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- Polynomial elements in one variable

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Available at: http//www.ictp.trieste.it IC/2003/85

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CANONICAL BASIS FOR TYPE A_4 (II) – Polynomial elements in one variable

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MIRAMARE – TRIESTE
December 2003

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Abstract

All the 62 monomial elements in the canonical basis **B** of the quantized enveloping algebra for type A_4 have been determined in [2]. According to Lusztig's idea [7], the elements in the canonical basis **B** consist of monomials and linear combinations of monomials (for convenience, we call them polynomials). In this note, we compute all the 144 polynomial elements in one variable in the canonical basis **B** of the quantized enveloping algebra for type A_4 based on our joint note [2]. We conjecture that there are other polynomial elements in two or three variables in the canonical basis **B**, which include independent variables and dependent variables. Moreover, it is conjectured that there are no polynomial elements in the canonical basis **B** with four or more variables.

1. Polynomial elements in one variable in the canonical basis of U^+

We shall freely use the notations in [2] without further comments.

1.1. In [2], we have determined the 62 monomial elements in the canonical basis $\mathbf{B}$ of the quantized enveloping algebra for type A_4. Each monomial element corresponds to a region which consists of six independent inequalities. Here independence of six inequalities implies that we can't deduce one inequality from the others. Moreover, the regions can't be described with less than six inequalities, and the interiors of any two regions are disjoint. These regions of monomial elements have "nice" forms, which will help us compute polynomial elements in the canonical basis $\mathbf{B}$.

All the 144 polynomial elements in one variable in the canonical basis $\mathbf{B}$ will be determined in this note based on our joint work [2]. We have also calculated polynomial elements in several variables in the canonical basis $\mathbf{B}$. And we found more than thirty polynomial elements in two "independent variables" u and w in the canonical basis $\mathbf{B}$. We are going to calculate all such elements, and determine completely the canonical basis $\mathbf{B}$ for type A_4 in future. These further results will appear elsewhere. Here by independent variables we mean the following: when these variables are summed, the multisum is independent of the order of the summations in these variables. Otherwise these variables are called dependent.

We conjecture that there are other polynomial elements in two or three variables in the canonical basis $\mathbf{B}$, which include independent variables and dependent variables. Moreover, it is conjectured that there are no polynomial elements in the canonical basis $\mathbf{B}$ with four or more variables.

1.2. Let us compute polynomial elements in one variable in the canonical basis $\mathbf{B}$ of the quantized enveloping algebra for type A_4. According to Lusztig's idea, these polynomial elements should be linear combinations of the monomial elements. We shall describe how one can compute them and begin with monomial element **1.**(1) in [2] as an example. Let $A = (a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}) \in \mathbb{N}^{10}$, we have known from [2] P. 242 that in order to make the linear form $l(x_1, x_2, \cdots, x_{14})$ and the unit form $q(x_1, x_2, \cdots, x_{14})$ to be non-negative for any $x = (x_1, x_2, \cdots, x_{14}) \in \mathbb{N}^{14}$, the following inequalities

$$(\sharp)\quad \begin{array}{l} a_5 + a_6 + a_7 \geq a_1 + a_2 + a_3, \qquad a_7 + a_9 \geq a_3 + a_6, \\ a_7 + a_9 + a_{10} \geq a_3 + a_6 + a_8, \qquad a_6 + a_7 \geq a_2 + a_3, \\ a_7 + a_8 + a_9 \geq a_2 + a_3 + a_5, \qquad a_2 + a_5 \geq a_6 + a_8, \\ a_8 + a_9 \geq a_5 + a_6, \qquad a_9 + a_{10} \geq a_6 + a_8, \\ a_1 \geq a_5, \quad a_2 \geq a_6, \quad a_5 \geq a_8, \quad a_{10} \geq a_8, \quad a_9 \geq a_6, \end{array}$$

equivalently,

$$(*)\qquad a_5 \ge a_8, \qquad a_{10} \ge a_8, \qquad a_2 \ge a_6, \qquad a_1 \ge a_5,$$
$$a_8 + a_9 \ge a_5 + a_6, \qquad a_5 + a_6 + a_7 \ge a_1 + a_2 + a_3$$

must hold, and then the monomial

$$(**)\qquad e_2^{(a_3)} e_3^{(a_2+a_3)} e_4^{(a_1+a_2+a_3)} e_2^{(a_6)} e_3^{(a_5+a_6)} e_2^{(a_8)} e_1^{(a_4+a_7+a_9+a_{10})} \times$$
$$\times\, e_2^{(a_4+a_7+a_9)} e_3^{(a_4+a_7)} e_4^{(a_4)}$$

belongs to **B**. Because none of the six inequalities in $(*)$ are a consequence of the other ones, the six inequalities in $(*)$ are independent. $(*)$ is called the region of the monomial $(**)$, and the six independent inequalities in $(*)$ are called the defining inequalities of the region $(*)$.

We now consider the linear combination of the monomial $(**)$ that could become a new member in **B**. Because there is a one-to-one correspondence between $\mathbb{N}^{10}$ and **B**, we have to observe the regions similar to $(*)$. First, we reverse the first defining inequality $a_5 \ge a_8$ in $(*)$, and get a new inequality $a_8 \ge a_5$. In order to make sure that the inequalities in $(\sharp)$ other than $a_5 \ge a_8$ hold, we have to replace the remaining five defining inequalities in $(*)$ by the following five defining inequalities

$$a_{10} \ge a_8, \qquad a_2 + a_5 \ge a_6 + a_8, \qquad a_1 \ge a_5,$$
$$a_9 \ge a_6, \qquad a_5 + a_6 + a_7 \ge a_1 + a_2 + a_3.$$

The five new defining inequalities together with $a_8 \ge a_5$ form a new region, which corresponds to a linear combination of the monomial $(**)$. Similarly, we can deal with the two other cases that the second defining inequality and the third defining inequality in $(*)$ are reversed, respectively. Secondly, we reverse the fourth defining inequality $a_1 \ge a_5$ in $(*)$, and get a new inequality $a_5 \ge a_1$. In order to make sure that the inequalities in $(\sharp)$ other than $a_1 \ge a_5$ hold, we have to replace the remaining five defining inequalities in $(*)$ by the following five defining inequalities

$$a_6 + a_7 \ge a_2 + a_3, \qquad a_8 + a_9 \ge a_5 + a_6,$$
$$a_2 \ge a_6, \qquad a_1 \ge a_8, \qquad a_{10} \ge a_8.$$

The five new defining inequalities together with $a_5 \ge a_1$ form a new region, which corresponds to a linear combination of the monomial $(**)$. Similarly, we can deal with the two other cases that the fifth defining inequality and the sixth defining inequality in

(∗) are reversed, respectively. By many computations, we have noticed that the coefficient of the linear combination of the monomial (∗∗) is closely related to the reversed inequality. Therefore, corresponding to the six defining inequalities in (∗), we get the following six possibilities, respectively.

$$\text{(i)}\ a_8 \geq a_5, \qquad \text{(ii)}\ a_8 \geq a_{10}, \qquad \text{(iii)}\ a_6 \geq a_2, \qquad \text{(iv)}\ a_5 \geq a_1,$$
$$\text{(v)}\ a_5 + a_6 \geq a_8 + a_9, \qquad \text{(vi)}\ a_1 + a_2 + a_3 \geq a_5 + a_6 + a_7,$$

Then we get six polynomial elements in one variable, each corresponds to one of the above six cases. Also, the region of each of the six polynomial elements in one variable consists of six independent inequalities. In this way, corresponding to each of the 62 monomial elements, we compute all polynomial elements in one variable and their regions.

Recall that from every one of the 62 monomials, we get six polynomial elements in a single variable and these six polynomial elements correspond to six regions. Among the six regions there are at least three regions which already appear in the set $\mathcal{S}$, where $\mathcal{S}$ is the set of the 62 regions determined by the 62 monomial elements. Moreover, there are at most two other regions, which are the same as those of the polynomial elements we already computed before. These polynomial elements will not contribute to the canonical basis $\mathbf{B}$.

In the above example, the regions corresponding to (i), (ii), (iii) have already occurred in $\mathcal{S}$, so the corresponding three polynomial elements in one variable don't make a contribution to the canonical basis $\mathbf{B}$, although they are combinations of the monomials (∗∗). On the other hand, the regions corresponding to (iv), (v), (vi) don't occur in $\mathcal{S}$, and the corresponding three polynomial elements in one variable become new members in the canonical basis $\mathbf{B}$ because this is the first consideration for the non-monomial case.

Repeating the above procedure for all 62 monomials one by one, we obtain all the 144 polynomial elements in one variable, which belong to the canonical basis $\mathbf{B}$. The main result concerning the polynomial elements in one variable in the canonical basis $\mathbf{B}$ of the quantized enveloping algebra for type A_4 is the following theorem.

Theorem 1.3. *Let* $A = (a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}) \in \mathbb{N}^{10}$. *Then*

1. *corresponding to* 62 *equivalence classes for* $\sim$, *the* 72 *polynomial elements in one variable in the canonical basis* $\mathbf{B}$

$$\theta(A) = \theta(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}) \in \mathbf{B}$$

are given by the following

1. (1). $$\sum_{0\le u\le a_4+a_7} (-1)^u \begin{bmatrix} a_5+a_6-a_8-a_9-1+u \\ u \end{bmatrix} e_2^{(a_3)} e_3^{(a_2+a_3)} e_4^{(a_1+a_2+a_3)} e_2^{(a_6)}$$
$$\times\, e_3^{(a_5+a_6+u)} e_2^{(a_8)} e_1^{(a_4+a_7+a_9+a_{10})} e_2^{(a_4+a_7+a_9)} e_3^{(a_4+a_7-u)} e_4^{(a_4)}$$
if $a_5+a_6+a_7 \ge a_1+a_2+a_3, \quad a_5+a_6 \ge a_8+a_9,$

$a_9 \ge a_6, \quad a_2 \ge a_6, \quad a_1 \ge a_5, \quad a_{10} \ge a_8.$

(2). $$\sum_{0\le u\le a_4} (-1)^u \begin{bmatrix} a_1+a_2+a_3-a_5-a_6-a_7-1+u \\ u \end{bmatrix} e_2^{(a_3)} e_3^{(a_2+a_3)} e_4^{(a_1+a_2+a_3+u)}$$
$$\times\, e_2^{(a_6)} e_3^{(a_5+a_6)} e_2^{(a_8)} e_1^{(a_4+a_7+a_9+a_{10})} e_2^{(a_4+a_7+a_9)} e_3^{(a_4+a_7)} e_4^{(a_4-u)}$$
if $a_1+a_2+a_3 \ge a_5+a_6+a_7, \quad a_8+a_9 \ge a_5+a_6, \quad a_2 \ge a_6,$

$a_6+a_7 \ge a_2+a_3, \quad a_5 \ge a_8, \quad a_{10} \ge a_8.$

(3). $$\sum_{0\le u\le a_2+a_3} (-1)^u \begin{bmatrix} a_5-a_1-1+u \\ u \end{bmatrix} e_2^{(a_3)} e_3^{(a_2+a_3-u)} e_4^{(a_1+a_2+a_3)} e_2^{(a_6)}$$
$$\times\, e_3^{(a_5+a_6+u)} e_2^{(a_8)} e_1^{(a_4+a_7+a_9+a_{10})} e_2^{(a_4+a_7+a_9)} e_3^{(a_4+a_7)} e_4^{(a_4)}$$
if $a_6+a_7 \ge a_2+a_3, \quad a_8+a_9 \ge a_5+a_6, \quad a_2 \ge a_6,$

$a_5 \ge a_1 \ge a_8, \quad a_{10} \ge a_8.$

2. (1). $$\sum_{0\le u\le a_4+a_7+a_9} (-1)^u \begin{bmatrix} a_{10}-a_8-1+u \\ u \end{bmatrix} e_3^{(a_2)} e_2^{(a_3+a_6)} e_1^{(a_4+a_7+a_9-u)} e_3^{(a_3)}$$
$$\times\, e_2^{(a_4+a_7)} e_3^{(a_4)} e_4^{(a_1+a_2+a_3+a_4)} e_3^{(a_5+a_6+a_7)} e_2^{(a_8+a_9)} e_1^{(a_{10}+u)}$$
if $a_1+a_2+a_3 \ge a_5+a_6+a_7, \quad a_5+a_6 \ge a_8+a_9,$

$a_9 \ge a_6 \ge a_2, \quad a_{10} \ge a_8, \quad a_7 \ge a_3.$

(2). $$\sum_{0\le u\le a_4+a_7} (-1)^u \begin{bmatrix} a_8+a_9-a_5-a_6-1+u \\ u \end{bmatrix} e_3^{(a_2)} e_2^{(a_3+a_6)} e_1^{(a_4+a_7+a_9)} e_3^{(a_3)}$$
$$\times\, e_2^{(a_4+a_7-u)} e_3^{(a_4)} e_4^{(a_1+a_2+a_3+a_4)} e_3^{(a_5+a_6+a_7)} e_2^{(a_8+a_9+u)} e_1^{(a_{10})}$$
if $a_1+a_2+a_3 \ge a_5+a_6+a_7, \quad a_8+a_9 \ge a_5+a_6,$

$a_5 \ge a_8 \ge a_{10}, \quad a_6 \ge a_2, \quad a_7 \ge a_3.$

(3). $$\sum_{0\le u\le a_4+a_7} (-1)^u \begin{bmatrix} a_6-a_9-1+u \\ u \end{bmatrix} e_3^{(a_2)} e_2^{(a_3+a_6+u)} e_1^{(a_4+a_7+a_9)} e_3^{(a_3)}$$
$$\times\, e_2^{(a_4+a_7-u)} e_3^{(a_4)} e_4^{(a_1+a_2+a_3+a_4)} e_3^{(a_5+a_6+a_7)} e_2^{(a_8+a_9)} e_1^{(a_{10})}$$
if $a_1+a_2+a_3 \ge a_5+a_6+a_7, \quad a_5 \ge a_8 \ge a_{10},$

$a_6 \ge a_9 \ge a_2, \quad a_7 \ge a_3.$

3. (1). $$\sum_{0\le u\le a_3+a_4+a_6} (-1)^u \begin{bmatrix} a_8-a_5-1+u \\ u \end{bmatrix} e_1^{(a_4)} e_3^{(a_2)} e_2^{(a_3+a_4+a_6-u)} e_4^{(a_1+a_2)}$$
$$\times\, e_1^{(a_7+a_9)} e_3^{(a_3+a_4+a_5+a_6)} e_2^{(a_7+a_8+a_9+u)} e_4^{(a_3+a_4)} e_1^{(a_{10})} e_3^{(a_7)}$$
if $\quad a_5+a_6 \ge a_1+a_2, \quad a_3+a_6 \ge a_7+a_9, \quad a_1 \ge a_5,$
$$a_8 \ge a_5 \ge a_{10}, \quad a_9 \ge a_6.$$

(2). $$\sum_{0\le u\le a_7} (-1)^u \begin{bmatrix} a_5+a_6-a_8-a_9-1+u \\ u \end{bmatrix} e_1^{(a_4)} e_3^{(a_2)} e_2^{(a_3+a_4+a_6)} e_4^{(a_1+a_2)}$$
$$\times\, e_1^{(a_7+a_9)} e_3^{(a_3+a_4+a_5+a_6+u)} e_2^{(a_7+a_8+a_9)} e_4^{(a_3+a_4)} e_1^{(a_{10})} e_3^{(a_7-u)}$$
if $\quad a_5+a_6 \ge a_8+a_9 \ge a_1+a_2, \quad a_3+a_6 \ge a_7+a_9,$
$$a_8 \ge a_{10}, \quad a_1 \ge a_5, \quad a_9 \ge a_6.$$

(3). $$\sum_{0\le u\le a_2} (-1)^u \begin{bmatrix} a_5-a_1-1+u \\ u \end{bmatrix} e_1^{(a_4)} e_3^{(a_2-u)} e_2^{(a_3+a_4+a_6)} e_4^{(a_1+a_2)} e_1^{(a_7+a_9)}$$
$$\times\, e_3^{(a_3+a_4+a_5+a_6+u)} e_2^{(a_7+a_8+a_9)} e_4^{(a_3+a_4)} e_1^{(a_{10})} e_3^{(a_7)}$$
if $\quad a_8+a_9 \ge a_5+a_6, \quad a_3+a_6 \ge a_7+a_9,$
$$a_8 \ge a_1 \ge a_5 \ge a_{10}, \quad a_6 \ge a_2.$$

4. (1). $$\sum_{0\le u\le a_2+a_3+a_4} (-1)^u \begin{bmatrix} a_5-a_1-1+u \\ u \end{bmatrix} e_1^{(a_4)} e_2^{(a_3+a_4)} e_3^{(a_2+a_3+a_4-u)} e_2^{(a_6)}$$
$$\times\, e_1^{(a_7+a_9)} e_4^{(a_1+a_2+a_3+a_4)} e_2^{(a_7)} e_3^{(a_5+a_6+a_7+u)} e_2^{(a_8+a_9)} e_1^{(a_{10})}$$
if $\quad a_3+a_6 \ge a_7+a_9, \quad a_1+a_6 \ge a_8+a_9, \quad a_2 \ge a_6,$
$$a_8 \ge a_{10}, \quad a_5 \ge a_1, \quad a_9 \ge a_6.$$

(2). $$\sum_{0\le u\le a_7+a_9} (-1)^u \begin{bmatrix} a_{10}-a_8-1+u \\ u \end{bmatrix} e_1^{(a_4)} e_2^{(a_3+a_4)} e_3^{(a_2+a_3+a_4)} e_2^{(a_6)} e_1^{(a_7+a_9-u)}$$
$$\times\, e_4^{(a_1+a_2+a_3+a_4)} e_2^{(a_7)} e_3^{(a_5+a_6+a_7)} e_2^{(a_8+a_9)} e_1^{(a_{10}+u)}$$
if $\quad a_3+a_6+a_8 \ge a_7+a_9+a_{10}, \quad a_5+a_6 \ge a_8+a_9,$
$$a_{10} \ge a_8, \quad a_9 \ge a_6, \quad a_2 \ge a_6, \quad a_1 \ge a_5.$$

(3). $$\sum_{0\le u\le a_4} (-1)^u \begin{bmatrix} a_7+a_9-a_3-a_6-1+u \\ u \end{bmatrix} e_1^{(a_4-u)} e_2^{(a_3+a_4)} e_3^{(a_2+a_3+a_4)} e_2^{(a_6)}$$
$$\times\, e_1^{(a_7+a_9+u)} e_4^{(a_1+a_2+a_3+a_4)} e_2^{(a_7)} e_3^{(a_5+a_6+a_7)} e_2^{(a_8+a_9)} e_1^{(a_{10})}$$
if $\quad a_7+a_9 \ge a_3+a_6, \quad a_5+a_6 \ge a_8+a_9, \quad a_1 \ge a_5,$
$$a_8 \ge a_{10}, \quad a_2 \ge a_6, \quad a_3 \ge a_7.$$

5. (1). $$\sum_{0\le u\le a_3+a_4} (-1)^u \begin{bmatrix} a_5+a_6-a_1-a_2-1+u \\ u \end{bmatrix} e_1^{(a_4)} e_3^{(a_2)} e_2^{(a_3+a_4+a_6)} e_1^{(a_7+a_9)}$$
$$\times e_3^{(a_3+a_4-u)} e_4^{(a_1+a_2+a_3+a_4)} e_2^{(a_7)} e_3^{(a_5+a_6+a_7+u)} e_2^{(a_8+a_9)} e_1^{(a_{10})}$$
if $\quad a_5+a_6 \ge a_1+a_2 \ge a_8+a_9, \quad a_3+a_6 \ge a_7+a_9,$
$$a_8 \ge a_{10}, \quad a_9 \ge a_6, \quad a_1 \ge a_5.$$

(2). $$\sum_{0\le u\le a_7+a_9} (-1)^u \begin{bmatrix} a_{10}-a_8-1+u \\ u \end{bmatrix} e_1^{(a_4)} e_3^{(a_2)} e_2^{(a_3+a_4+a_6)} e_1^{(a_7+a_9-u)}$$
$$\times e_3^{(a_3+a_4)} e_4^{(a_1+a_2+a_3+a_4)} e_2^{(a_7)} e_3^{(a_5+a_6+a_7)} e_2^{(a_8+a_9)} e_1^{(a_{10}+u)}$$
if $\quad a_3+a_6+a_8 \ge a_7+a_9+a_{10}, \quad a_9 \ge a_6 \ge a_2,$
$$a_1+a_2 \ge a_5+a_6 \ge a_8+a_9, \quad a_{10} \ge a_8.$$

(3). $$\sum_{0\le u\le a_7} (-1)^u \begin{bmatrix} a_6-a_9-1+u \\ u \end{bmatrix} e_1^{(a_4)} e_3^{(a_2)} e_2^{(a_3+a_4+a_6+u)} e_1^{(a_7+a_9)} e_3^{(a_3+a_4)}$$
$$\times e_4^{(a_1+a_2+a_3+a_4)} e_2^{(a_7-u)} e_3^{(a_5+a_6+a_7)} e_2^{(a_8+a_9)} e_1^{(a_{10})}$$
if $\quad a_1+a_2 \ge a_5+a_6, \quad a_6 \ge a_9 \ge a_2, \quad a_3 \ge a_7,$
$$a_5 \ge a_8 \ge a_{10}.$$

6. (1). $$\sum_{0\le u\le a_6} (-1)^u \begin{bmatrix} a_8-a_5-1+u \\ u \end{bmatrix} e_1^{(a_4)} e_2^{(a_3+a_4)} e_3^{(a_2+a_3+a_4)} e_2^{(a_6-u)}$$
$$\times e_4^{(a_1+a_2+a_3+a_4)} e_3^{(a_5+a_6)} e_1^{(a_7+a_9)} e_2^{(a_7+a_8+a_9+u)} e_3^{(a_7)} e_1^{(a_{10})}$$
if $\quad a_3+a_6 \ge a_7+a_9, \quad a_2+a_5 \ge a_6+a_8, \quad a_1 \ge a_5,$
$$a_8 \ge a_5 \ge a_{10}, \quad a_9 \ge a_6.$$

(2). $$\sum_{0\le u\le a_4} (-1)^u \begin{bmatrix} a_7+a_9-a_3-a_6-1+u \\ u \end{bmatrix} e_1^{(a_4-u)} e_2^{(a_3+a_4)} e_3^{(a_2+a_3+a_4)} e_2^{(a_6)}$$
$$\times e_4^{(a_1+a_2+a_3+a_4)} e_3^{(a_5+a_6)} e_1^{(a_7+a_9+u)} e_2^{(a_7+a_8+a_9)} e_3^{(a_7)} e_1^{(a_{10})}$$
if $\quad a_3+a_5+a_6 \ge a_7+a_8+a_9, \quad a_7+a_9 \ge a_3+a_6, \quad a_1 \ge a_5,$
$$a_8+a_9 \ge a_5+a_6, \quad a_8 \ge a_{10}, \quad a_2 \ge a_6.$$

(3). $$\sum_{0\le u\le a_2+a_3+a_4} (-1)^u \begin{bmatrix} a_5-a_1-1+u \\ u \end{bmatrix} e_1^{(a_4)} e_2^{(a_3+a_4)} e_3^{(a_2+a_3+a_4-u)} e_2^{(a_6)}$$
$$\times e_4^{(a_1+a_2+a_3+a_4)} e_3^{(a_5+a_6+u)} e_1^{(a_7+a_9)} e_2^{(a_7+a_8+a_9)} e_3^{(a_7)} e_1^{(a_{10})}$$
if $\quad a_8+a_9 \ge a_5+a_6, \quad a_3+a_6 \ge a_7+a_9,$
$$a_5 \ge a_1 \ge a_8 \ge a_{10}, \quad a_2 \ge a_6.$$

7. (1). $$\sum_{0\le u\le a_7}(-1)^u\begin{bmatrix}a_5+a_6-a_8-a_9-1+u\\u\end{bmatrix}e_3^{(a_2)}e_2^{(a_3+a_6)}e_1^{(a_4+a_7+a_9)}e_2^{(a_4)}$$
$$\times\, e_4^{(a_1+a_2)}e_3^{(a_3+a_4+a_5+a_6+u)}e_2^{(a_7+a_8+a_9)}e_4^{(a_3+a_4)}e_1^{(a_{10})}e_3^{(a_7-u)}$$
if $a_5+a_6\ge a_8+a_9\ge a_1+a_2,\quad a_7+a_9\ge a_3+a_6,$

$a_1\ge a_5,\quad a_3\ge a_7,\quad a_8\ge a_{10}.$

(2). $$\sum_{0\le u\le a_2}(-1)^u\begin{bmatrix}a_5-a_1-1+u\\u\end{bmatrix}e_3^{(a_2-u)}e_2^{(a_3+a_6)}e_1^{(a_4+a_7+a_9)}e_2^{(a_4)}e_4^{(a_1+a_2)}$$
$$\times\, e_3^{(a_3+a_4+a_5+a_6+u)}e_2^{(a_7+a_8+a_9)}e_4^{(a_3+a_4)}e_1^{(a_{10})}e_3^{(a_7)}$$
if $a_1+a_3+a_6\ge a_7+a_8+a_9,\quad a_7+a_9\ge a_3+a_6,\quad a_5\ge a_1,$

$a_8+a_9\ge a_5+a_6,\quad a_6\ge a_2,\quad a_8\ge a_{10}.$

(3). $$\sum_{0\le u\le a_4+a_7+a_9}(-1)^u\begin{bmatrix}a_{10}-a_8-1+u\\u\end{bmatrix}e_3^{(a_2)}e_2^{(a_3+a_6)}e_1^{(a_4+a_7+a_9-u)}e_2^{(a_4)}$$
$$\times\, e_4^{(a_1+a_2)}e_3^{(a_3+a_4+a_5+a_6)}e_2^{(a_7+a_8+a_9)}e_4^{(a_3+a_4)}e_1^{(a_{10}+u)}e_3^{(a_7)}$$
if $a_3+a_5+a_6\ge a_7+a_8+a_9,\quad a_7+a_9\ge a_3+a_6,\quad a_1\ge a_5,$

$a_8+a_9\ge a_5+a_6\ge a_1+a_2,\quad a_{10}\ge a_8.$

8. (1). $$\sum_{0\le u\le a_4}(-1)^u\begin{bmatrix}a_7+a_9+a_{10}-a_3-a_6-a_8-1+u\\u\end{bmatrix}e_3^{(a_2)}e_1^{(a_4-u)}e_4^{(a_1+a_2)}$$
$$\times\, e_2^{(a_3+a_4+a_6)}e_3^{(a_3+a_4+a_5+a_6)}e_4^{(a_3+a_4)}e_2^{(a_8)}e_1^{(a_7+a_9+a_{10}+u)}e_2^{(a_7+a_9)}e_3^{(a_7)}$$
if $a_7+a_9+a_{10}\ge a_3+a_6+a_8,\quad a_3+a_6\ge a_7+a_9,$

$a_8+a_9\ge a_5+a_6\ge a_1+a_2,\quad a_1\ge a_5\ge a_8.$

(2). $$\sum_{0\le u\le a_2}(-1)^u\begin{bmatrix}a_5-a_1-1+u\\u\end{bmatrix}e_3^{(a_2-u)}e_1^{(a_4)}e_4^{(a_1+a_2)}e_2^{(a_3+a_4+a_6)}$$
$$\times\, e_3^{(a_3+a_4+a_5+a_6+u)}e_4^{(a_3+a_4)}e_2^{(a_8)}e_1^{(a_7+a_9+a_{10})}e_2^{(a_7+a_9)}e_3^{(a_7)}$$
if $a_3+a_6+a_8\ge a_7+a_9+a_{10},\quad a_8+a_9\ge a_5+a_6,$

$a_5\ge a_1\ge a_8,\quad a_6\ge a_2,\quad a_{10}\ge a_8.$

(3). $$\sum_{0\le u\le a_7}(-1)^u\begin{bmatrix}a_5+a_6-a_8-a_9-1+u\\u\end{bmatrix}e_3^{(a_2)}e_1^{(a_4)}e_4^{(a_1+a_2)}e_2^{(a_3+a_4+a_6)}$$
$$\times\, e_3^{(a_3+a_4+a_5+a_6+u)}e_4^{(a_3+a_4)}e_2^{(a_8)}e_1^{(a_7+a_9+a_{10})}e_2^{(a_7+a_9)}e_3^{(a_7-u)}$$
if $a_3+a_6+a_8\ge a_7+a_9+a_{10},\quad a_1\ge a_5,\quad a_{10}\ge a_8,$

$a_5+a_6\ge a_8+a_9\ge a_1+a_2,\quad a_9\ge a_6.$

9. (1). $$\sum_{0\le u\le a_6+a_7}(-1)^u\begin{bmatrix} a_8-a_5-1+u\\ u\end{bmatrix} e_1^{(a_4)}e_2^{(a_3+a_4)}e_3^{(a_2+a_3+a_4)}e_1^{(a_7)}e_2^{(a_6+a_7-u)}$$
$$\times\, e_1^{(a_9)}e_4^{(a_1+a_2+a_3+a_4)}e_3^{(a_5+a_6+a_7)}e_2^{(a_8+a_9+u)}e_1^{(a_{10})}$$
$$\text{if}\quad a_2+a_5\ge a_6+a_8,\quad a_1\ge a_5\ge a_{10},$$
$$a_8\ge a_5,\quad a_3\ge a_7,\quad a_6\ge a_9.$$

(2). $$\sum_{0\le u\le a_2+a_3+a_4}(-1)^u\begin{bmatrix} a_5-a_1-1+u\\ u\end{bmatrix} e_1^{(a_4)}e_2^{(a_3+a_4)}e_3^{(a_2+a_3+a_4-u)}e_1^{(a_7)}$$
$$\times\, e_2^{(a_6+a_7)}e_1^{(a_9)}e_4^{(a_1+a_2+a_3+a_4)}e_3^{(a_5+a_6+a_7+u)}e_2^{(a_8+a_9)}e_1^{(a_{10})}$$
$$\text{if}\quad a_5\ge a_1\ge a_8\ge a_{10},\quad a_2\ge a_6\ge a_9,\quad a_3\ge a_7.$$

(3). $$\sum_{0\le u\le a_3+a_4}(-1)^u\begin{bmatrix} a_6-a_2-1+u\\ u\end{bmatrix} e_1^{(a_4)}e_2^{(a_3+a_4-u)}e_3^{(a_2+a_3+a_4)}e_1^{(a_7)}$$
$$\times\, e_2^{(a_6+a_7+u)}e_1^{(a_9)}e_4^{(a_1+a_2+a_3+a_4)}e_3^{(a_5+a_6+a_7)}e_2^{(a_8+a_9)}e_1^{(a_{10})}$$
$$\text{if}\quad a_1+a_2\ge a_5+a_6,\quad a_5\ge a_8\ge a_{10},$$
$$a_6\ge a_2\ge a_9,\quad a_3\ge a_7.$$

10. (1). $$\sum_{0\le u\le a_3+a_4}(-1)^u\begin{bmatrix} a_6-a_2-1+u\\ u\end{bmatrix} e_1^{(a_4)}e_2^{(a_3+a_4-u)}e_3^{(a_2+a_3+a_4)}e_1^{(a_7)}$$
$$\times\, e_2^{(a_6+a_7+u)}e_4^{(a_1+a_2+a_3+a_4)}e_3^{(a_5+a_6+a_7)}e_2^{(a_8)}e_1^{(a_9+a_{10})}e_2^{(a_9)}$$
$$\text{if}\quad a_1+a_2\ge a_5+a_6,\quad a_2+a_8\ge a_9+a_{10},$$
$$a_{10}\ge a_8,\quad a_5\ge a_8,\quad a_6\ge a_2,\quad a_3\ge a_7.$$

(2). $$\sum_{0\le u\le a_7}(-1)^u\begin{bmatrix} a_9+a_{10}-a_6-a_8-1+u\\ u\end{bmatrix} e_1^{(a_4)}e_2^{(a_3+a_4)}e_3^{(a_2+a_3+a_4)}e_1^{(a_7-u)}$$
$$\times\, e_2^{(a_6+a_7)}e_4^{(a_1+a_2+a_3+a_4)}e_3^{(a_5+a_6+a_7)}e_2^{(a_8)}e_1^{(a_9+a_{10}+u)}e_2^{(a_9)}$$
$$\text{if}\quad a_3+a_6+a_8\ge a_7+a_9+a_{10},\quad a_9+a_{10}\ge a_6+a_8,$$
$$a_1\ge a_5\ge a_8,\quad a_2\ge a_6\ge a_9.$$

(3). $$\sum_{0\le u\le a_2+a_3+a_4}(-1)^u\begin{bmatrix} a_5-a_1-1+u\\ u\end{bmatrix} e_1^{(a_4)}e_2^{(a_3+a_4)}e_3^{(a_2+a_3+a_4-u)}e_1^{(a_7)}$$
$$\times\, e_2^{(a_6+a_7)}e_4^{(a_1+a_2+a_3+a_4)}e_3^{(a_5+a_6+a_7+u)}e_2^{(a_8)}e_1^{(a_9+a_{10})}e_2^{(a_9)}$$
$$\text{if}\quad a_6+a_8\ge a_9+a_{10},\quad a_5\ge a_1\ge a_8,$$
$$a_{10}\ge a_8,\quad a_3\ge a_7,\quad a_2\ge a_6.$$

11. (1). $$\sum_{0\le u\le a_3+a_6} (-1)^u \begin{bmatrix} a_5-a_8-1+u \\ u \end{bmatrix} e_4^{(a_1)} e_3^{(a_2+a_5+u)} e_2^{(a_3+a_6+a_8)} e_4^{(a_2)}$$
$$\times e_3^{(a_3+a_6-u)} e_1^{(a_4+a_7+a_9+a_{10})} e_2^{(a_4+a_7+a_9)} e_3^{(a_4)} e_4^{(a_3+a_4)} e_3^{(a_7)}$$
$$\text{if} \quad a_7+a_8+a_9 \ge a_3+a_5+a_6, \quad a_5 \ge a_8,$$
$$a_{10} \ge a_8 \ge a_1, \quad a_6 \ge a_2, \quad a_3 \ge a_7.$$

(2). $$\sum_{0\le u\le a_3+a_4} (-1)^u \begin{bmatrix} a_2-a_6-1+u \\ u \end{bmatrix} e_4^{(a_1)} e_3^{(a_2+a_5)} e_2^{(a_3+a_6+a_8)} e_4^{(a_2+u)}$$
$$\times e_3^{(a_3+a_6)} e_1^{(a_4+a_7+a_9+a_{10})} e_2^{(a_4+a_7+a_9)} e_3^{(a_4)} e_4^{(a_3+a_4-u)} e_3^{(a_7)}$$
$$\text{if} \quad a_7+a_9 \ge a_3+a_6, \quad a_6+a_8 \ge a_2+a_5, \quad a_3 \ge a_7,$$
$$a_{10} \ge a_8, \quad a_2 \ge a_6, \quad a_5 \ge a_1.$$

(3). $$\sum_{0\le u\le a_4+a_7+a_9} (-1)^u \begin{bmatrix} a_8-a_{10}-1+u \\ u \end{bmatrix} e_4^{(a_1)} e_3^{(a_2+a_5)} e_2^{(a_3+a_6+a_8+u)} e_4^{(a_2)}$$
$$\times e_3^{(a_3+a_6)} e_1^{(a_4+a_7+a_9+a_{10})} e_2^{(a_4+a_7+a_9-u)} e_3^{(a_4)} e_4^{(a_3+a_4)} e_3^{(a_7)}$$
$$\text{if} \quad a_7+a_9 \ge a_3+a_6, \quad a_6 \ge a_2,$$
$$a_8 \ge a_{10} \ge a_5 \ge a_1, \quad a_3 \ge a_7.$$

12. (1). $$\sum_{0\le u\le a_4+a_7} (-1)^u \begin{bmatrix} a_5+a_6-a_8-a_9-1+u \\ u \end{bmatrix} e_3^{(a_2)} e_2^{(a_3+a_6)} e_1^{(a_4+a_7+a_9)} e_4^{(a_1+a_2)}$$
$$\times e_3^{(a_3+a_5+a_6+u)} e_2^{(a_4+a_7+a_8+a_9)} e_4^{(a_3)} e_3^{(a_4+a_7-u)} e_4^{(a_4)} e_1^{(a_{10})}$$
$$\text{if} \quad a_5+a_6 \ge a_8+a_9 \ge a_1+a_2, \quad a_7 \ge a_3,$$
$$a_8 \ge a_{10}, \quad a_1 \ge a_5, \quad a_9 \ge a_6.$$

(2). $$\sum_{0\le u\le a_2} (-1)^u \begin{bmatrix} a_5-a_1-1+u \\ u \end{bmatrix} e_3^{(a_2-u)} e_2^{(a_3+a_6)} e_1^{(a_4+a_7+a_9)} e_4^{(a_1+a_2)}$$
$$\times e_3^{(a_3+a_5+a_6+u)} e_2^{(a_4+a_7+a_8+a_9)} e_4^{(a_3)} e_3^{(a_4+a_7)} e_4^{(a_4)} e_1^{(a_{10})}$$
$$\text{if} \quad a_8+a_9 \ge a_5+a_6, \quad a_7 \ge a_3,$$
$$a_5 \ge a_1 \ge a_8 \ge a_{10}, \quad a_6 \ge a_2.$$

(3). $$\sum_{0\le u\le a_3+a_6} (-1)^u \begin{bmatrix} a_8-a_5-1+u \\ u \end{bmatrix} e_3^{(a_2)} e_2^{(a_3+a_6-u)} e_1^{(a_4+a_7+a_9)} e_4^{(a_1+a_2)}$$
$$\times e_3^{(a_3+a_5+a_6)} e_2^{(a_4+a_7+a_8+a_9+u)} e_4^{(a_3)} e_3^{(a_4+a_7)} e_4^{(a_4)} e_1^{(a_{10})}$$
$$\text{if} \quad a_5+a_6 \ge a_1+a_2, \quad a_1 \ge a_5 \ge a_{10},$$
$$a_8 \ge a_5, \quad a_7 \ge a_3, \quad a_9 \ge a_6.$$

13. (1). $$\sum_{0\le u\le a_4+a_6+a_7} (-1)^u \begin{bmatrix} a_8-a_5-1+u \\ u \end{bmatrix} e_2^{(a_3)} e_3^{(a_2+a_3)} e_4^{(a_1+a_2+a_3)} e_1^{(a_4+a_7)}$$
$$\times\, e_2^{(a_4+a_6+a_7-u)} e_3^{(a_4+a_5+a_6+a_7)} e_1^{(a_9)} e_2^{(a_8+a_9+u)} e_1^{(a_{10})} e_4^{(a_4)}$$
$$\text{if}\quad a_5+a_6+a_7 \ge a_1+a_2+a_3,\quad a_2+a_5 \ge a_6+a_8,$$
$$a_1 \ge a_5 \ge a_{10},\quad a_6 \ge a_9,\quad a_8 \ge a_5.$$

(2). $$\sum_{0\le u\le a_3} (-1)^u \begin{bmatrix} a_6-a_2-1+u \\ u \end{bmatrix} e_2^{(a_3-u)} e_3^{(a_2+a_3)} e_4^{(a_1+a_2+a_3)} e_1^{(a_4+a_7)}$$
$$\times\, e_2^{(a_4+a_6+a_7+u)} e_3^{(a_4+a_5+a_6+a_7)} e_1^{(a_9)} e_2^{(a_8+a_9)} e_1^{(a_{10})} e_4^{(a_4)}$$
$$\text{if}\quad a_5+a_6+a_7 \ge a_1+a_2+a_3,\quad a_1+a_2 \ge a_5+a_6,$$
$$a_5 \ge a_8 \ge a_{10},\quad a_6 \ge a_2 \ge a_9.$$

(3). $$\sum_{0\le u\le a_2+a_3} (-1)^u \begin{bmatrix} a_5-a_1-1+u \\ u \end{bmatrix} e_2^{(a_3)} e_3^{(a_2+a_3-u)} e_4^{(a_1+a_2+a_3)} e_1^{(a_4+a_7)}$$
$$\times\, e_2^{(a_4+a_6+a_7)} e_3^{(a_4+a_5+a_6+a_7+u)} e_1^{(a_9)} e_2^{(a_8+a_9)} e_1^{(a_{10})} e_4^{(a_4)}$$
$$\text{if}\quad a_6+a_7 \ge a_2+a_3,\quad a_2 \ge a_6 \ge a_9,$$
$$a_5 \ge a_1 \ge a_8 \ge a_{10}.$$

14. (1). $$\sum_{0\le u\le a_4+a_7+a_9} (-1)^u \begin{bmatrix} a_{10}-a_8-1+u \\ u \end{bmatrix} e_3^{(a_2)} e_2^{(a_3+a_6)} e_1^{(a_4+a_7+a_9-u)} e_2^{(a_4)}$$
$$\times\, e_3^{(a_3+a_4)} e_4^{(a_1+a_2+a_3+a_4)} e_3^{(a_5+a_6)} e_2^{(a_7+a_8+a_9)} e_1^{(a_{10}+u)} e_3^{(a_7)}$$
$$\text{if}\quad a_3+a_5+a_6 \ge a_7+a_8+a_9,\quad a_7+a_9 \ge a_3+a_6,\quad a_6 \ge a_2,$$
$$a_8+a_9 \ge a_5+a_6,\quad a_1+a_2 \ge a_5+a_6,\quad a_{10} \ge a_8.$$

(2). $$\sum_{0\le u\le a_4} (-1)^u \begin{bmatrix} a_7+a_8+a_9-a_3-a_5-a_6-1+u \\ u \end{bmatrix} e_3^{(a_2)} e_2^{(a_3+a_6)} e_1^{(a_4+a_7+a_9)}$$
$$\times\, e_2^{(a_4-u)} e_3^{(a_3+a_4)} e_4^{(a_1+a_2+a_3+a_4)} e_3^{(a_5+a_6)} e_2^{(a_7+a_8+a_9+u)} e_1^{(a_{10})} e_3^{(a_7)}$$
$$\text{if}\quad a_7+a_8+a_9 \ge a_3+a_5+a_6,\quad a_1+a_2 \ge a_5+a_6,\quad a_3 \ge a_7,$$
$$a_5 \ge a_8 \ge a_{10},\quad a_6 \ge a_2.$$

15. (1). $$\sum_{0\le u\le a_4} (-1)^u \begin{bmatrix} a_7+a_9+a_{10}-a_3-a_6-a_8-1+u \\ u \end{bmatrix} e_3^{(a_2)} e_1^{(a_4-u)} e_2^{(a_3+a_4+a_6)}$$
$$\times\, e_3^{(a_3+a_4)} e_4^{(a_1+a_2+a_3+a_4)} e_3^{(a_5+a_6)} e_2^{(a_8)} e_1^{(a_7+a_9+a_{10}+u)} e_2^{(a_7+a_9)} e_3^{(a_7)}$$
$$\text{if}\quad a_7+a_9+a_{10} \ge a_3+a_6+a_8,\quad a_1+a_2 \ge a_5+a_6,\quad a_6 \ge a_2,$$
$$a_8+a_9 \ge a_5+a_6,\quad a_3+a_6 \ge a_7+a_9,\quad a_5 \ge a_8.$$

(2). $$\sum_{0\le u\le a_3+a_4+a_6} (-1)^u \begin{bmatrix} a_8-a_5-1+u \\ u \end{bmatrix} e_3^{(a_2)} e_1^{(a_4)} e_2^{(a_3+a_4+a_6-u)} e_3^{(a_3+a_4)}$$
$$\times\, e_4^{(a_1+a_2+a_3+a_4)} e_3^{(a_5+a_6)} e_2^{(a_8+u)} e_1^{(a_7+a_9+a_{10})} e_2^{(a_7+a_9)} e_3^{(a_7)}$$
if $a_3+a_6+a_8 \ge a_7+a_9+a_{10}, \quad a_1+a_2 \ge a_5+a_6,$

$a_9 \ge a_6 \ge a_2, \quad a_{10} \ge a_8 \ge a_5.$

16. (1). $$\sum_{0\le u\le a_3+a_4} (-1)^u \begin{bmatrix} a_5+a_6-a_1-a_2-1+u \\ u \end{bmatrix} e_3^{(a_2)} e_2^{(a_3+a_6)} e_1^{(a_4+a_7+a_9)} e_2^{(a_4)}$$
$$\times\, e_3^{(a_3+a_4-u)} e_4^{(a_1+a_2+a_3+a_4)} e_2^{(a_7)} e_3^{(a_5+a_6+a_7+u)} e_2^{(a_8+a_9)} e_1^{(a_{10})}$$
if $a_5+a_6 \ge a_1+a_2 \ge a_8+a_9, \quad a_7+a_9 \ge a_3+a_6,$

$a_8 \ge a_{10}, \quad a_3 \ge a_7, \quad a_1 \ge a_5.$

(2). $$\sum_{0\le u\le a_4+a_7+a_9} (-1)^u \begin{bmatrix} a_{10}-a_8-1+u \\ u \end{bmatrix} e_3^{(a_2)} e_2^{(a_3+a_6)} e_1^{(a_4+a_7+a_9-u)} e_2^{(a_4)}$$
$$\times\, e_3^{(a_3+a_4)} e_4^{(a_1+a_2+a_3+a_4)} e_2^{(a_7)} e_3^{(a_5+a_6+a_7)} e_2^{(a_8+a_9)} e_1^{(a_{10}+u)}$$
if $a_1+a_2 \ge a_5+a_6 \ge a_8+a_9, \quad a_7+a_9 \ge a_3+a_6,$

$a_{10} \ge a_8, \quad a_3 \ge a_7, \quad a_6 \ge a_2.$

17. (1). $$\sum_{0\le u\le a_4} (-1)^u \begin{bmatrix} a_1+a_2+a_3-a_5-a_6-a_7-1+u \\ u \end{bmatrix} e_2^{(a_3)} e_3^{(a_2+a_3)} e_4^{(a_1+a_2+a_3+u)}$$
$$\times\, e_3^{(a_5)} e_2^{(a_6+a_8)} e_1^{(a_4+a_7+a_9+a_{10})} e_3^{(a_6)} e_2^{(a_4+a_7+a_9)} e_3^{(a_4+a_7)} e_4^{(a_4-u)}$$
if $a_1+a_2+a_3 \ge a_5+a_6+a_7, \quad a_2+a_5 \ge a_6+a_8, \quad a_9 \ge a_6,$

$a_6+a_7 \ge a_2+a_3, \quad a_{10} \ge a_8 \ge a_5.$

(2). $$\sum_{0\le u\le a_4+a_7+a_9} (-1)^u \begin{bmatrix} a_8-a_{10}-1+u \\ u \end{bmatrix} e_2^{(a_3)} e_3^{(a_2+a_3)} e_4^{(a_1+a_2+a_3)} e_3^{(a_5)}$$
$$\times\, e_2^{(a_6+a_8+u)} e_1^{(a_4+a_7+a_9+a_{10})} e_3^{(a_6)} e_2^{(a_4+a_7+a_9-u)} e_3^{(a_4+a_7)} e_4^{(a_4)}$$
if $a_5+a_6+a_7 \ge a_1+a_2+a_3, \quad a_2+a_5 \ge a_6+a_8,$

$a_1 \ge a_5, \quad a_8 \ge a_{10} \ge a_5, \quad a_9 \ge a_6.$

18. (1). $$\sum_{0\le u\le a_3} (-1)^u \begin{bmatrix} a_6-a_2-1+u \\ u \end{bmatrix} e_2^{(a_3-u)} e_3^{(a_2+a_3)} e_4^{(a_1+a_2+a_3)} e_1^{(a_4+a_7)}$$
$$\times\, e_2^{(a_4+a_6+a_7+u)} e_3^{(a_4+a_5+a_6+a_7)} e_2^{(a_8)} e_1^{(a_9+a_{10})} e_4^{(a_4)} e_2^{(a_9)}$$
if $a_5+a_6+a_7 \ge a_1+a_2+a_3, \quad a_2+a_8 \ge a_9+a_{10}, \quad a_5 \ge a_8,$

$a_1+a_2 \ge a_5+a_6, \quad a_{10} \ge a_8, \quad a_6 \ge a_2.$

$$(2).\quad \sum_{0\le u\le a_2+a_3} (-1)^u \begin{bmatrix} a_5-a_1-1+u \\ u \end{bmatrix} e_2^{(a_3)} e_3^{(a_2+a_3-u)} e_4^{(a_1+a_2+a_3)} e_1^{(a_4+a_7)}$$

$$\times\, e_2^{(a_4+a_6+a_7)} e_3^{(a_4+a_5+a_6+a_7+u)} e_2^{(a_8)} e_1^{(a_9+a_{10})} e_4^{(a_4)} e_2^{(a_9)}$$

$$\text{if} \quad a_6+a_7 \ge a_2+a_3, \quad a_6+a_8 \ge a_9+a_{10},$$

$$a_5 \ge a_1 \ge a_8, \quad a_2 \ge a_6, \quad a_{10} \ge a_8.$$

19. $$(1).\quad \sum_{0\le u\le a_6+a_7} (-1)^u \begin{bmatrix} a_5-a_8-1+u \\ u \end{bmatrix} e_2^{(a_3)} e_4^{(a_1)} e_1^{(a_4+a_7)} e_2^{(a_4)} e_3^{(a_2+a_3+a_4+a_5+u)}$$

$$\times\, e_2^{(a_6+a_7+a_8)} e_1^{(a_9+a_{10})} e_4^{(a_2+a_3+a_4)} e_3^{(a_6+a_7-u)} e_2^{(a_9)}$$

$$\text{if} \quad a_2+a_3 \ge a_6+a_7, \quad a_6+a_8 \ge a_9+a_{10}, \quad a_5 \ge a_8,$$

$$a_{10} \ge a_8 \ge a_1, \quad a_7 \ge a_3.$$

$$(2).\quad \sum_{0\le u\le a_9} (-1)^u \begin{bmatrix} a_8-a_{10}-1+u \\ u \end{bmatrix} e_2^{(a_3)} e_4^{(a_1)} e_1^{(a_4+a_7)} e_2^{(a_4)} e_3^{(a_2+a_3+a_4+a_5)}$$

$$\times\, e_2^{(a_6+a_7+a_8+u)} e_1^{(a_9+a_{10})} e_4^{(a_2+a_3+a_4)} e_3^{(a_6+a_7)} e_2^{(a_9-u)}$$

$$\text{if} \quad a_2+a_3+a_5 \ge a_6+a_7+a_8, \quad a_6 \ge a_9,$$

$$a_8 \ge a_{10} \ge a_5 \ge a_1, \quad a_7 \ge a_3.$$

20. $$(1).\quad \sum_{0\le u\le a_4+a_7+a_9} (-1)^u \begin{bmatrix} a_{10}-a_8-1+u \\ u \end{bmatrix} e_2^{(a_3)} e_3^{(a_2+a_3)} e_2^{(a_6)} e_1^{(a_4+a_7+a_9-u)}$$

$$\times\, e_2^{(a_4+a_7)} e_3^{(a_4)} e_4^{(a_1+a_2+a_3+a_4)} e_3^{(a_5+a_6+a_7)} e_2^{(a_8+a_9)} e_1^{(a_{10}+u)}$$

$$\text{if} \quad a_1+a_2+a_3 \ge a_5+a_6+a_7, \quad a_6+a_7 \ge a_2+a_3, \quad a_2 \ge a_6,$$

$$a_5+a_6 \ge a_8+a_9, \quad a_{10} \ge a_8, \quad a_9 \ge a_6.$$

$$(2).\quad \sum_{0\le u\le a_4+a_7} (-1)^u \begin{bmatrix} a_8+a_9-a_5-a_6-1+u \\ u \end{bmatrix} e_2^{(a_3)} e_3^{(a_2+a_3)} e_2^{(a_6)} e_1^{(a_4+a_7+a_9)}$$

$$\times\, e_2^{(a_4+a_7-u)} e_3^{(a_4)} e_4^{(a_1+a_2+a_3+a_4)} e_3^{(a_5+a_6+a_7)} e_2^{(a_8+a_9+u)} e_1^{(a_{10})}$$

$$\text{if} \quad a_1+a_2+a_3 \ge a_5+a_6+a_7, \quad a_6+a_7 \ge a_2+a_3, \quad a_2 \ge a_6,$$

$$a_8+a_9 \ge a_5+a_6, \quad a_5 \ge a_8 \ge a_{10}.$$

21. $$(1).\, \sum_{0\le u\le a_4} (-1)^u \begin{bmatrix} a_1+a_2+a_3-a_5-a_6-a_7-1+u \\ u \end{bmatrix} e_3^{(a_2)} e_2^{(a_3+a_6)} e_3^{(a_3)} e_4^{(a_1+a_2+a_3+u)}$$

$$\times\, e_3^{(a_5+a_6)} e_2^{(a_8)} e_1^{(a_4+a_7+a_9+a_{10})} e_2^{(a_4+a_7+a_9)} e_3^{(a_4+a_7)} e_4^{(a_4-u)}$$

$$\text{if} \quad a_1+a_2+a_3 \ge a_5+a_6+a_7, \quad a_8+a_9 \ge a_5+a_6, \quad a_5 \ge a_8,$$

$$a_7 \ge a_3, \quad a_6 \ge a_2, \quad a_{10} \ge a_8.$$

(2). $$\sum_{0\le u\le a_4+a_7} (-1)^u \begin{bmatrix} a_5+a_6-a_8-a_9-1+u \\ u \end{bmatrix} e_3^{(a_2)} e_2^{(a_3+a_6)} e_3^{(a_3)} e_4^{(a_1+a_2+a_3)}$$
$$\times\, e_3^{(a_5+a_6+u)} e_2^{(a_8)} e_1^{(a_4+a_7+a_9+a_{10})} e_2^{(a_4+a_7+a_9)} e_3^{(a_4+a_7-u)} e_4^{(a_4)}$$
$$\text{if}\quad a_5+a_6+a_7 \ge a_1+a_2+a_3, \quad a_{10}\ge a_8,$$
$$a_1+a_2 \ge a_5+a_6 \ge a_8+a_9, \quad a_9 \ge a_6 \ge a_2.$$

22. (1). $$\sum_{0\le u\le a_3} (-1)^u \begin{bmatrix} a_5+a_6-a_1-a_2-1+u \\ u \end{bmatrix} e_3^{(a_2)} e_2^{(a_3+a_6)} e_1^{(a_4+a_7+a_9)} e_3^{(a_3-u)}$$
$$\times\, e_4^{(a_1+a_2+a_3)} e_2^{(a_4+a_7)} e_3^{(a_4+a_5+a_6+a_7+u)} e_2^{(a_8+a_9)} e_4^{(a_4)} e_1^{(a_{10})}$$
$$\text{if}\quad a_5+a_6 \ge a_1+a_2 \ge a_8+a_9, \quad a_7 \ge a_3,$$
$$a_9 \ge a_6, \quad a_8 \ge a_{10}, \quad a_1 \ge a_5.$$

(2). $$\sum_{0\le u\le a_4+a_7} (-1)^u \begin{bmatrix} a_6-a_9-1+u \\ u \end{bmatrix} e_3^{(a_2)} e_2^{(a_3+a_6+u)} e_1^{(a_4+a_7+a_9)} e_3^{(a_3)}$$
$$\times\, e_4^{(a_1+a_2+a_3)} e_2^{(a_4+a_7-u)} e_3^{(a_4+a_5+a_6+a_7)} e_2^{(a_8+a_9)} e_4^{(a_4)} e_1^{(a_{10})}$$
$$\text{if}\quad a_5+a_6+a_7 \ge a_1+a_2+a_3, \quad a_1+a_2 \ge a_5+a_6,$$
$$a_5 \ge a_8 \ge a_{10}, \quad a_6 \ge a_9 \ge a_2.$$

23. (1). $$\sum_{0\le u\le a_2+a_3} (-1)^u \begin{bmatrix} a_5-a_1-1+u \\ u \end{bmatrix} e_2^{(a_3)} e_3^{(a_2+a_3-u)} e_4^{(a_1+a_2+a_3)} e_2^{(a_6)}$$
$$\times\, e_1^{(a_4+a_7+a_9)} e_3^{(a_5+a_6+u)} e_2^{(a_4+a_7+a_8+a_9)} e_3^{(a_4+a_7)} e_1^{(a_{10})} e_4^{(a_4)}$$
$$\text{if}\quad a_6+a_7 \ge a_2+a_3, \quad a_8+a_9 \ge a_5+a_6,$$
$$a_5 \ge a_1 \ge a_8 \ge a_{10}, \quad a_2 \ge a_6.$$

(2). $$\sum_{0\le u\le a_6} (-1)^u \begin{bmatrix} a_8-a_5-1+u \\ u \end{bmatrix} e_2^{(a_3)} e_3^{(a_2+a_3)} e_4^{(a_1+a_2+a_3)} e_2^{(a_6-u)}$$
$$\times\, e_1^{(a_4+a_7+a_9)} e_3^{(a_5+a_6)} e_2^{(a_4+a_7+a_8+a_9+u)} e_3^{(a_4+a_7)} e_1^{(a_{10})} e_4^{(a_4)}$$
$$\text{if}\quad a_5+a_6+a_7 \ge a_1+a_2+a_3, \quad a_2+a_5 \ge a_6+a_8,$$
$$a_1 \ge a_5 \ge a_{10}, \quad a_9 \ge a_6, \quad a_8 \ge a_5.$$

24. (1). $$\sum_{0\le u\le a_3+a_4} (-1)^u \begin{bmatrix} a_1+a_2-a_5-a_6-1+u \\ u \end{bmatrix} e_3^{(a_2)} e_2^{(a_3+a_6)} e_4^{(a_1+a_2+u)} e_3^{(a_3+a_5+a_6)}$$
$$\times\, e_2^{(a_8)} e_1^{(a_4+a_7+a_9+a_{10})} e_2^{(a_4+a_7+a_9)} e_3^{(a_4)} e_4^{(a_3+a_4-u)} e_3^{(a_7)}$$
$$\text{if}\quad a_7+a_8+a_9 \ge a_3+a_5+a_6, \quad a_1+a_2 \ge a_5+a_6,$$
$$a_3 \ge a_7, \quad a_5 \ge a_8, \quad a_{10} \ge a_8, \quad a_6 \ge a_2.$$

$$(2).\quad \sum_{0\le u\le a_2} (-1)^u \begin{bmatrix} a_5-a_1-1+u \\ u \end{bmatrix} e_3^{(a_2-u)} e_2^{(a_3+a_6)} e_4^{(a_1+a_2)} e_3^{(a_3+a_5+a_6+u)}$$
$$\times e_2^{(a_8)} e_1^{(a_4+a_7+a_9+a_{10})} e_2^{(a_4+a_7+a_9)} e_3^{(a_4)} e_4^{(a_3+a_4)} e_3^{(a_7)}$$
$$\text{if} \quad a_7+a_8+a_9 \ge a_3+a_5+a_6, \quad a_5 \ge a_1 \ge a_8,$$
$$a_3 \ge a_7, \quad a_{10} \ge a_8, \quad a_6 \ge a_2.$$

25. $$(1).\quad \sum_{0\le u\le a_4+a_7} (-1)^u \begin{bmatrix} a_9+a_{10}-a_6-a_8-1+u \\ u \end{bmatrix} e_2^{(a_3)} e_3^{(a_2+a_3)} e_1^{(a_4+a_7-u)} e_2^{(a_4+a_6+a_7)}$$
$$\times e_3^{(a_4)} e_4^{(a_1+a_2+a_3+a_4)} e_3^{(a_5+a_6+a_7)} e_2^{(a_8)} e_1^{(a_9+a_{10}+u)} e_2^{(a_9)}$$
$$\text{if} \quad a_1+a_2+a_3 \ge a_5+a_6+a_7, \quad a_9+a_{10} \ge a_6+a_8,$$
$$a_6+a_7 \ge a_2+a_3, \quad a_2 \ge a_6 \ge a_9, \quad a_5 \ge a_8.$$

$$(2).\quad \sum_{0\le u\le a_3} (-1)^u \begin{bmatrix} a_6-a_2-1+u \\ u \end{bmatrix} e_2^{(a_3-u)} e_3^{(a_2+a_3)} e_1^{(a_4+a_7)} e_2^{(a_4+a_6+a_7+u)}$$
$$\times e_3^{(a_4)} e_4^{(a_1+a_2+a_3+a_4)} e_3^{(a_5+a_6+a_7)} e_2^{(a_8)} e_1^{(a_9+a_{10})} e_2^{(a_9)}$$
$$\text{if} \quad a_1+a_2+a_3 \ge a_5+a_6+a_7, \quad a_2+a_8 \ge a_9+a_{10},$$
$$a_7 \ge a_3, \quad a_6 \ge a_2, \quad a_{10} \ge a_8, \quad a_5 \ge a_8.$$

26. $$(1).\quad \sum_{0\le u\le a_9} (-1)^u \begin{bmatrix} a_8-a_{10}-1+u \\ u \end{bmatrix} e_1^{(a_4)} e_2^{(a_3+a_4)} e_3^{(a_2+a_3+a_4)} e_4^{(a_1+a_2+a_3+a_4)}$$
$$\times e_3^{(a_5)} e_1^{(a_7)} e_2^{(a_6+a_7+a_8+u)} e_1^{(a_9+a_{10})} e_3^{(a_6+a_7)} e_2^{(a_9-u)}$$
$$\text{if} \quad a_2+a_5 \ge a_6+a_8, \quad a_8 \ge a_{10} \ge a_5,$$
$$a_1 \ge a_5, \quad a_3 \ge a_7, \quad a_6 \ge a_9.$$

$$(2).\quad \sum_{0\le u\le a_3+a_4} (-1)^u \begin{bmatrix} a_6+a_8-a_2-a_5-1+u \\ u \end{bmatrix} e_1^{(a_4)} e_2^{(a_3+a_4-u)} e_3^{(a_2+a_3+a_4)}$$
$$\times e_4^{(a_1+a_2+a_3+a_4)} e_3^{(a_5)} e_1^{(a_7)} e_2^{(a_6+a_7+a_8+u)} e_1^{(a_9+a_{10})} e_3^{(a_6+a_7)} e_2^{(a_9)}$$
$$\text{if} \quad a_6+a_8 \ge a_2+a_5 \ge a_9+a_{10}, \quad a_1 \ge a_5,$$
$$a_{10} \ge a_8, \quad a_3 \ge a_7, \quad a_2 \ge a_6.$$

27. $$(1).\quad \sum_{0\le u\le a_3+a_4+a_6} (-1)^u \begin{bmatrix} a_5-a_8-1+u \\ u \end{bmatrix} e_4^{(a_1)} e_3^{(a_2+a_5+u)} e_2^{(a_3+a_6+a_8)}$$
$$\times e_1^{(a_4+a_7+a_9+a_{10})} e_2^{(a_4)} e_4^{(a_2)} e_3^{(a_3+a_4+a_6-u)} e_4^{(a_3+a_4)} e_2^{(a_7+a_9)} e_3^{(a_7)}$$
$$\text{if} \quad a_7+a_9+a_{10} \ge a_3+a_6+a_8, \quad a_3+a_6 \ge a_7+a_9,$$
$$a_8+a_9 \ge a_5+a_6, \quad a_5 \ge a_8 \ge a_1, \quad a_6 \ge a_2.$$

$$(2).\sum_{0\le u\le a_7}(-1)^u\begin{bmatrix}a_6-a_9-1+u\\u\end{bmatrix}e_4^{(a_1)}e_3^{(a_2+a_5)}e_2^{(a_3+a_6+a_8)}e_1^{(a_4+a_7+a_9+a_{10})}$$
$$\times e_2^{(a_4)}e_4^{(a_2)}e_3^{(a_3+a_4+a_6+u)}e_4^{(a_3+a_4)}e_2^{(a_7+a_9)}e_3^{(a_7-u)}$$
$$\text{if}\quad a_7+a_9+a_{10}\ge a_3+a_6+a_8,\quad a_3\ge a_7,$$
$$a_6\ge a_9\ge a_2,\quad a_8\ge a_5\ge a_1.$$

28. (1). $$\sum_{0\le u\le a_4+a_7+a_9}(-1)^u\begin{bmatrix}a_8-a_{10}-1+u\\u\end{bmatrix}e_3^{(a_2)}e_4^{(a_1+a_2)}e_3^{(a_5)}e_2^{(a_3+a_6+a_8+u)}$$
$$\times e_3^{(a_3+a_6)}e_1^{(a_4+a_7+a_9+a_{10})}e_2^{(a_4+a_7+a_9-u)}e_3^{(a_4)}e_4^{(a_3+a_4)}e_3^{(a_7)}$$
$$\text{if}\quad a_5+a_6\ge a_1+a_2,\quad a_8\ge a_{10}\ge a_5,\quad a_1\ge a_5,$$
$$a_7+a_9\ge a_3+a_6,\quad a_3\ge a_7.$$

(2). $$\sum_{0\le u\le a_3+a_4}(-1)^u\begin{bmatrix}a_1+a_2-a_5-a_6-1+u\\u\end{bmatrix}e_3^{(a_2)}e_4^{(a_1+a_2+u)}e_3^{(a_5)}e_2^{(a_3+a_6+a_8)}$$
$$\times e_3^{(a_3+a_6)}e_1^{(a_4+a_7+a_9+a_{10})}e_2^{(a_4+a_7+a_9)}e_3^{(a_4)}e_4^{(a_3+a_4-u)}e_3^{(a_7)}$$
$$\text{if}\quad a_7+a_9\ge a_3+a_6,\quad a_{10}\ge a_8\ge a_5,\quad a_6\ge a_2,$$
$$a_1+a_2\ge a_5+a_6,\quad a_3\ge a_7.$$

29. (1). $$\sum_{0\le u\le a_3+a_6}(-1)^u\begin{bmatrix}a_8-a_5-1+u\\u\end{bmatrix}e_3^{(a_2)}e_2^{(a_3+a_6-u)}e_1^{(a_4+a_7+a_9)}e_3^{(a_3)}$$
$$\times e_4^{(a_1+a_2+a_3)}e_3^{(a_5+a_6)}e_2^{(a_4+a_7+a_8+a_9+u)}e_3^{(a_4+a_7)}e_4^{(a_4)}e_1^{(a_{10})}$$
$$\text{if}\quad a_5+a_6+a_7\ge a_1+a_2+a_3,\quad a_1+a_2\ge a_5+a_6,$$
$$a_8\ge a_5\ge a_{10},\quad a_9\ge a_6\ge a_2.$$

30. (1). $$\sum_{0\le u\le a_2+a_3}(-1)^u\begin{bmatrix}a_5-a_1-1+u\\u\end{bmatrix}e_2^{(a_3)}e_3^{(a_2+a_3-u)}e_4^{(a_1+a_2+a_3)}e_2^{(a_6)}$$
$$\times e_1^{(a_4+a_7+a_9)}e_2^{(a_4+a_7)}e_3^{(a_4+a_5+a_6+a_7+u)}e_2^{(a_8+a_9)}e_1^{(a_{10})}e_4^{(a_4)}$$
$$\text{if}\quad a_6+a_7\ge a_2+a_3,\quad a_1+a_6\ge a_8+a_9,\quad a_2\ge a_6,$$
$$a_5\ge a_1,\quad a_9\ge a_6,\quad a_8\ge a_{10}.$$

31. (1). $$\sum_{0\le u\le a_3+a_4+a_6}(-1)^u\begin{bmatrix}a_8-a_5-1+u\\u\end{bmatrix}e_1^{(a_4)}e_3^{(a_2)}e_2^{(a_3+a_4+a_6-u)}e_1^{(a_7+a_9)}$$
$$\times e_3^{(a_3+a_4)}e_4^{(a_1+a_2+a_3+a_4)}e_3^{(a_5+a_6)}e_2^{(a_7+a_8+a_9+u)}e_1^{(a_{10})}e_3^{(a_7)}$$
$$\text{if}\quad a_1+a_2\ge a_5+a_6,\quad a_3+a_6\ge a_7+a_9,$$
$$a_9\ge a_6\ge a_2,\quad a_8\ge a_5\ge a_{10}.$$

2. *Applying the map $\Gamma\circ\Psi\circ\Phi$ defined in [2] to cases* **1–31**, *we get another* 72 *poly-*

nomial elements in one variable in the canonical basis **B**.

2. Proof of Theorem 1.3

2.1. In order to prove Theorem 1.3. We first need the following identity shown in [9]. Assume that $m \geq k \geq 0,\ \delta \in \mathbb{N}$. Then

$$\text{(i)} \qquad \sum_{0 \leq i \leq \delta} (-1)^i \begin{bmatrix} k-1+i \\ i \end{bmatrix} \begin{bmatrix} m \\ \delta - i \end{bmatrix} v^{i(m-k)} = \begin{bmatrix} m-k \\ \delta \end{bmatrix} v^{-k\delta}.$$

Also, we need the following identity shown in [10]. Assume that $m \geq k \geq 0,\ \delta, n \in \mathbb{N}$. Then

$$\text{(ii)} \qquad \sum_{0 \leq i \leq \delta} (-1)^i \begin{bmatrix} k-1+i \\ i \end{bmatrix} \begin{bmatrix} m+n \\ \delta - i \end{bmatrix} v^{i(m-k-n)}$$

$$= \sum_{0 \leq t \leq \delta, n} \begin{bmatrix} m-k \\ \delta - t \end{bmatrix} \begin{bmatrix} n \\ t \end{bmatrix} v^{-k(\delta-t)-n\delta+t(m+n)}.$$

2.2. Now we prove 1 of Theorem 1.3. It is obvious that all the elements from case **1** to case **31** in Theorem 1.3 are fixed by the involution $\bar{\cdot}$. So we only need to check that these elements lie in $\mathcal{L}$. Moreover, just like what we have done in [2] we only prove the most complicated case here, i.e., case **1**. First of all, we observe the monomial corresponding to (1) – (3) in case **1**. Using the commutative relations in [2, §1.3], we have

$$\text{(iii)} \qquad e_2^{(a_3)} e_3^{(a_2+a_3)} e_4^{(a_1+a_2+a_3)} e_2^{(a_6)} e_3^{(a_5+a_6)} e_2^{(a_8)} e_1^{(a_4+a_7+a_9+a_{10})}$$

$$\times\, e_2^{(a_4+a_7+a_9)} e_3^{(a_4+a_7)} e_4^{(a_4)} = \sum_{\omega \in \Omega} v^{\mathcal{A}(\omega)} \times \mathcal{B}(\omega) \times E^{\omega},$$

where

$$\mathcal{A}(\omega) := -\,(a_4+a_7+a_9+a_{10}-i)(a_4+a_7+a_9-i) - (a_4+a_7-j)(i-j)$$

$$-\,(a_1+a_2+a_3-r)(k-r) - (a_3-k+a_6-l)(a_5+a_6-l)$$

$$-\,(a_3-k+a_6-l+a_8+a_4+a_7+a_9-i-m)(a_4+a_7-j-m)$$

$$-\,(a_3-k)(a_2+a_3-k) - (a_2+a_3-k-t)(a_1+a_2+a_3-r-t)$$

$$- (a_2+ a_3-k -t +a_5+ a_6- l +a_4+ a_7- j- m -p)(a_4- n- s -p)$$

$$- (a_4 - n)(j - n) - (a_4 - n - s)(l + m + k - r - s)$$

$$+ (a_4 + a_7 - j - m)l + (a_5 + a_6 - l + a_4 + a_7 - j - m)(k - r)$$

$$+ (a_4 - n - s - p)(r + t) + pr,$$

and

$$\mathcal{B}(\omega) := \begin{bmatrix} a_8 + a_4 + a_7 + a_9 - i \\ a_8 \end{bmatrix} \begin{bmatrix} a_3 - k + a_6 - l + a_8 + a_4 + a_7 + a_9 - i \\ a_3 - k + a_6 - l \end{bmatrix}$$

$$\times \begin{bmatrix} a_3 - k + a_6 \\ a_6 \end{bmatrix} \begin{bmatrix} l + m \\ l \end{bmatrix} \begin{bmatrix} l + m + k - r \\ k - r \end{bmatrix} \begin{bmatrix} p + t \\ p \end{bmatrix} \begin{bmatrix} r + s \\ r \end{bmatrix}$$

$$\times \begin{bmatrix} a_1 + a_2 + a_3 - r - t + a_4 - n - s - p \\ a_4 - n - s - p \end{bmatrix}$$

$$\times \begin{bmatrix} a_5 + a_6 - l + a_4 + a_7 - j - m \\ a_4 + a_7 - j - m \end{bmatrix}$$

$$\times \begin{bmatrix} a_2 + a_3 - k - t + a_5 + a_6 - l + a_4 + a_7 - j - m \\ a_2 + a_3 - k - t \end{bmatrix}.$$

Note that the last two factors in $\mathcal{B}(\omega)$ can also be equivalently represented as follows

$$\begin{bmatrix} a_2 + a_3 - k - t + a_5 + a_6 - l \\ a_2 + a_3 - k - t \end{bmatrix}$$

$$\times \begin{bmatrix} a_2 + a_3 - k - t + a_5 + a_6 - l + a_4 + a_7 - j - m \\ a_4 + a_7 - j - m \end{bmatrix}.$$

Moreover, we have

$$E^\omega := e_4^{(a_1+a_2+a_3-r-t+a_4-n-s-p)} e_{34}^{(p+t)} e_{24}^{(r+s)} e_{14}^{(n)}$$

$$\times e_3^{(a_2+a_3-k-t+a_5+a_6-l+a_4+a_7-j-m-p)} e_{23}^{(l+m+k-r-s)} e_{13}^{(j-n)}$$

$$\times e_2^{(a_3-k+a_6-l+a_8+a_4+a_7+a_9-i-m)} e_{12}^{(i-j)} e_1^{(a_4+a_7+a_9+a_{10}-i)},$$

and

$$\Omega = \{\omega = (i, j, k, l, m, n, r, s, t, p)\} \subset \mathbb{N}^{10},$$

where integers $i, j, k, l, m, n, r, s, t, p$ satisfy the following inequalities:

$$0 \le i \le a_4 + a_7 + a_9; \quad 0 \le j \le a_4 + a_7, i; \quad 0 \le k \le a_3;$$

$$0 \le m \le a_4 + a_7 - j, a_3 - k + a_6 - l + a_8 + a_4 + a_7 + a_9 - i;$$

$$(\flat) \quad 0 \le l \le a_3 - k + a_6, a_5 + a_6; \quad 0 \le n \le a_4, j; \quad 0 \le r \le k;$$

$$0 \le s \le a_4 - n, l + m + k - r; \quad 0 \le t \le a_2 + a_3 - k;$$

$$0 \le p \le a_4 - n - s, a_2 + a_3 - k - t + a_5 + a_6 - l + a_4 + a_7 - j - m.$$

It should be mentioned here that in the above argument we use 10 different parameters to describe Ω in order to simplify the proof of polynomial elements **1.**(1), (2), (3) in one variable, however we use 14 different parameters to describe Ω in the proof of the monomial element **1.**(1) in [2]. It is only because the commutative order we use here is somewhat different from that in [2].

Set

$$(\natural) \quad \begin{aligned} x_1 &= a_4 + a_7 + a_9 - i, & x_2 &= a_4 + a_7 - j - m, \\ x_3 &= a_3 - k, & x_4 &= a_3 - k + a_6 - l, \\ x_5 &= a_4 - n - s - p, & x_6 &= k - r, \\ x_7 &= a_2 + a_3 - k - t. \end{aligned}$$

Then the degree (with respect to v) of the coefficient $v^{\mathcal{A}(\omega)} \times \mathcal{B}(\omega)$ of E^ω in the sum expression of Formula (iii) is

$$D_M := -L_M(x_1, x_2, \cdots, x_7, m, s, p) - Q_M(x_1, x_2, \cdots, x_7, m, s, p),$$

where $L_M(x_1, x_2, \cdots, x_7, m, s, p)$ is a linear form in non-negative integers $x_1, x_2, \cdots, x_7, m, s, p$, and $Q_M(x_1, x_2, \cdots, x_7, m, s, p)$ is a unit form in non-negative integers $x_1, x_2, \cdots, x_7, m, s, p$. Moreover, we have

$$L_M(x_1, x_2, \cdots, x_7, m, s, p):$$

$$= (a_{10} - a_8)x_1 + (a_8 + a_9 - a_5 - a_6)x_2 + (a_2 - a_6)x_3 + (a_5 - a_8)x_4$$

$$+ (a_5 + a_6 + a_7 - a_1 - a_2 - a_3)x_5 + (a_1 + a_2 - a_5 - a_6)x_6$$

$$+ (a_1 - a_5)x_7 + (a_9 - a_6)m + (a_7 - a_3)s + (a_6 + a_7 - a_2 - a_3)p,$$

and

$$Q_M(x_1, x_2, \cdots, x_7, m, s, p):$$

$$= x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 + x_7^2 + m^2 + s^2 + p^2$$

$$+ x_2x_4 + x_2m + x_3x_6 + x_3x_7 + x_3s + x_3p + x_4m + x_5x_6 + x_5x_7$$

$$+ x_5s + x_5p + x_6x_7 + x_6s + 2x_6p + x_7p + sp - x_1x_4 - x_1m - x_2x_6$$

$$- x_2x_7 - x_2s - x_2p - x_3x_4 - x_3m - x_4x_7 - x_4p - x_6m - ms.$$

By the BDDP-algorithm (see [1] or [2] §3.3) and Lemma 3.4 in [2], we know that the unit form $Q_M(x_1, x_2, \cdots, x_7, m, s, p)$ is weakly positive, i.e.,

$$Q_M(x_1, x_2, \cdots, x_7, m, s, p) \geq 0 \quad \text{for any} \quad (x_1, x_2, \cdots, x_7, m, s, p) \in \mathbb{N}^{10}.$$

Therefore, we have

$$\sum_{0\leq u\leq a_4+a_7} (-1)^u \begin{bmatrix} a_5+a_6-a_8-a_9-1+u \\ u \end{bmatrix} e_2^{(a_3)} e_3^{(a_2+a_3)} e_4^{(a_1+a_2+a_3)} e_2^{(a_6)}$$

$$\times e_3^{(a_5+a_6+u)} e_2^{(a_8)} e_1^{(a_4+a_7+a_9+a_{10})} e_2^{(a_4+a_7+a_9)} e_3^{(a_4+a_7-u)} e_4^{(a_4)}$$

$$= \sum_{\omega\in\Omega_1} \left(\sum_{0\leq u\leq a_4+a_7-j-m} (-1)^u \begin{bmatrix} a_5+a_6-a_8-a_9-1+u \\ u \end{bmatrix} \right.$$

(iv) $$\left. \times \begin{bmatrix} a_4+a_5+a_6+a_7-j-m-l \\ a_4+a_7-j-m-u \end{bmatrix} \times v^{(a_4+a_7+a_8+a_9-j-m-l)u} \right)$$

$$\times v^{\mathcal{A}(\omega)} \times \mathcal{B}_1(\omega) \times E^{\omega}$$

$$= \sum_{\omega\in\Omega_1} v^{\mathcal{A}(\omega)-(a_5+a_6-a_8-a_9)(a_4+a_7-j-m)} \times \mathcal{B}_1(\omega)$$

$$\times \begin{bmatrix} a_4+a_7+a_8+a_9-j-m-l \\ a_4+a_7-j-m \end{bmatrix} \times E^{\omega},$$

where the last equality comes from Formula 2.1 (i), Ω_1 is obtained from Ω by replacing the defining inequality "$0 \leq l \leq a_3 - k + a_6, a_5 + a_6$" in ($\flat$) by "$0 \leq l \leq a_3 - k + a_6, a_5 + a_6 + u$", and $\mathcal{B}_1(\omega)$ is obtained from $\mathcal{B}(\omega)$ by deleting the factor $\begin{bmatrix} a_5 + a_6 - l + a_4 + a_7 - j - m \\ a_4 + a_7 - j - m \end{bmatrix}$. Note that $a_5 + a_6$ is always less than or equal to $a_5 + a_6 + u$, and relations ($\natural$) are independent of u, even if u occurs in the upper boundary of the defining inequality of l, we can still conclude that the expressions behind the second equal sign in (iv) are independent of u.

Using relations ($\natural$), we can get the degree D_{P_1} (with respect to v) of the coefficient of E^ω in the last sum expression of (iv), i.e.,

$$\begin{aligned} D_{P_1} = D_M &- (a_5 + x_4 - x_3)x_2 + (a_8 + a_9 - a_6 + x_4 - x_3)x_2 \\ &- (a_5 + a_6 - a_8 - a_9)x_2 \\ &= -L_{P_1}(x_1, x_2, \cdots, x_7, m, s, p) - Q_{P_1}(x_1, x_2, \cdots, x_7, m, s, p), \end{aligned}$$

where

$$\begin{aligned} &L_{P_1}(x_1, x_2, \cdots, x_7, m, s, p) \\ =&(a_{10} - a_8)x_1 + (a_5 + a_6 - a_8 - a_9)x_2 + (a_2 - a_6)x_3 + (a_5 - a_8)x_4 \\ &+(a_5 + a_6 + a_7 - a_1 - a_2 - a_3)x_5 + (a_1 + a_2 - a_5 - a_6)x_6 \\ &+(a_1 - a_5)x_7 + (a_9 - a_6)m + (a_7 - a_3)s + (a_6 + a_7 - a_2 - a_3)p, \end{aligned}$$

and

$$Q_{P_1}(x_1, x_2, \cdots, x_7, m, s, p) = Q_M(x_1, x_2, \cdots, x_7, m, s, p).$$

When

$$a_5 + a_6 + a_7 \geq a_1 + a_2 + a_3, \quad a_5 + a_6 \geq a_8 + a_9,$$

$$a_1 \geq a_5, \quad a_{10} \geq a_8, \quad a_2 \geq a_6, \quad a_9 \geq a_6,$$

we have $D_{P_1} \leq 0$. Moreover, $D_{P_1} = 0 \Leftrightarrow x_1 = \cdots = x_7 = m = s = p = 0$, and

$E^\omega = E^A$. Therefore, we have

$$\sum_{0\le u\le a_4+a_7} (-1)^u \begin{bmatrix} a_5+a_6-a_8-a_9-1+u \\ u \end{bmatrix} e_2^{(a_3)} e_3^{(a_2+a_3)} e_4^{(a_1+a_2+a_3)} e_2^{(a_6)}$$
$$\times e_3^{(a_5+a_6+u)} e_2^{(a_8)} e_1^{(a_4+a_7+a_9+a_{10})} e_2^{(a_4+a_7+a_9)} e_3^{(a_4+a_7-u)} e_4^{(a_4)}$$

$$\equiv e_4^{(a_1)} e_{34}^{(a_2)} e_{24}^{(a_3)} e_{14}^{(a_4)} e_3^{(a_5)} e_{23}^{(a_6)} e_{13}^{(a_7)} e_2^{(a_8)} e_{12}^{(a_9)} e_1^{(a_{10})} \qquad (\operatorname{mod} v^{-1}\mathcal{L})$$

$$\text{if} \quad a_5+a_6+a_7 \ge a_1+a_2+a_3, \quad a_5+a_6 \ge a_8+a_9,$$
$$a_1 \ge a_5, \quad a_{10} \ge a_8, \quad a_2 \ge a_6, \quad a_9 \ge a_6.$$

So we have proved (1) of case **1**. Let us consider

$$\sum_{0\le u\le a_4} (-1)^u \begin{bmatrix} a_1+a_2+a_3-a_5-a_6-a_7-1+u \\ u \end{bmatrix} e_2^{(a_3)} e_3^{(a_2+a_3)} e_4^{(a_1+a_2+a_3+u)}$$
$$\times e_2^{(a_6)} e_3^{(a_5+a_6)} e_2^{(a_8)} e_1^{(a_4+a_7+a_9+a_{10})} e_2^{(a_4+a_7+a_9)} e_3^{(a_4+a_7)} e_4^{(a_4-u)}$$

(v)
$$= \sum_{\omega\in\Omega} \left(\sum_{0\le u\le a_4-n-s-p} (-1)^u \begin{bmatrix} a_1+a_2+a_3-a_5-a_6-a_7-1+u \\ u \end{bmatrix} \right.$$
$$\times \begin{bmatrix} a_1+a_2+a_3+a_4-r-t-n-s-p \\ a_4-n-s-p-u \end{bmatrix}$$
$$\left. \times v^{(a_4+a_5+a_6+a_7-r-t-n-s-p)u} \right) \times v^{\mathcal{A}(\omega)} \times \mathcal{B}_2(\omega) \times E^\omega$$

$$= \sum_{\omega\in\Omega} v^{\mathcal{A}(\omega)-(a_1+a_2+a_3-a_5-a_6-a_7)(a_4-n-s-p)} \times \mathcal{B}_2(\omega)$$
$$\times \begin{bmatrix} a_4+a_5+a_6+a_7-r-t-n-s-p \\ a_4-n-s-p \end{bmatrix} \times E^\omega,$$

where the last equality comes from Formula 2.1 (i), $\mathcal{B}_2(\omega)$ is obtained from $\mathcal{B}(\omega)$ by deleting the factor $\begin{bmatrix} a_1+a_2+a_3-r-t+a_4-n-s-p \\ a_4-n-s-p \end{bmatrix}$.

Using relations (♮), we can get the degree D_{P_2} (with respect to v) of the coefficient of E^ω in the last sum expression of (v), i.e.,

$$D_{P_2} = D_M - (a_1 + x_6 + x_7)x_5 + (a_5 + a_6 + a_7 - a_2 - a_3 + x_6 + x_7)x_5$$

$$- (a_1 + a_2 + a_3 - a_5 - a_6 - a_7)x_5$$

$$= -L_{P_2}(x_1, x_2, \cdots, x_7, m, s, p) - Q_{P_2}(x_1, x_2, \cdots, x_7, m, s, p),$$

where

$$L_{P_2}(x_1, x_2, \cdots, x_7, m, s, p)$$

$$=(a_{10} - a_8)x_1 + (a_8 + a_9 - a_5 - a_6)x_2 + (a_2 - a_6)x_3 + (a_5 - a_8)x_4$$

$$+(a_1 + a_2 + a_3 - a_5 - a_6 - a_7)x_5 + (a_1 + a_2 - a_5 - a_6)x_6$$

$$+(a_1 - a_5)x_7 + (a_9 - a_6)m + (a_7 - a_3)s + (a_6 + a_7 - a_2 - a_3)p,$$

and

$$Q_{P_2}(x_1, x_2, \cdots, x_7, m, s, p) = Q_M(x_1, x_2, \cdots, x_7, m, s, p).$$

When

$$a_1 + a_2 + a_3 \geq a_5 + a_6 + a_7, \quad a_8 + a_9 \geq a_5 + a_6, \quad a_2 \geq a_6,$$
$$a_6 + a_7 \geq a_2 + a_3, \quad a_{10} \geq a_8, \quad a_5 \geq a_8,$$

we have $D_{P_2} \leq 0$. Moreover, $D_{P_2} = 0 \Leftrightarrow x_1 = \cdots = x_7 = m = s = p = 0$, and $E^\omega = E^A$. Therefore, we have

$$\sum_{0 \leq u \leq a_4} (-1)^u \begin{bmatrix} a_1+a_2+a_3-a_5-a_6-a_7-1+u \\ u \end{bmatrix} e_2^{(a_3)} e_3^{(a_2+a_3)} e_4^{(a_1+a_2+a_3+u)} e_2^{(a_6)}$$
$$\times e_3^{(a_5+a_6)} e_2^{(a_8)} e_1^{(a_4+a_7+a_9+a_{10})} e_2^{(a_4+a_7+a_9)} e_3^{(a_4+a_7-u)} e_4^{(a_4)}$$

$$\equiv e_4^{(a_1)} e_{34}^{(a_2)} e_{24}^{(a_3)} e_{14}^{(a_4)} e_3^{(a_5)} e_{23}^{(a_6)} e_{13}^{(a_7)} e_2^{(a_8)} e_{12}^{(a_9)} e_1^{(a_{10})} \qquad (\mathrm{mod}\, v^{-1}\mathcal{L})$$

$$\text{if} \quad a_1 + a_2 + a_3 \geq a_5 + a_6 + a_7, \quad a_8 + a_9 \geq a_5 + a_6, \quad a_2 \geq a_6,$$
$$a_6 + a_7 \geq a_2 + a_3, \quad a_{10} \geq a_8, \quad a_5 \geq a_8.$$

This gives (2) of case **1**. Finally, we consider

$$\sum_{0\le u\le a_2+a_3} (-1)^u \begin{bmatrix} a_5-a_1-1+u \\ u \end{bmatrix} e_2^{(a_3)} e_3^{(a_2+a_3-u)} e_4^{(a_1+a_2+a_3)} e_2^{(a_6)}$$
$$\times e_3^{(a_5+a_6+u)} e_2^{(a_8)} e_1^{(a_4+a_7+a_9+a_{10})} e_2^{(a_4+a_7+a_9)} e_3^{(a_4+a_7)} e_4^{(a_4)}$$

$$= \sum_{\omega\in\Omega_1} \Bigg(\sum_{0\le u\le a_2+a_3-k-t} (-1)^u \begin{bmatrix} a_5-a_1-1+u \\ u \end{bmatrix}$$
$$\times \begin{bmatrix} a_2+a_3+a_5-r-t+a_6-l-k+r \\ a_2+a_3-k-t-u \end{bmatrix}$$

(vi) $$\times v^{(a_1+a_2+a_3-a_6+l+k-2r-t)u} \Bigg) \times v^{\mathcal{A}(\omega)} \times \mathcal{B}_3(\omega) \times E^\omega$$

$$= \sum_{\omega\in\Omega_2} v^{\mathcal{A}(\omega)-(a_5-a_1)(a_2+a_3-k-t-w)-(a_6-l-k+r)(a_2+a_3-k-t)}$$
$$\times v^{(a_2+a_3-k-t+a_5+a_6-l)w} \times \mathcal{B}_3(\omega) \times \begin{bmatrix} a_1+a_2+a_3-r-t \\ a_2+a_3-k-t-w \end{bmatrix}$$
$$\times \begin{bmatrix} a_6-l-k+r \\ w \end{bmatrix} \times E^\omega,$$

where the last equality comes from the Formula 2.1 (ii), Ω_2 is obtained from Ω_1 by adding "$0\le w\le a_2+a_3-k-t, a_6-l-k+r$" to its defining inequalities, and $\mathcal{B}_3(\omega)$ is obtained from $\mathcal{B}(\omega)$ by deleting the factor $\begin{bmatrix} a_2+a_3-k-t+a_5+a_6-l \\ a_2+a_3-k-t \end{bmatrix}$. Note that the expressions behind the second equal sign in (vi) are also independent of u, just like (1) of case **1**.

Using relations (♮), we can get the degree D_{P_3} (with respect to v) of the coefficient of E^ω in the last sum expression of (vi), i.e.,

$$D_{P_3} = D_M - (a_5+x_4-x_3)x_7 - (a_5-a_1)(x_7-w) - (x_4-x_3-x_6)x_7$$
$$+ (a_5+x_7+x_4-x_3)w + (a_1+x_6+w)(x_7-w)$$
$$+ (x_4-x_3-x_6-w)w$$
$$= -L_{P_3}(x_1,x_2,\cdots,x_7,m,s,p,w) - Q_{P_3}(x_1,x_2,\cdots,x_7,m,s,p,w),$$

where

$$L_{P_3}(x_1, x_2, \cdots, x_7, m, s, p, w)$$

$$=(a_{10} - a_8)x_1 + (a_8 + a_9 - a_5 - a_6)x_2 + (a_2 + a_5 - a_1 - a_6)x_3$$

$$+ (a_1 - a_8)x_4 + (a_5 + a_6 + a_7 - a_1 - a_2 - a_3)x_5 + (a_2 - a_6)x_6$$

$$+ (a_5 - a_1)(x_7 - w) + (a_9 - a_6)m + (a_7 - a_3)s$$

$$+ (a_6 + a_7 - a_2 - a_3)p + (a_5 - a_1)(x_4 - x_3 - x_6 - w),$$

and

$$Q_{P_3}(x_1, x_2, \cdots, x_7, m, s, p, w)$$

$$=Q_M(x_1, x_2, \cdots, x_7, m, s, p) + 2(x_4 - x_3 - x_6 - w)(x_7 - w).$$

By the definition of Ω_2, we have $(x_4 - x_3 - x_6 - w) \geq 0$, and $(x_7 - w) \geq 0$. When

$$a_8 + a_9 \geq a_5 + a_6, \quad a_6 + a_7 \geq a_2 + a_3, \quad a_2 \geq a_6,$$
$$a_5 \geq a_1 \geq a_8, \quad a_{10} \geq a_8,$$

we have $D_{P_3} \leq 0$. Moreover, $D_{P_3} = 0 \Leftrightarrow x_1 = \cdots = x_7 = m = s = p = w = 0$, and $E^\omega = E^A$. Then we have

$$\sum_{0 \leq u \leq a_2 + a_3} (-1)^u \begin{bmatrix} a_5 - a_1 - 1 + u \\ u \end{bmatrix} e_2^{(a_3)} e_3^{(a_2+a_3-u)} e_4^{(a_1+a_2+a_3)} e_2^{(a_6)}$$
$$\times e_3^{(a_5+a_6+u)} e_2^{(a_8)} e_1^{(a_4+a_7+a_9+a_{10})} e_2^{(a_4+a_7+a_9)} e_3^{(a_4+a_7)} e_4^{(a_4)}$$

$$\equiv e_4^{(a_1)} e_{34}^{(a_2)} e_{24}^{(a_3)} e_{14}^{(a_4)} e_3^{(a_5)} e_{23}^{(a_6)} e_{13}^{(a_7)} e_2^{(a_8)} e_{12}^{(a_9)} e_1^{(a_{10})} \qquad (\operatorname{mod} v^{-1}\mathcal{L})$$

$$\text{if} \quad a_8 + a_9 \geq a_5 + a_6, \quad a_6 + a_7 \geq a_2 + a_3, \quad a_2 \geq a_6,$$
$$a_5 \geq a_1 \geq a_8, \quad a_{10} \geq a_8,$$

and (3) of case **1** is proved.

3. Polynomial elements in several variables

Finally, we conclude our note with the following three remarks:

Remark 1. The proof of all the other cases in Theorem 1.3 is quite similar to that of the above case. Also, the proof of a polynomial element has close relations to that of the corresponding monomial element.

Remark 2. We see that the regions of 62 monomial elements and 144 polynomial elements in one variable in the canonical basis **B** do not fill the space $\mathbb{N}^{10}$. Actually, we need only to fill the space $\mathbb{N}^9$ because the regions which we consider are independent of a_4. We believe that in addition to monomial elements and polynomial elements in one variable we have worked out, there exist many polynomial elements in two or more variables in the canonical basis **B**. We are going to calculate all such elements, and determine completely the canonical basis **B** for type A_4 in future. We found more than thirty polynomial elements in two independent variables in the canonical basis **B**. Here, so-called "independent variables" means that the summing of the following two polynomials is independent of the order of u and w. For example, corresponding to monomial element **4.**(1) in Theorem 3.1 in [2], we list the following two polynomial elements in two independent variables u and w:

(a)
$$\sum_{\substack{0\le u\le a_7+a_9\\ 0\le w\le a_2+a_3+a_4}} (-1)^{u+w} \begin{bmatrix} a_{10}-a_8-1+u \\ u \end{bmatrix} \begin{bmatrix} a_5-a_1-1+w \\ w \end{bmatrix} e_1^{(a_4)} e_2^{(a_3+a_4)}$$
$$\times\, e_3^{(a_2+a_3+a_4-w)} e_2^{(a_6)} e_1^{(a_7+a_9-u)} e_4^{(a_1+a_2+a_3+a_4)}$$
$$\times\, e_2^{(a_7)} e_3^{(a_5+a_6+a_7+w)} e_2^{(a_8+a_9)} e_1^{(a_{10}+u)},$$

$$\text{if}\quad a_3+a_6+a_8 \ge a_7+a_9+a_{10},\quad a_1+a_6\ge a_8+a_9,$$
$$a_{10}\ge a_8,\quad a_9\ge a_6,\quad a_2\ge a_6,\quad a_5\ge a_1,$$

and

(b)
$$\sum_{\substack{0\le u\le a_2+a_3+a_4\\ 0\le w\le a_4}} (-1)^{u+w} \begin{bmatrix} a_5-a_1-1+u \\ u \end{bmatrix} \begin{bmatrix} a_7+a_9-a_3-a_6-1+w \\ w \end{bmatrix} e_1^{(a_4-w)}$$
$$\times\, e_2^{(a_3+a_4)} e_3^{(a_2+a_3+a_4-u)} e_2^{(a_6)} e_1^{(a_7+a_9+w)}$$
$$\times\, e_4^{(a_1+a_2+a_3+a_4)} e_2^{(a_7)} e_3^{(a_5+a_6+a_7+u)} e_2^{(a_8+a_9)} e_1^{(a_{10})},$$

$$\text{if}\quad a_7+a_9\ge a_3+a_6,\quad a_1+a_6\ge a_8+a_9,\quad a_5\ge a_1,$$
$$a_8\ge a_{10},\quad a_2\ge a_6,\quad a_3\ge a_7.$$

Remark 3. Recently, Zelevinsky and Marsh-Reineke have independently conjectured that there should be 672 regions of the canonical basis **B** of the quantized enveloping algebra of type A_4 (see [8]). Zelevinsky's approach is based on the dual canonical basis- he has conjectured that this is given by the basis of a corresponding cluster algebra of type D_6 with 672 naturally occurring regions consisting of quasi-commuting monomials. Marsh and Reineke's approach is based on the representation theory of the preprojective algebra of type A_4, which has 672 tilting modules.

Acknowledgments

This work is supported in part by the National Natural Science Foundation of China (10271088) and the Natural Science Foundation of Henan Province (0311010100). The first named author would like to thank Professor Nanhua Xi and Professor Kaiming Zhao for financial support and for their valuable advice and comments during his visit the Morning Side Centre of Mathematics in the Academy of Mathematics and System Sciences in Beijing in May–September 2001. The second named author is also grateful to the Abdus Salam International Centre for Theoretical Physics for financial support and hospitability during his visit. This work was done within the framework of the Associateshop Scheme of the Abdus Salam ICTP, Trieste, Italy. Finally, both authors would like to thank Professor Robert Marsh for helpful communications, and for sending us [1] and other papers.

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