \mathbf{Q}^* = -i\mathbf{E} \times \mathbf{B} \; , \tag{20c}$$

which lead to the definitions of the following physical quantities of the electromagnetic field:

Electromagnetic Energy Density ($\mathcal{E}_{em} = \frac{1}{8\pi}(\mathbf{E}^2 + \mathbf{B}^2)$):

$$\mathcal{E}_{em} = \frac{1}{4\pi}\mathbf{Q}^* \cdot \mathbf{Q} \qquad (21a)$$

Electromagnetic Lagrangian Density ($\mathcal{L}_{em} = \frac{1}{8\pi}(\mathbf{E}^2 - \mathbf{B}^2)$):

$$\mathcal{L}_{em} = \frac{1}{8\pi}(\mathbf{Q}^{*2} + \mathbf{Q}^2) \qquad (21b)$$

The Poynting Vector ($\mathbf{S} = \frac{c}{4\pi}\mathbf{E} \times \mathbf{B}$):

$$\mathbf{S} = \frac{ic}{4\pi}\mathbf{Q} \times \mathbf{Q}^* \qquad (21c)$$

Electromagnetic Linear Momentun Density $\mathbf{P}_{em}$:

$$\mathbf{P}_{em} = \frac{\mathbf{S}}{c^2} \qquad (21d)$$

We also obtain an invariant electromagnetic field quantity $\mathcal{I}_{em} = \frac{1}{4\pi}\mathbf{E} \cdot \mathbf{B}$:

$$\mathcal{I}_{em} = \frac{i}{8\pi}(\mathbf{Q}^{*2} - \mathbf{Q}^2) \qquad (21e)$$

Use of eqs. (16a)-(16b) and (19a)-(19b) gives,

$$\mathbf{Q}^* \cdot \mathbf{Z} + \mathbf{Q} \cdot \mathbf{Z}^* = \mathbf{E} \cdot \mathbf{J} + \mathbf{B} \cdot \mathbf{K} \ , \qquad (22)$$

which defines the rate at which the electromagnetic field intensity performs work on the complex currents (i.e., electric and magnetic currents). We also find,

$$\Lambda\mathbf{Q}^* + \Lambda^*\mathbf{Q} = \rho\mathbf{E} + \varrho\mathbf{B} \ ; \qquad \frac{i}{c}(\mathbf{Z}^* \times \mathbf{Q} - \mathbf{Z} \times \mathbf{Q}^*) = \frac{1}{c}(\mathbf{K} \times \mathbf{E} - \mathbf{J} \times \mathbf{B}) \ , \qquad (23)$$

from which the definition of the Lorentz force $\mathbf{F}_L$ follows in the form,

$$\mathbf{F}_L = (\Lambda\mathbf{Q}^* + \frac{i}{c}\mathbf{Z} \times \mathbf{Q}^*) + (\Lambda^*\mathbf{Q} - \frac{i}{c}\mathbf{Z}^* \times \mathbf{Q}) \qquad (24)$$

3.2 Complex Field Potential Four-vector

Multiplying eq. (15c) by $\frac{1}{\sqrt{2}}$, eq. (15d) by $\frac{i}{\sqrt{2}}$ and then adding the results gives,

$$\mathbf{Q} = -\nabla Y_0 - \frac{1}{c}\frac{\partial \mathbf{Y}}{\partial t} + i\nabla \times \mathbf{Y} \qquad (25)$$

after introducing a complex four-vector field potential $Y = (Y_0 \ , \ \mathbf{Y})$ defined by,

$$Y_0 = \frac{1}{\sqrt{2}}(A_0 + iW_0) \ ; \qquad \mathbf{Y} = \frac{1}{\sqrt{2}}(\mathbf{A} + i\mathbf{W}) \ ; \qquad Y = \frac{1}{\sqrt{2}}(A + iW) \qquad (26)$$

Taking the curl of eq. (25) and using Maxwell's eq. (17b), we obtain the wave equation for $\mathbf{Y}$,

$$\frac{1}{c^2}\frac{\partial^2 \mathbf{Y}}{\partial t^2} - \nabla^2\mathbf{Y} = \frac{4\pi}{c}\mathbf{Z} \qquad (27a)$$

under the Lorentz gauge condition,

$$\frac{1}{c}\frac{\partial Y_0}{\partial t} + \nabla \cdot \mathbf{Y} = 0 \tag{27b}$$

Taking the divergence of eq. (25), using Maxwell's eq. (17a) and applying eq. (27b) gives the wave equation for Y_0,

$$\frac{1}{c^2}\frac{\partial^2 Y_0}{\partial t^2} - \nabla^2 Y_0 = 4\pi\Lambda \tag{27c}$$

The complex conjugates of eqs. (25)-(27c) are obtained through taking complex conjugation.

4 The Duality Transformation

It would be useful at this stage to define the duality transformation at this stage. The definition of $\mathbf{Q}$ in terms of Y in eq. (25), provides a good starting point. Taking the imaginary number i common, we write eq. (25) in the form,

$$(-i\mathbf{Q}) = -\nabla(-iY_0) - \frac{1}{c}\frac{\partial(-i\mathbf{Y})}{\partial t} + i\nabla \times (-i\mathbf{Y}) \tag{28a}$$

This motivates the introduction of new field intensity $\tilde{\mathbf{Q}}$ and field potential four-vector $\tilde{Y} = (\tilde{Y}_0\ ,\ \tilde{\mathbf{Y}})$, defined by,

$$\tilde{\mathbf{Q}} = -i\mathbf{Q}\ ; \qquad \tilde{Y}_0 = -iY_0\ , \quad \tilde{\mathbf{Y}} = -i\mathbf{Y}\ , \quad \tilde{Y} = -iY \tag{28b}$$

so that eq. (28a) becomes,

$$\tilde{\mathbf{Q}} = -\nabla\tilde{Y}_0 - \frac{1}{c}\frac{\partial\tilde{\mathbf{Y}}}{\partial t} + i\nabla \times \tilde{\mathbf{Y}} \tag{29}$$

We find,

$$\tilde{\mathbf{Q}} = \frac{1}{\sqrt{2}}(\mathbf{B} - i\mathbf{E}) = \frac{1}{\sqrt{2}}(\mathbf{B} + i(-\mathbf{E})) \tag{30a}$$

$$\tilde{Y}_0 = \frac{1}{\sqrt{2}}(W_0 - iA_0) = \frac{1}{\sqrt{2}}(W_0 + i(-A_0))\ ; \qquad \tilde{\mathbf{Y}} = \frac{1}{\sqrt{2}}(\mathbf{W} - i\mathbf{A}) = \frac{1}{\sqrt{2}}(\mathbf{W} + i(-\mathbf{A})) \tag{30b}$$

It is clear that $\tilde{\mathbf{Q}}$, $\tilde{Y}_0$, $\tilde{\mathbf{Y}}$ and $\tilde{Y}$ are obtained from $\mathbf{Q}$, Y_0, $\mathbf{Y}$ and Y, respectively through the set of transformations,

$$\tilde{\mathbf{Q}}: \qquad \mathbf{E} \to \mathbf{B}\ , \quad \mathbf{B} \to -\mathbf{E} \tag{31a}$$

$$\tilde{Y}: \quad A_0 \to W_0\ , \quad W_0 \to -A_0\ ; \quad \mathbf{A} \to \mathbf{W}\ , \quad \mathbf{W} \to -\mathbf{A}\ ; \quad A \to W\ , \quad W \to -A \tag{31b}$$

The set of transformations defined in eqs. (31a)-(31b) constitute the duality transformation. In this respect, $\tilde{\mathbf{Q}}$, $\tilde{Y}_0$, $\tilde{\mathbf{Y}}$ and $\tilde{Y}$ as defined in eqs. (28b) and (30a)-(30b) are the duals of $\mathbf{Q}$, Y_0, $\mathbf{Y}$ and Y, respectively.

Multiplying Maxwell's eqs. (17a)-(17b) by $-i$, applying the definitions of duals from eq. (28b) and defining dual charge and current densities by,

$$\tilde{\Lambda} = -i\Lambda = \frac{1}{\sqrt{2}}(\varrho - i\rho); \qquad \tilde{\mathbf{Z}} = -i\mathbf{Z} = \frac{1}{\sqrt{2}}(\mathbf{K} - i\mathbf{J})\ ; \qquad \tilde{Z} = -iZ\ , \tag{32a}$$

which satisfy the duality transformations

$$\rho \to \varrho\ , \quad \varrho \to -\rho\ ; \qquad \mathbf{J} \to \mathbf{K}\ , \quad \mathbf{K} \to -\mathbf{J}\ ; \qquad J \to K\ , \quad K \to -J\ , \tag{32b}$$

we obtain,

$$\nabla \cdot \tilde{\mathbf{Q}} = 4\pi\tilde{\Lambda} \tag{33a}$$

$$\nabla \times \tilde{\mathbf{Q}} = \frac{i}{c}\frac{\partial \tilde{\mathbf{Q}}}{\partial t} + \frac{4\pi i}{c}\tilde{\mathbf{Z}} \tag{33b}$$

Taking the divergence of eq. (33b) and applying eq. (33a) leads to the continuity equation,

$$\frac{\partial \tilde{\Lambda}}{\partial t} + \nabla \cdot \tilde{\mathbf{Z}} = 0 \tag{33c}$$

It is easy to use eq. (29) together with Maxwell's eqs. (33a)-(33b) for the dual field intensity to obtain the wave equations,

$$\frac{1}{c^2}\frac{\partial^2 \tilde{Y}_0}{\partial t^2} - \nabla^2 \tilde{Y}_0 = 4\pi\tilde{\Lambda} \tag{34a}$$

$$\frac{1}{c^2}\frac{\partial^2 \tilde{\mathbf{Y}}}{\partial t^2} - \nabla^2 \tilde{\mathbf{Y}} = \frac{4\pi}{c}\tilde{\mathbf{Z}} \tag{34b}$$

under the Lorentz gauge condition,

$$\frac{1}{c}\frac{\partial \tilde{Y}_0}{\partial t} + \nabla \cdot \tilde{\mathbf{Y}} = 0 \tag{34c}$$

According to eqs. (29) and (33a)-(34c), the electromagnetic field equations are invariant under the duality transformation. It is easy to confirm using eq. (25) that the complex conjugates of the duals are given by,

$$\tilde{\mathbf{Q}}^* = i\mathbf{Q}^*\ ; \qquad \tilde{Y}_0^* = iY_0^*\ ; \qquad \tilde{\mathbf{Y}}^* = i\mathbf{Y}^*\ ; \qquad \tilde{Y}^* = iY^*$$

4.1 The Duality Transformation Operator

To determine the form of the duality transformation operator which we denote by $\mathcal{D}$, we consider its action on the complex field intensity column matrix,

$$\mathbf{\Omega} = \begin{pmatrix} \mathbf{Q} \\ \mathbf{Q}^* \end{pmatrix}\ , \tag{35a}$$

such that,

$$\mathcal{D}\mathbf{\Omega} = \mathcal{D}\begin{pmatrix} \mathbf{Q} \\ \mathbf{Q}^* \end{pmatrix} = \begin{pmatrix} -i\mathbf{Q} \\ i\mathbf{Q}^* \end{pmatrix} = \begin{pmatrix} \tilde{\mathbf{Q}} \\ \tilde{\mathbf{Q}}^* \end{pmatrix} \tag{35b}$$

The solution of eq. (35b) is simple:

$$\mathcal{D} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \tag{36a}$$

We can write,

$$\mathcal{D} = -i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = -i\sigma_3\ , \tag{36b}$$

where σ_3 is the third component of the Pauli matrix:

$$\sigma = (\sigma_1, \sigma_2, \sigma_3) \; ; \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \; ; \quad \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \; ; \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad (36c)$$

It is easy to find,

$$\mathcal{D}^2 \mathbf{\Omega} = \mathcal{D}\mathcal{D}\mathbf{\Omega} = -\mathbf{\Omega} \qquad (37a)$$

Defining the matrix for the dual field intensities by,

$$\tilde{\mathbf{\Omega}} = \begin{pmatrix} -i\mathbf{Q} \\ i\mathbf{Q}^* \end{pmatrix} = \begin{pmatrix} \tilde{\mathbf{Q}} \\ \tilde{\mathbf{Q}}^* \end{pmatrix} , \qquad (37b)$$

we write eqs. (35*b*) and (37*a*) in the form,

$$\mathcal{D}\mathbf{\Omega} = \tilde{\mathbf{\Omega}} \; ; \qquad \mathcal{D}^2 \mathbf{\Omega} = \mathcal{D}\tilde{\mathbf{\Omega}} = -\mathbf{\Omega} \qquad (37c)$$

We observe that σ_3 is a generator of rotations about the $x^3 (= z)$-axis. The duality transformation operator is therefore associated with rotations which transform electric into magnetic and magnetic into electric components within the electromagnetic medium. It is the symmetry operation relating the electric and magnetic components of the electrodynamic field. If we define charge space as the space in which a given type of charge, together with its associated field potential and field intensity appear on the real axis, then we understand the duality operator to effect a transformation from the electric charge space to the magnetic charge space and vice versa. In this respect, we identify J, A and $\mathbf{E}$ as the basic elements of the electric charge space, while K, W, and $\mathbf{B}$ are the basic elements of the magnetic charge space. The two charge spaces are related by a duality transformation and their coupling through Maxwell's equations constitutes a unified electromagnetic field.

Finally, we generalize the duality transformation by introducing a transformation parameter θ to exponentiate the duality transformation operator in the form,

$$\mathcal{T}(\theta) = e^{\theta \mathcal{D}} = e^{-i\theta\sigma_3} , \qquad (38a)$$

which on using a standard identity takes the form,

$$\mathcal{T}(\theta) = I \cos\theta - i\sigma_3 \sin\theta = I \cos\theta + \mathcal{D} \cos\theta \; ; \qquad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad (38b)$$

We can now consider any basic electric charge space component ϕ_e and any basic magnetic charge space component ϕ_m with complex electromagnetic quantity defined within the electric charge space as usual by,

$$\phi = \frac{1}{\sqrt{2}} (\phi_e + i\phi_m) \qquad (39a)$$

and satisfying the duality transformation,

$$\tilde{\phi} = -i\phi \; ; \qquad \tilde{\phi}^* = i\phi^* \qquad (39b)$$

The corresponding electromagnetic matrix

$$\phi_{em} = \begin{pmatrix} \phi \\ \phi^* \end{pmatrix} \tag{39c}$$

satisfies the duality transformation,

$$\mathcal{D}\phi_{em} = \tilde{\phi}_{em} = \begin{pmatrix} -i\phi \\ i\phi^* \end{pmatrix} \tag{39d}$$

Application of the general transformation operator $\mathcal{T}(\theta)$ on ϕ_{em} and using the above results as appropriate gives the final result,

$$\mathcal{T}\phi_{em} = e^{\theta\mathcal{D}}\phi_{em} = \tilde{\phi}_{em}(\theta) \ , \tag{40a}$$

where

$$\tilde{\phi}_{em}(\theta) = \begin{pmatrix} \phi(\theta) \\ \phi^*(\theta) \end{pmatrix} = \begin{pmatrix} \phi_e\cos\theta + \phi_m\sin\theta + i(-\phi_e\sin\theta + \phi_m\cos\theta) \\ \phi_e\cos\theta + \phi_m\sin\theta - i(-\phi_e\sin\theta + \phi_m\cos\theta) \end{pmatrix} \tag{40b}$$

From this result, we conclude that under the general transformation represented by the operator $\mathcal{T}(\theta)$, the basic electric and magnetic charge space components transform according to,

$$\phi_e \to \phi_e\cos\theta + \phi_m\sin\theta \tag{40c}$$

$$\phi_m \to -\phi_e\sin\theta + \phi_m\cos\theta \ , \tag{40d}$$

which is understood to be a rotation through angle θ. We have thus obtained a generalization of the duality transformation. The duality transformation represented by the operator $\mathcal{D}$ is a special case defined by $\theta = \frac{1}{2}\pi$. Further details on the dual nature of the electromagnetic field is presented elsewhere.

5 The Field Equations in Covariant Form

We can now define the coordinate system,

$$x^0 = ct \ , \ x^1 = x \ , \ x^2 = y \ , \ x^3 = z \ ; \ x_0 = x^0 \ , \ x_j = -x^j \ , j = 1,2,3 \ ,$$

with,

$$\partial_0 = \frac{\partial}{\partial x^0} = \partial^0 = \frac{\partial}{\partial x_0} \ ; \qquad \partial_j = \frac{\partial}{\partial x^j} \ , \ \partial^j = \frac{\partial}{\partial x_j} = -\partial_j \ , \ j = 1,2,3$$

and

$$V^0 = V_t \ , \ V^1 = V_x \ , \ V^2 = V_y \ , \ V^3 = V_z \ ; \ V_0 = V^0 \ , \ V_j = -V^j \ , \ j = 1,2,3 \ ,$$

in usual notation to express the wave equations for the complex field potentials in covariant form.

In the electric charge space, eqs. (27a)-(27c) are then expressed in the covariant form,

$$\partial_\mu Y^{\mu\nu} = \frac{4\pi}{c}Z^\nu \ ; \qquad Y^{\mu\nu} = \partial^\mu Y^\nu - \partial^\nu Y^\mu \ ; \quad \mu,\nu = 0,1,2,3 \ , \tag{42a}$$

$$\partial_\mu Z^\mu = 0 \; ; \qquad \partial_\mu Y^\mu = 0 \tag{42b}$$

Following the definition of the complex field intensity $\mathbf{Q}$ in eq. (25), which on application of eq. (28b) can be written in the form,

$$\mathbf{Q} = -\nabla Y_0 - \frac{1}{c}\frac{\partial \mathbf{Y}}{\partial t} - \nabla \times \tilde{\mathbf{Y}} \; , \tag{25$'$}$$

we introduce a field strength tensor $F^{\mu\nu}$ defined by,

$$F^{\mu\nu} = Y^{\mu\nu} + \tilde{Y}^{\mu\nu} \; ; \qquad \tilde{Y}^{\mu\nu} = -iY^{\mu\nu} \tag{43a}$$

The field intensity is then defined through,

$$F^{j0} = Q^j \; ; \qquad F^{jk} = -\tilde{Q}^l \; , \quad j,k,l = 1,2,3 \; , \tag{43b}$$

where j, k, l are taken in cyclic order. Noting that in eq. (43a) we can also use the totally antisymmetric tensor $\epsilon^{\mu\nu\alpha\beta}$ to define

$$\tilde{Y}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}Y_{\alpha\beta}, \tag{43c}$$

we obtain,

$$\partial_\mu F^{\mu\nu} = \partial_\mu Y^{\mu\nu} \; , \tag{44}$$

since

$$\partial_\mu \tilde{Y}^{\mu\nu} = \frac{1}{2}\partial_\mu \epsilon^{\mu\nu\alpha\beta}Y_{\alpha\beta} = 0 \tag{45}$$

The result in eq. (45) also follows from the fact that the dual current density four-vector $\tilde{Z}$ is not defined within the electric charge space and $\tilde{Y}^{\mu\nu}$ must therefor satisfy eq.(45) within the electric charge space. Applying eq. (42a) in eq. (44), we finally write the field equations within the electric charge space in the form,

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c}Z^\nu \tag{46}$$

We observe that the covariant field equations presented in eq. (46) (or eq. (42a)) yield Maxwell's equations for $\mathbf{Q}$ given in eqs. (17a)-(17b). This easy to establish by considering for a start the cases $\nu = 0$ and $\nu = 1$ in eq. (46) and applying eq. (43b). Alternatively, we set $\nu = 0$ and $\nu = 1$ in eq. (42a) and then using eq. (25) to make appropriate substitutions for the field intensity components Q^j, with extra terms vanishing. In either approach, we apply the definition of the complex current density four-vector Z from eq. (16c) to obtain the results,

$$\nabla \cdot \mathbf{Q} = 4\pi\Lambda \tag{17a'}$$

$$(\nabla \times \mathbf{Q})_x = \frac{i}{c}\frac{\partial}{\partial t}Q_x + \frac{4\pi i}{c}Z_x \tag{17b'}$$

The second of these results, taken together with the results for $\nu = 2$ and $\nu = 3$ gives eq. (17b).

Application of the duality transformation leads to the covariant field equations in the magnetic charge space given by (see eqs. (34a)-(34c)),

$$\partial_\mu \tilde{Y}^{\mu\nu} = \frac{4\pi}{c}\tilde{Z}^\nu \; ; \qquad \tilde{Y}^{\mu\nu} = \partial^\mu \tilde{Y}^\nu - \partial^\nu \tilde{Y}^\mu \tag{47a}$$

$$\partial_\mu \tilde{Z}^\mu = 0 \; ; \qquad \partial_\mu \tilde{Y}^\mu = 0 \tag{47b}$$

or

$$\partial_\mu \tilde{F}^{\mu\nu} = \frac{4\pi}{c}\tilde{Z}^\nu \; , \tag{48a}$$

where now the dual field strength tensor is defined by,

$$\tilde{F}^{\mu\nu} = -iF^{\mu\nu} = \tilde{Y}^{\mu\nu} - Y^{\mu\nu} \tag{48b}$$

We shall present the details of the covariant formulation in the next paper where the Lagrangian (Hamiltonian) formalism is developed.

6 Complex Charged Particle Interactions

This is now the right stage to describe the mode of interaction between the complex charges, composed here of electric charge as the real part and magnetic charge as the imaginary part. In our general formulation, the complex charges generate the electromagnetic field. The Lorentz force, which we have defined in eq. (24) should be understood to represent an interaction between complex charges, which at the basic level is an interaction between electric and magnetic charges, either separately or jointly. The fact is that both electric and magnetic charges create and respond to the electromagnetic field. This is clearly reflected in the coupling of the four-vector potentials A and W according to eqs. (15c) and (15d). The electric field intensity $\mathbf{E}$ and the magnetic field intensity $\mathbf{B}$ both receive contributions from the moving electric and magnetic charges in accordance with the wave equations. On the other hand, static electric charges produce $\mathbf{E}$, while static magnetic charges produce $\mathbf{B}$, both depending only on position in such a special (nonrelativistic) case. As a consequence, a static complex charge produces a solely position dependent complex field intensity $\mathbf{Q}(\mathbf{r})$ in an electrodynamic medium in which both electric and magnetic charges exist.

To define the complex charges explicitly, we note that electric and magnetic charges are understood to be pointlike, with the charge densities ρ and ϱ being zero everywhere except at the space points where the charges are located. In this respect, we simplify matters by considering the electric and magnetic charge densities to be given by discrete electric charge e and magnetic charge g. A complex charged particle moving with velocity $\mathbf{v}$ then has charge and current densities,

$$\rho = e \; , \qquad \mathbf{J} = e\mathbf{v} \; ; \qquad \varrho = g \; , \qquad \mathbf{K} = g\mathbf{v} \tag{49a}$$

$$\Lambda = \frac{1}{\sqrt{2}}(e + ig) \; , \qquad \mathbf{Z} = \Lambda\mathbf{v} \tag{49b}$$

Using eq. (49*b*) in eq. (24), we express the Lorentz force on a particle of complex charge Λ moving with velocity $\mathbf{v}$ in an electromagnetic field of complex field intensity $\mathbf{Q}$ in the form,

$$\mathbf{F}_L = \Lambda(\mathbf{Q}^* + \frac{i}{c}\mathbf{v} \times \mathbf{Q}^*) + \Lambda^*(\mathbf{Q} - \frac{i}{c}\mathbf{v} \times \mathbf{Q}) \ , \tag{50}$$

where Λ is defined in eq. (49*b*). If the complex charged particle has mass m and therefore linear momentum $\mathbf{p} = m\mathbf{v}$, then the equation of motion under the Lorentz force is given by,

$$\frac{d\mathbf{p}}{dt} = \Lambda(\mathbf{Q}^* + \frac{i}{c}\mathbf{v} \times \mathbf{Q}^*) + \Lambda^*(\mathbf{Q} - \frac{i}{c}\mathbf{v} \times \mathbf{Q}) \ , \tag{51}$$

This is the basic equation governing the interaction between complex charged particles in the electromagnetic field. It describes the interaction between the particle of complex charge Λ and the complex charged particle(s) which generate the complex field intensity $\mathbf{Q}$. In this case, the particles generating $\mathbf{Q}$ are considered to be at the origin of the coordinate system. But the field source(s) can still be located at an arbitrary point so that we consider the motion of the particle relative to the field source(s).

6.1 Nonrelativistic interaction between two complex charges

Let us specialise to the case of nonrelativistic interaction between two complex charged particles. The physical quantities of the two interacting complex charges are generally labelled by 1 and 2. Specifically, we consider a particle of complex charge Λ_1 and mass m moving with velocity $\mathbf{v}$ in the field of a static particle of complex charge Λ_2 located (fixed) at the origin of the coordinate system. The position vector of particle 1 relative to particle 2 is therefore given by $\mathbf{r}$, with its velocity defined by,

$$\mathbf{v} = \frac{d\mathbf{r}}{dt}$$

The field intensity created by the static particle 2 at the position $\mathbf{r}$ is labelled by $\mathbf{Q}_2(\mathbf{r})$. Using eq. (44), the Lorentz force governing the nonrelativistic interaction between the two complex charges takes the form,

$$\frac{d\mathbf{p}}{dt} = (\Lambda_1\mathbf{Q}_2^* + \Lambda_1^*\mathbf{Q}_2) + \frac{i}{c}\mathbf{v} \times (\Lambda_1\mathbf{Q}_2^* - \Lambda_1^* \times \mathbf{Q}_2) \ , \tag{52}$$

In the static case, application of Gauss' theorem to the Maxwell's equations,

$$\nabla \cdot \mathbf{Q}_2 = 4\pi\Lambda_2 \ ; \qquad \nabla \cdot \mathbf{Q}_2^* = 4\pi\Lambda_2^* \tag{53a}$$

gives the standard solutions,

$$\mathbf{Q}_2(\mathbf{r}) = \Lambda_2\frac{\mathbf{r}}{r^3} \ ; \qquad \mathbf{Q}_2^*(\mathbf{r}) = \Lambda_2^*\frac{\mathbf{r}}{r^3} \ ; \qquad r = |\mathbf{r}| \tag{53b}$$

Using $\mathbf{Q}_2(\mathbf{r})$ and $\mathbf{Q}_2^*(\mathbf{r})$ from eq. (53*b*) in eq. (52), we obtain the Lorentz force equation for a particle of complex charge Λ_1 and mass m moving with velocity $\mathbf{v}$ in the field of a static particle of complex charge Λ_2 in the form,

$$\frac{d\mathbf{p}}{dt} = (\Lambda_1\Lambda_2^* + \Lambda_1^*\Lambda_2)\frac{\mathbf{r}}{r^3} + \frac{i}{c}(\Lambda_1\Lambda_2^* - \Lambda_1^*\Lambda_2)\mathbf{v} \times \frac{\mathbf{r}}{r^3} \tag{54}$$

This is the general form of the nonrelativistic interaction between two complex charged particles in an electrodynamic medium.

According to the definitions of complex charge and current density given in eq. (49b), we have

$$\Lambda_1 = \frac{1}{\sqrt{2}}(e_1 + ig_1) \ ; \qquad \Lambda_2 = \frac{1}{\sqrt{2}}(e_2 + ig_2) \tag{55a}$$

$$\Lambda_1\Lambda_2^* = \frac{1}{2}(e_1e_2 + g_1g_2) + \frac{i}{2}(e_2g_1 - e_1g_2) \ ; \qquad \Lambda_1^*\Lambda_2 = \frac{1}{2}(e_1e_2 + g_1g_2) - \frac{i}{2}(e_2g_1 - e_1g_2) \tag{55b}$$

$$\Lambda_1\Lambda_2^* + \Lambda_1^*\Lambda_2 = e_1e_2 + g_1g_2 \ ; \qquad \Lambda_1\Lambda_2^* - \Lambda_1^*\Lambda_2 = -i(e_1g_2 - e_2g_1) \ , \tag{55c}$$

Substituting the results from eq. (55c) in eq. (52), we obtain,

$$\frac{d\mathbf{p}}{dt} = (e_1e_2 + g_1g_2)\frac{\mathbf{r}}{r^3} + \frac{1}{c}(e_1g_2 - e_2g_1)\mathbf{v} \times \frac{\mathbf{r}}{r^3} \tag{56}$$

This is exactly the equation of motion first derived by Schwinger (Schwinger, 1969) on the nonrelativistic dyon-dyon interaction. We now observe that Schwinger suggested the name dyon for a particle carrying both electric and magnetic charges. Within the framework of Schwinger's model, our complex charges are dyons. But a dyon in our case carries an electric charge in the real axis and a magnetic charge in the imaginary axis when defined within the electric charge space, and vice versa when defined within the magnetic charge space. The approach in Schwinger's theory does not distinguish between the electric and magnetic charge spaces as we have done it here. Schwinger's dyon can therefore have both electric and magnetic charges appearing on the real axis of charge space.

Taking the cross product of eq. (56) with the position vector $\mathbf{r}$ and applying standard vector analysis results, we obtain,

$$\frac{d}{dt}(\mathbf{r} \times \mathbf{p}) = \frac{1}{c}(e_1g_2 - e_2g_1)\frac{d}{dt}(\frac{\mathbf{r}}{r}) \tag{57a}$$

Defining the orbital angular momentum $\mathbf{l}$ as usual by,

$$\mathbf{l} = \mathbf{r} \times \mathbf{p} \ , \tag{57b}$$

and introducing a convenient notation

$$\mathbf{q} = -\frac{1}{c}(e_1g_2 - e_2g_1)\frac{\mathbf{r}}{r} \ , \tag{57c}$$

we write eq. (57a) in the form,

$$\frac{d}{dt}(\mathbf{l} + \mathbf{q}) = 0 \tag{58}$$

This leads to the definition of a total angular momentum $\mathbf{j}$ by,

$$\mathbf{j} = \mathbf{l} + \mathbf{q} \tag{59a}$$

According to eq. (58),

$$\mathbf{j} = \mathbf{l} + \mathbf{q} = \text{constant} \ , \tag{59b}$$

which is the desired angular momentum conservation. The quantization of the total angular momentum along $\mathbf{r}$ is obtained through,

$$\mathbf{j} \cdot \hat{\mathbf{r}} = n\hbar \ , \qquad n = \text{number} \ ; \qquad \hat{\mathbf{r}} = \frac{\mathbf{r}}{r} \ , \tag{60}$$

where we have denoted $\hbar = \frac{h}{2\pi}$ as usual, h being Planck's constant. Using eqs. (57b)-(57c), (59a) and (60), we obtain the charge quantization condition,

$$\frac{e_1 g_2 - e_2 g_1}{c} = n\hbar \ , \tag{61}$$

already obtained earlier in more elaborate quantum mechanical treatment of the nonrelativistic dyon-dyon interaction (Schwinger et al, 1976; Barut, 1971; Barut and Bornzin, 1971).

The result presented in eqs.(54)-(56) provides a generalization of nonrelativistic interactions between two charged particles. Complex charges are more general than particles carrying electric or magnetic charge only. If we set either $g_1 = g_2 = 0$ or $e_1 = e_2 = 0$ in eq. (56), then we obtain the Coulomb force governing the interaction between two electric or two magnetic charges, respectively. On the other hand, if we set $g_1 = e_2 = 0$ then eq. (56) reduces to the equation governing the interaction between an electric and a magnetic charge, which is the standard equation normally used to derive the Dirac quantization condition through imposing the angular momentum conservation and quantization rule as presented above. A detailed study of the important physical features of this special case can be found in the works of Wu and Yang (1976), Kazama et al (1977) and Jackiw (1980).

Finally, we observe that the Lorentz force as defined in eq. (24) and eqs. (50)-(51) provides a suitable model for studying the more general case of interactions between moving complex charges (moving dyons). Specifically, eq. (51) can be used to study the motion of a complex charged particle in the field of a moving complex charged particle. The solution procedure is then the same as in standard electrodynamics. We also note that the Lorentz force equations will take exactly the same form within the magnetic charge space.

6.2 Some Physical Implications

Complex charge as defined here is flexible enough to embrace all particles of matter. It provides for every particle to have either an electric charge or a magnetic charge or both electric and magnetic charges or no charge at all. Charge neutrality then occurs at three levels: a particle can be neutral with respect to either electric or magnetic or both electric and magnetic charges. A particle which is neutral with respect to both electric and magnetic charges is considered to be strictly neutral. The net electric and magnetic charges on a strictly neutral particle is zero. There are two charge spaces within which complex charge is defined. In the electric charge space, the electric charge appears on the real axis and magnetic charge appears on the imaginary axis. The electric charge takes prominence and is the more readily observed in the electric charge space. Similarly, in the magnetic charge space, magnetic charge appears on the real axis and the electric charge on the imaginary axis. The magnetic charge takes prominence and would be the more readily observed in the magnetic charge space. The two charge spaces are related through a duality transformation as we established above.

Within this framework, matter is to be understood to be made up of particles carrying complex charges in general terms. In the electric charge space of the electromagnetic medium readily accessible to us, we may treat atoms as generally composed of complex charges (or dyons), with electric charge in the real axis and magnetic charge in the imaginary axis of the charge space. The electrons, protons and neutrons in atoms or nuclei are then to be understood to be dyons. The neutron may be a dyon with zero electric charge in the real axis, but a nonzero magnetic charge in the imaginary axis of its charge space. In such a case, a neutron would interact with a proton or electron through its magnetic charge in accordance with eq. (54) (or eq. (56)). In general, we may consider magnetic charge in protons and neutrons (the nucleons) to be carried by the quarks which make up these nucleons. Each quark in a neutron or proton is then a dyon carrying electric charge in the real axis and a proportionate amount of magnetic charge of appropriate sign ($\pm$) in the imaginary axis of charge space to give the net magnetic charge inside a neucleon. If the net magnetic charge in a nucleon is zero, then the neucleon is neutral with respect to magnetic charge. Hence, a neutron which carries zero net magnetic charge would be stricly neutral in the sense that it is neutral with respect to both electric and magnetic charges. However, a neutron carrying nonzero net magnetic charge would interact with both electrons and protons in accordance with eq. (54). The electron, having no internal structure, may be treated as the fundamental unit of the electric charge space, coupling to a corresponding fundamental unit of the magnetic charge space appearing on the imaginary axis in this respect. A strictly neutral atom would be one in which both electric and magnetic charges have been balanced off so that the net charge is zero. Charge neutrality would be governed by charge conservation principle, expressed by the continuity equation for both electric and magnetic charges.

The magnetic charge assignment for both electrons and nucleons should provide for their antiparticles. Hence, to each electric charge there corresponds the right proportion of magnetic charge, one positive and the other negative in sign, to provide for the antiparticle to the electric charge. In this respect, antiquarks also carry electric charge in the real axis and magnetic charge of appropriate proportion and sign ($\pm$) in the imaginary axis of charge space such that particles (hadrons) composed of quark and antiquark combinations can have zero or nonzero net magnetic charge. Magnetic charge would therefore contribute to the interaction and hence, the binding of quarks and antiquarks inside neutrons, protons and hadrons in general. Magnetic charge carried by quarks and antiquarks would remain confined as much as the quarks and antiquarks remain confined inside the hadrons. Schwinger proposed magnetic charge assignments which adequately accounted for the observed electric charges up to the quark level (Schwinger, 1969, 1970). We shall not take up such details in our speculations in the present paper.

7 Conclusion

The mathematical definition of the electric and magnetic field intensities in accordance with Maxwell's equations provides for a second field potential four-vector together with an associated magnetic current density four-vector, leading to a system of coupled electric and magnet field intensities on the one hand, and field potential four-vectors on the other hand, with the electric and magnetic current density four-vectors acting as field sources. The form of the couplings facilitates a reformulation of electrodynamics in complex form as a theory of complex charges moving in a complex force field specified by a complex field potential four-vector. The Lorentz force expressed in complex form is suitable for studying the motion of complex charged particles in the fields of static or moving complex charged particles, leading to a generalization of Schwinger's theory of dyon-dyon interactions.

Within the present framework, every particle is considered to possess complex charge, with electric charge in the real axis and magnetic charge in the imaginary axis of the electric charge space. Charge neutrality may then be considered at three levels, according as to whether the net electric or magnetic or both electric and magnetic charges is zero. A particle can then be neutral with respect to electric charge, but still have magnetic charge or vice versa determining its interaction with other charged particles within the electrodynamic medium. A particle is strictly neutral if its net electric and magnetic charges are both zero. The concept of complex charge is therefore all embrasing, since every particle in nature must carry either zero or nonzero net charge. Matter in any state is formed through the interactions of complex charges. The electric and magnetic charge assignments ensures neutrality as appropriate in accordance with charge conservation principle for both electric and magnetic charges.

Duality transformation helps to define and distinguish between the electric and magnetic charge spaces. The electric charge space constitutes the electrodynamic medium in which the basic electric components $(J\ ,\ \mathbf{E}\ ,\ A)$ appear on the real axis, while the basic magnetic components $(K\ ,\ \mathbf{B}\ ,\ W)$ appear on the imaginary axis. Similarly, the magnetic charge space constitutes the electrodynamic medium in which the basic magnetic components appear on the real axis, while the basic electric components appear on the imaginary axis. The electric and magnetic charge spaces are related through a duality transformation, implemented through a duality transformation operator whose net effect is the rotation of one charge space into the other charge space. As we established, the global duality transformation leaves all the basic electrodynamic field equations invariant.

The definition of charge space based on the duality transformation means that electric charge takes prominence within the electric charge space where magnetic charge is hidden in the imaginary axis. Likewise, magnetic charge takes prominence within the magnetic charge space where the electric charge is hidden in the imaginary axis. The duality transformation connects the two charge spaces and gives a picture of what happens in each charge space. The coupling of the basic electric and magnetic components through Maxwell's equations constitutes a unified electromagnetic force field. Electrodynamics can therefore be formulated either within the electric charge space or within the magnetic charge space. We seem to be living within the electric charge space where the electric charge takes prominence and is therefore the more readily observed to characterize the electrodynamic medium. A duality transformation would take us to the magnetic charge space where magnetic charge takes prominence and would therefore be the more readily observed to characterize the electrodynamic medium.

From the above, we derive a fundamental question: *can one expect to find magnetic charge within our electric charge space?*. This is a question which we shy away from answering at this preliminary stage of the reformulation of electrodynamics. More elaborate investigation must follow after completing the Lagrangian (Hamiltonian) formulation which will form a suitable framework for the classical and quantum theory of complex charges interacting with complex photons. The only thing which is certain within our framework is that electric and magnetic charge do not coexist on the same real axis of charge space.

8 Acknowledgements

Part of this work was done during the author's visit to the Abdus Salam ICTP as a Regular Associate. I am thankful to the ICTP for hosting me in the period 15 Jan.-15 April, 2003 and SIDA for sponsoring my associateship.

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