3. 3 Analytic Description of Tokamak Equilibrium Sustained by High Fraction Bootstrap Current[1)]

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Key words: Bootstrap current, Equilibrium

Recently, to save the current drive power and to obtain more favorable confinement merit for a tokamak reactor, large fraction of bootstrap current sustained equilibrium has attracted great interests both theoretically and experimentally. We use a powerful expanding technique[1] and the tokamak ordering to expand the GSh-E to obtain a series of ordinary differential equations, which allow for different sets of input profiles. The fully bootstrap current sustained tokamak equilibria are then solved analytically.

We discuss the toroidally axisymmetric equilibrium, assume an isotropic plasma pressure, correspondingly, the Grad-Shafranov equation in the cylindrical co-ordinates can be written as:

$$\Delta^{*}\Psi = R^{2}\nabla\cdot\left(\frac{\nabla\Psi}{R^{2}}\right) = \mu_0 R j_{\phi}(R,\ \Psi) \tag{1}$$

where Ψ is the poloidal magnetic flux while the total magnetic field is defined as:

$$\boldsymbol{B} = \nabla\Psi\times\nabla\phi\times F\nabla\phi \tag{2}$$

The toroidal current j_{ϕ} is related with the plasma pressure and the poloidal current flux F by the relation

$$j_{\phi} = -R\frac{\mathrm{d}p}{\mathrm{d}\Psi} - \frac{F}{\mu_0 R}\frac{\mathrm{d}F}{\mathrm{d}\Psi} \tag{3}$$

Introduce the parallel current averaged over the magnetic surface, i. e., the term

$$\langle \boldsymbol{j}\cdot\boldsymbol{B}\rangle = \langle \boldsymbol{j}\cdot\boldsymbol{B}\rangle_{\Omega} + \langle \boldsymbol{j}\cdot\boldsymbol{B}\rangle_{\mathrm{CD}} + \langle \boldsymbol{j}\cdot\boldsymbol{B}\rangle_{\mathrm{BS}} \tag{4}$$

we can relate the toroidal current with realistic driven mechanisms. In Eq. (4), the

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1) Sponsored by the National Science Foundation of China (19975015) and the Nuclear Industry Science Foundation of China (Y7100C0303).

subscripts Ω, CD, BS represent the ohmic drive, wave or beam drive and the bootstrap current, respectively. In this way, the j_ϕ component can be written as:

$$j_\phi = -R\frac{\mathrm{d}p}{\mathrm{d}\Psi}\left[1-\frac{F^2}{R^2\langle B^2\rangle}\right]+\frac{F}{R\langle B^2\rangle}\langle \boldsymbol{j}\cdot\boldsymbol{B}\rangle \tag{5}$$

This current can be distinguished as two parts further

$$j_\phi = \frac{R}{R_0}j_1(\Psi)+\frac{R_0}{R}j_2(\Psi) \tag{6}$$

where

$$j_1(\Psi) = -R_0\frac{\mathrm{d}p}{\mathrm{d}\Psi}$$

$$j_2(\Psi) = \frac{F}{R_0}\left[\frac{F^2}{\langle B^2\rangle}\frac{\mathrm{d}p}{\mathrm{d}\Psi}+\frac{\langle \boldsymbol{j}\cdot\boldsymbol{B}\rangle}{\langle B^2\rangle}\right] \tag{7}$$

$$\langle B^2\rangle = F^2\left\langle\frac{1}{B^2}\right\rangle+\left\langle\left(\frac{\nabla\Psi}{R}\right)^2\right\rangle \tag{8}$$

The bracket $\langle\cdots\rangle$ means an averaging over a magnetic surface and R_0 is the major radius of the magnetic axis. Using the technique developed in Ref. [1], we make the following co-ordinate transform

$$R = R_0 + r(\rho,\ \theta)\cos\theta$$

$$Z=r(\rho,\ \theta)\ \sin\theta \tag{9}$$

$$r = \rho + \sum_{n=1} a_n(\rho)\cos n\theta \tag{10}$$

So, that

$$\rho = \frac{1}{2\pi}\int_0^{2\pi} r\mathrm{d}\theta \tag{11}$$

The coefficients $a_n(\rho)$ correspond to an elliptic deformation for n =2 and a triangular one for n = 3, etc., however, the usual elongation $e = [\rho + a_2(\rho)] / [\rho - a_2(\rho)]$ can be relatively larger than the coefficient a_2 itself. This might be a merit of this expansion. Besides, the coefficient a_1 is equivalent to the Shafranov shift. In the following, we assume the following tokamak ordering

$$\rho/R_0 \approx a_n/\rho \approx 0(\varepsilon)$$

$$(\nabla\Psi)^2/F^2 \approx (2\mu_0 P/B^2) \approx 0(\varepsilon^2) \tag{12}$$

In the (ρ, θ, ϕ) coordinates, the Grad-Shafranov equation becomes

$$\Delta^*\Psi = \frac{1}{(\partial r/\partial\rho)^2}\left[1+\left(\frac{1}{r}\frac{\partial r}{\partial\theta}\right)^2\right]\frac{\mathrm{d}^2\Psi}{\mathrm{d}\rho^2}+$$

$$\left\{\frac{1}{2}\frac{\partial}{\partial\rho}\left[\left(1+\frac{1}{r^2}\left(\frac{\partial r}{\partial\theta}\right)^2\right)\Big/\left(\frac{\partial r}{\partial\theta}\right)^2\right]+\left(\frac{1}{r\partial r/\partial\rho}\right)-\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{\partial r/\partial\theta}{r\partial r/\partial\rho}\right)-\frac{\cos\theta+\frac{1}{r}\frac{\partial r}{\partial\theta}\sin\theta}{R\partial r/\partial\rho}\right\} \tag{13}$$

$$\frac{\mathrm{d}\,\Psi}{\mathrm{d}\rho} = \mu_0 R\left[\frac{R}{R_0}j_1(\Psi)+\frac{R_0}{R}j_2(\Psi)\right]$$

Multiply the above equation by $\left[\frac{(\partial r/\partial\rho)}{1+(\partial r/r\partial\theta)^2}\right]\cos n\theta$, then averaging it over θ, i. e., carrying out the integration $\int_0^{2\pi}\mathrm{d}\theta$ for n= 0, 1, 2, 3, we can obtain a series of coupled nonlinear ordinary differential equations about variables Ψ, a_1, a_2, a_3,⋯. The first equation serves to determine $\Psi(\rho)$, it reads

$$\begin{aligned}&\frac{\mathrm{d}^2\Psi}{\mathrm{d}\rho^2}+A_0(\rho, a_n, a_n', a_n'')\frac{\mathrm{d}\Psi}{\mathrm{d}\rho}\\&=B_1(\rho, a_n, a_n')j_1(\Psi)+B_2(\rho, a_n, a_n')j_2(\Psi)\end{aligned} \tag{14}$$

here A_0, B_1, B_2 not only depend on ρ, but also on $a_n(\rho)$(n = 1, 2, 3⋯) and their first and second derivatives. The subsequent equations do not include the $\mathrm{d}^2\Psi/\mathrm{d}\rho^2$ term and serves to determine the $a_n(\rho)$, they usually have the following form:

$$\begin{aligned}&A_n(\rho, a_n, a_n', a_n'')\frac{\mathrm{d}\Psi}{\mathrm{d}\rho}\\&=B_{1n}(\rho, a_n, a_n')j_1+B_{2n}(\rho, a_n, a_n')j_2\end{aligned} \tag{15}$$

Though the above sets of equations are ordinary differential in nature, they are coupled and nonlinear so that they do not describe the equilibrium in simple way. Now we use the tokamak ordering of Eq. (12) to simplify them to linear ones. We expand

$$\Psi = \Psi_0 + \varepsilon\Psi_1 + \varepsilon^2\Psi_2 + \cdots \tag{16}$$

$$a_n = \varepsilon a_{n0} + \varepsilon^2 a_{n1} + \cdots \tag{17}$$

Substitute them into Eqs. (14)~(15), collecting terms of the same order will result in a linear set of equations. This procedure is rather tedious but straight forward and the results take the following form[1]:

$$\frac{\mathrm{d}^2\Psi_0}{\mathrm{d}\rho^2} + \frac{1}{\rho}\frac{\mathrm{d}\Psi_0}{\mathrm{d}\rho} = \mu_0 R_0 (j_1 + j_2) \tag{18}$$

$$\frac{\mathrm{d}^2 a_{10}}{\mathrm{d}\rho^2} + \left(\frac{1}{\rho} + \frac{2\Psi_0''}{\Psi_0'}\right)\frac{\mathrm{d}a_{10}}{\mathrm{d}\rho} = -\frac{1}{R_0} + 2\mu_0\rho\frac{j_1}{\Psi_0'} \tag{19}$$

$$\frac{\mathrm{d}^2 a_{n0}}{\mathrm{d}\rho^2} + \left(\frac{1}{\rho} + \frac{2\Psi_0''}{\Psi_0'}\right)\frac{\mathrm{d}a_{n0}}{\mathrm{d}\rho} + \frac{1-n^2}{\rho^2}a_{n0} = 0 \quad (n = 2,\ 3, \cdots) \tag{20}$$

It can be noted that Eq. (18) is the same for non-circular tokamak and for circular tokamak and the non-circular deformations are determined by the homogeneous Eq. (20) so that its solution is proportional to every deformation at the boundary: $a_{n0}(\rho_b) = \varepsilon_n$. The Shafranov shift a_{10} is determined by Eq. (19) that is inhomogeneous so that its solution is related with the j_1 part, i. e., with the pressure gradient. Then the lowest order quantities of and a_1 are as the same as for circular and non-circular tokamaks. The more complicated higher order equations are no use for the present purpose and we do not write their explicit forms. The boundary conditions are:

$$\Psi_0(0) = \Psi_0'(0) = 0,\ a_{10}(0) = a_{10}'(0) = 0$$

$$a_{n0}(0) = 0,\ \ a_{n0}(\rho_0) = \delta_n, \quad \text{for } n \leqslant 2 \tag{21}$$

It implies that we are treating a fixed boundary problem.

We note that there is a degree of freedom in the choice of the input parameter forms. In case of high fraction of bootstrap current, we assume that all density and temperature profile are known function of the minor radius.

At first, we consider the case that no other current drives exist. From the point of view of the equilibrium establishment, this situation is possible and its performance is of particularly interest. Considering the solution of Eq. (18) first, to the lowest order of the tokamak ordering we have (in the following we neglect the subscript 0)

$$j_1+ j_2=j_{bs} \tag{22}$$

Because our main purpose is to extract the method of solving the GSh-equation, we will use the simplest model for the bootstrap current. A suitable form can be written as[2]:

$$j_{bs} = \frac{p_e}{B_p}\left(\frac{\rho}{R_0}\right)^{1/2}\left[-2.44\left(1+\frac{T_i}{T_e}\right)\frac{d}{d\rho}(\ln n) - 0.69\frac{d}{d\rho}(\ln T_e) + \frac{T_i}{T_e}\frac{d}{d\rho}(\ln T_i)\right] \tag{23}$$

and $B_p = \Psi'/R_0$ is the poloidal magnetic field. Now we introduce normalized radius $x=\rho/\rho_0$, with ρ_0 being the averaged radius of the last closed magnetic surface. Rewrite

$$j_{bs} = (R_0\rho_0)^{1/2} p_{e0}\sqrt{x} f(x)/\Psi'(x) \tag{24}$$

where

$$\begin{aligned} f(x) = (p_e/p_{e0})[-2.44(1+T_i/T_e)\times \mathrm{d}\ln n/\mathrm{d}x - \\ 0.69\mathrm{d}\ln T_e/\mathrm{d}x + (T_i/T_e)\mathrm{d}\ln T_i/\mathrm{d}x \end{aligned} \tag{25}$$

that can be selected as an arbitrary function depending on the density and the temperature profiles. Now the Eq. (18) has the following form:

$$(x)'\Psi(x)'' + \Psi'(x)^2/x = \mu_0 R_0^{3/2}\rho_0^{5/2} p_{e0} f(x) \tag{26}$$

Then its general solution can be written as:

$$\Psi'(x) = \left[2\mu_0 p_{e0} R_0^{5/2}\rho_0^{1/2}\int dx' x'^{5/2} f(x')\right]^{1/2} \Big/ x \tag{27}$$

If we assume that near the magnetic axis the profiles of the density and the temperatures all have parabolic type, then the function $f(x)$ will be proportional to $x^{3/2}$, this gives $\Psi'(x) \propto x^{5/4}$ and the first non-zero term of the bootstrap current $j_{bs} \propto \rho^{1/4}$. Generally, we can assume

$$f(x) = \sum_{n=1} a_n x^n \tag{28}$$

Then we have

$$\Psi'(\rho) = (2\mu_0 p_{e0} R_0^{3/2} / \rho_0^{3/2})^{1/2} \rho^{5/4} \left[\sum_{i=1} \frac{\alpha_i}{i+7/2} x^{i-1} \right]^{1/2} \tag{29}$$

With this solution, we can express terms appearing in the Eqs. (19) ~ (20):

$$\begin{aligned} \frac{1}{\rho} + \frac{2\Psi''}{\Psi'} &= \frac{\rho^2 f(\rho)}{\int_0^\rho \mathrm{d}x x^2 f(x)} - \frac{1}{\rho} \\ &= \frac{1}{\rho}\left(1 + \frac{\sum_{i=1} (3+2i)\alpha_i x^{i+3/2} / (7+2i)}{\sum_{i=1} 2\alpha_i x^{i+3/2} / (7+2i)} \right) \end{aligned} \tag{30}$$

In the near magnetic axis region, this term becomes $\frac{7}{2\rho}$, the solution of the non-homogenous Eq. (19) has a solution of the following form:

$$a_1'(\rho) = \rho^{-7/2} \int_0^\rho \mathrm{d}x x^{7/2} \left[-\frac{1}{R_0} + 2\mu_0 x \frac{j_1(x)}{\Psi'(x)} \right] \tag{31}$$

The elliptic and the triangular deformations then are determined by Eq. (20) that is homogeneous in nature, in the near magnetic axis region, its solution can be expressed by

$$a_n(\rho) = \delta_n (\rho / \rho_0)^s \tag{32}$$

the power index s in above solution is determined by

$$s(s-1) + 7s/2 - n^2 + 1 = 0 \tag{33}$$

For more general density and temperature profile, the solution of Eq. (19) can still be expressed by an integral form like Eq. (31) while the Eq. (20) can be expressed by a power series

$$a_n(\rho) = \delta_n x^s \sum_{i=0} \beta_i x^i \tag{34}$$

The coefficients β_i are related with the $\alpha_i' s$.

The safety factor profile can be determined from its definition as:

$$q(\rho)=\frac{F(\rho)}{\Psi'(\rho)}\oint\frac{\mathrm{d}l}{R|\nabla\rho|}=\frac{F\rho}{R_0\Psi'(\rho)}[1+o(\varepsilon^2)]$$
$$\approx\frac{B_0}{\sqrt{2\mu_0 P_{\mathrm{e}0}}}\left(\frac{\rho_0}{R_0}\right)^{3/4}x^{-1/4}\left[\sum_{i=1}\frac{\alpha_i x^{i-1}}{i+7/2}\right]^{-1/2} \tag{35}$$

The edge safety factor related with the total current I_{p} is then determined by

$$q(l)=\frac{B_0}{[2\mu_0 P_{\mathrm{e}0}\sum_{i=1}\frac{\alpha_i}{i+7/2}]^{1/2}}\left(\frac{\rho_0}{R_0}\right)^{3/4} \tag{36}$$

The total toroidal current density is

$$j_\phi=\frac{R_0}{R}j_{\mathrm{bs}}+\frac{1}{\Psi'}\frac{\mathrm{d}p}{\mathrm{d}\rho}\left(\frac{R_0^2}{R}-R\right) \tag{37}$$

The poloidal current flux function F is

$$F^2=F^2(0)+2\mu_0 R_0^2[p(0)-P(x)]-2\mu_0(R_0\rho_0)^{3/2}p_{\mathrm{e}0}\int_0^x \mathrm{d}x'\sqrt{x'}f(x') \tag{38}$$

Now we set an example for equilibrium sustained by full bootstrap current, we assume a very simple input set for the density and the electron and ion temperature profiles, all these profiles are parabolic as:

$$n_{\mathrm{e}}=n_{\mathrm{e}0}(1-x^2) \tag{39}$$

$$T=T_0(1-x^2)$$

and $T_{\mathrm{e}}=T_{\mathrm{I}}$. The function $f(x)$ in Eq. (25) becomes

$$f(x)=9.14x(1-x^2) \tag{40}$$

It readily gives

$$\Psi'(x)=1.425(2\mu_0 p_{\mathrm{e}0})^{1/2}R_0^{5/4}\rho_0^{1/4}x^{5/4}(1-9x^2/13)^{1/2} \tag{41}$$

$$j_{\mathrm{bs}}(\rho)=\frac{6.414}{R_0}\left(\frac{p_{\mathrm{e}0}}{2\mu_0}\right)^{1/2}\left(\frac{\rho_0}{R_0}\right)^{1/4}\frac{x^{1/4}(1-x^2)}{\sqrt{1-9x^2/13}} \tag{42}$$

$$q(\rho)=\frac{0.702B_0}{\sqrt{2\mu_0 p_{e0}}}\left(\frac{\rho_0}{R_0}\right)^{3/4}x^{-1/4}(1-9x^2/13)^{-1/2} \tag{43}$$

In region $x \leqslant 0.537$, the magnetic shear is negative, the minimum value of q is $q_{\min} = 0.725q(l)$.

The Shafranov shift and the deformations of the cross-section can be obtained analytically or by a little numerical work.

REFERENCES

1 SHI Bingren. Kexue Tongbao, 1981, 26: 494

2 Hazeltine R D, Meiss J D. Plasma Confinement. Addison Wesley Publishing Company, 1992, 363

3. 4 Analytic Study of Propagation and Absorption of Nearly Perpendicular Injected Electron Cyclotron Ordinary Wave[1)]

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Key words: O-X mode coupling, Relativistic resonance, Nearly perpendicular propagation

Electron cyclotron resonance heating (ECRH), such as the fundamental heating and the second harmonic heating, is a basic and powerful method to heat the plasma in tokamak and stellarator devices. Theoretical studies of this heating have been done in rather early literatures[1~3], however, the understanding of some important problems is still uncertain. These include: the coupling of the O-mode and the E-mode and the role of this coupling in wave damping, the O-mode damping mechanism, the evolution of the electron distribution function during O-mode damping, the synergetic effect of the O-mode heating with other wave processes, etc. For the linear dispersion study, we have recently obtained a refined result for pure

1) Sponsored by the National Natural Science Foundation of China (19975015) and the Nuclear Industry Foundation of China (Y7100CD301).