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ERROR ESTIMATES FOR THE FOURIER-FINITE-ELEMENT APPROXIMATION OF THE LAMÉ SYSTEM IN NONSMOOTH AXISYMMETRIC DOMAINS

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Abstract

This paper is concerned with the effective implementation of the Fourier-finite-element method, which combines the approximating Fourier and the finite-element methods, for treating the Derichlet problem for the Lamé equations in axisymmetric domains $\hat{\Omega} \subset \mathbf{R}^3$ with conical vertices and reentrant edges. The partial Fourier decomposition reduces the three-dimensional boundary value problem to an infinite sequence of decoupled two-dimensional boundary value problems on the plane meridian domain $\Omega_a \subset R^2_+$ of $\hat{\Omega}$ with solutions $\mathbf{u}_n$ $(n = 0, 1, 2, \cdots)$ being the Fourier coefficients of the solution $\hat{\mathbf{u}}$ of the 3D problem. The asymptotic behavior of the Fourier coefficients near the angular points of Ω_a is described by appropriate singular vector-functions and treated numerically by linear finite elements on locally graded meshes. For the right-hand side function $\hat{\mathbf{f}} \in (L_2(\hat{\Omega}))^3$ it is proved that with appropriate mesh grading the rate of convergence of the combined approximations in $(W^1_2(\hat{\Omega}))^3$ is of the order $\mathcal{O}(h + N^{-1})$, where h and N are the parameters of the finite-element and Fourier approximations, respectively, with $h \to 0$ and $N \to \infty$.

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1 Introduction

The finite-element method (FEM) has proved to be the most efficient and flexible numerical method for treating elliptic boundary value problems in physics and engineering (cf. [5, 9, 20, 30]). However, the application of the FEM for treating three-dimensional boundary value problems, particularly in the theory of elasticity, requires the discretization of complex structures and the solution of very large systems of equations for which the cost, despite the advances in computational possibilities, may still be high. It is therefore still worthwhile to investigate approaches which simplify the solution process of three-dimensional problems, reduce the cost or admit an effective parallelization.

The Fourier-finite-element method (FFEM) is often applied in physics and engineering to elliptic boundary value problems in three-dimensional axisymmetric domains with nonaxisymmetric data (cf. [5, 7, 8, 30]). The underlining principle is the application of partial Fourier approximation (truncated partial Fourier series) with respect to the rotational angle for dimension reduction, as first step, and as second step, the discretization of a finite number of the reduced problems in the plane meridian domain Ω_a of $\hat{\Omega}$ by finite-element method.

The FFEM provides several advantages. The approximation of the boundary value problem in three dimensions is reduced to the approximation of a finite number of boundary value problems in two dimensions, where standard tools for pre- and post-processing are readily available. In the case of the Lamé operator, the 2D problems are decoupled and can be solved in parallel. Furthermore, some important physical futures such as the three-dimensional stress distributions near singular boundary points can be estimated by truncated Fourier series, with coefficients being the stress intensity factors of the corresponding singular vector-functions in two dimensions.

In papers of the engineering literature on FFEM for boundary value problems in the theory of elasticity (cf. [5, 11, 29]) one finds mainly the practical implementation and experimental demonstrations of the method. Much less work has been done to give the necessary mathematical framework for the method for treating the Lamé equations. However, some advances have been made in the papers [23] – [26].

In this paper we are mainly concerned with error estimates and convergence analysis of the FFEM applied to the Derichlet problem for the Lamé equations in three-dimensional axisymmetric domains with conical vertices and reentrant edges. It is well known that error estimates of the classical FEM for elliptic problems with boundary singularities show a lower rate of convergence than for problems on smooth domains. However, the optimal accuracy can be regained if appropriate graded mesh refinement near the singular boundary points is employed, see e.g., [2, 3, 4, 12, 16, 28] for related works. In this paper we describe the solutions of the reduced two-dimensional BVPs near angular points of the meridian domain Ω_a by appropriate tensor product type singular vector-functions and apply the FEM on locally graded meshes for their treatment.

It should be noted that for the Poisson's equation, a careful study on error analysis for the FFEM has been made (cf. [7, 8, 16, 17, 21]). Particularly, *Mercier/Raugel* in [21] used a mixed Lagrange interpolation and Clément's [10] L_2-projection operator of the Fourier coefficients to obtain error estimates in which the discretization parameters h and N are not coupled. In this Paper we employ the usual Lagrange interpolation of the Fourier coefficients into the finite-element subspaces, but we require additional smoothness of the function $\hat{\mathbf{f}}$ of the right-hand side of the differential equation with respect to the rotational angle φ, that is, for $\hat{\mathbf{f}} \in (L_2(\hat{\Omega}))^3$, we demand that additionally $\frac{\partial \hat{\mathbf{f}}}{\partial \varphi} \in (L_2(\hat{\Omega}))^3$ holds. With these assumptions, the main result in this paper obtained for piecewise linear triangular elements can be summerized by

$$\|\hat{\mathbf{u}} - \hat{\mathbf{u}}_{hN}\|_{(W_2^1(\hat{\Omega}))^3} \leq C\Big(\|\hat{\mathbf{f}}\|_{(L_2(\hat{\Omega}))^3} + \Big\|\frac{\partial \hat{\mathbf{f}}}{\partial \varphi}\Big\|_{(L_2(\hat{\Omega}))^3}\Big) \begin{cases} N^{-1} + h^{\beta_0/\kappa} & \text{if} \quad \kappa > \beta_0 \\ N^{-1} + h & \text{if} \quad \kappa < \beta_0 \end{cases}, \tag{1.1}$$

where β_0 is the minimum value of the singular exponents and κ $(0 < \kappa \leq 1)$ denotes some real grading parameter of the mesh near the corners.

This paper is organized in the following way. In Section 2, the BVP is considered and by means of Partial Fourier analysis, the 2D BVPs in the plane meridian domain Ω_a are obtain. Some a priori estimates for the solutions are given and the asymptotic behavior of the solutions near angular points of the domain Ω_a is described by suitable singular functions. The descriptions are based on the papers [23, 26].

In Section 3, the triangulation of the meridian domain Ω_a and the Fourier-finite-element subspaces with linear finite elements are introduced. In particular, graded local mesh refinements near angular points of Ω_a which generate edges and conical vertices on the axisymmetric domain are considered. Descriptions of graded mesh refinement in plane domains can be found, e.g., in [3, 4, 16, 28]. Here we use the approach presented in [3, 16] to derive estimates of the error of the Lagrange interpolation of the Fourier coefficients $\mathbf{u}_n = \mathbf{s}_n + \mathbf{w}_n$ $(n = 0, 1, \cdots, N)$ into the finite-element subspaces, where $\mathbf{s}_n$ and $\mathbf{w}_n$ represent the singular and regular parts of $\mathbf{u}_n$, respectively. The results obtained in [21] are fundamental for the estimates of the interpolation error for the regular part. Error estimates for the singular part $\mathbf{s}_n$ are obtained from direct computation using their explicit representations as given in Section 2. Furthermore, estimates of the error of the FFEM in the norm of $(W_2^1(\hat{\Omega}))^3$ on triangulations with and without local mesh grading are given. The main result presented in (1.1) is proved.

2 The BVP and analytical preliminaries

2.1 The BVP

Let $\hat{\Omega} \subset \mathbf{R}^3$ be a bounded simply connected open set with at least Lipschitz continuous and piecewise twice continuously differentiable boundary $\hat{\Gamma} := \partial\hat{\Omega}$ $(\hat{\Gamma} \in C^{0,1} \cap PC^2)$. Let $\hat{\mathbf{u}}$ denote the displacement vector field, and let $\hat{\varepsilon}_{ij}(\hat{\mathbf{u}})$ and $\hat{\sigma}_{ij}(\hat{\mathbf{u}})$ $(i, j = 1, 2, 3)$ denote the linearized strain and stress tensors, respectively, for an isotropic and homogeneous elastic body $\hat{\Omega}$. For a given volume force $\hat{\mathbf{f}} \in \left(L_2(\hat{\Omega})\right)^3$, we consider the homogeneous Derichlet problem for the Lamé equations,

$$\sum_{j=1}^{3} \frac{\partial \hat{\sigma}_{ij}(\hat{\mathbf{u}})}{\partial x_j} + \hat{f}_i = 0 \quad \text{in} \quad \hat{\Omega}, \quad \hat{u}_i = 0 \quad \text{on} \quad \hat{\Gamma}, \quad i = 1,2,3. \tag{2.1}$$

Let (x_1, x_2, x_3) denote the Cartesian coordinates of the point $\mathbf{x} \in \mathbf{R}^3$. Suppose that the domain $\hat{\Omega}$ is axisymmetric with respect to the x_3-axis, and that the set $\hat{\Omega} \setminus \Gamma_0$ (Γ_0 is the part of the x_3-axis contained in $\hat{\Omega}$) is generated by rotation of a bounded plane meridian domain Ω_a about the x_3-axis. Let $\partial\Omega_a$ denote the boundary of Ω_a and let $\Gamma_a := \partial\Omega_a \setminus \bar{\Gamma}_0$. For simplicity of the presentation of the finite-element approach we assume that Γ_a is polygonal and that the angles at the points of intersection $\bar{\Gamma}_a \cap \bar{\Gamma}_0$ are not necessarily right angles, that is they may be conical points of the boundary $\hat{\Gamma}$ of $\hat{\Omega}$.

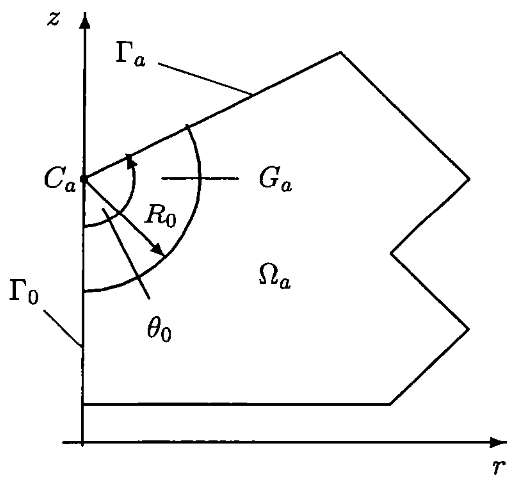

Figure 2.1

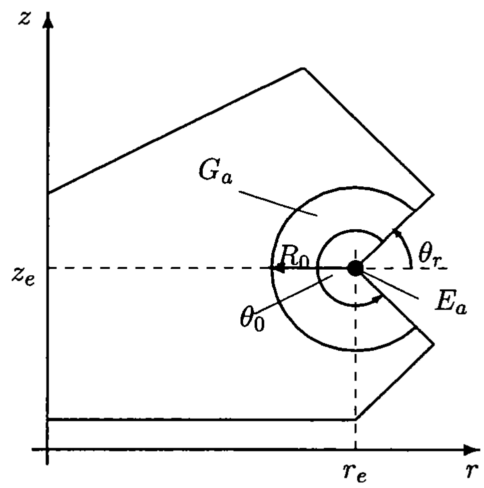

Figure 2.2

Let r, φ, z ($x_1 = r\cos\varphi$, $x_2 = r\sin\varphi$, $x_3 = z$) with $\varphi \in (-\pi, \pi]$ denote the cylindrical coordinates. Then the mappings $\hat{\Omega} \setminus \Gamma_0 \to \Omega := \Omega_a \times (-\pi, \pi]$ and $\hat{\Gamma} \to \Gamma_a \times (-\pi, \pi]$ are one-to-one. For each function $\hat{u}(\mathbf{x})$, $\mathbf{x} \in \hat{\Omega} \setminus \Gamma_0$, some function u on Ω, is uniquely defined by

$$u(r, \varphi, z) := \hat{u}(r\cos\varphi, r\sin\varphi, z), \tag{2.2}$$

and for any vector field $\hat{\mathbf{u}}(\mathbf{x}) = (\hat{u}_1(\mathbf{x}), \hat{u}_2(\mathbf{x}), \hat{u}_3(\mathbf{x}))^T$, $\mathbf{x} \in \hat{\Omega} \setminus \Gamma_0$, some vector field $\mathbf{u} = (u_r(r,\varphi,z), u_\varphi(r,\varphi,z), u_z(r,\varphi,z))^T$, $(r,\varphi,z) \in \Omega$ is uniquely defined by

$$u_r = \hat{u}_1 \cos\varphi + \hat{u}_2 \sin\varphi, \quad u_\varphi = -\hat{u}_1 \sin\varphi + \hat{u}_2 \cos\varphi, \quad u_z = \hat{u}_3. \tag{2.3}$$

We consider some corner point $E_a = (r_e, z_e) \in \Gamma_a$ with the interior angle θ_0 and introduce in the (r,z)-plane local polar coordinates (R, θ) with respect to E_a by $r - r_e = R\cos(\theta + \theta_r)$, $\quad z - z_e = R\sin(\theta + \theta_r)$. For a corner $C_a = (0, z_c) \in \bar{\Gamma}_a \cap \bar{\Gamma}_0$ with interior angle $\theta_0 \neq \pi/2$ we introduce local polar coordinates (R, θ) with respect to this point by $r = R\sin(\theta)$, $\quad z - z_c = -R\cos(\theta)$. The rotation of $E_a \in \Gamma_a$ about the x_3-axis yields and axisymmetric edge on $\hat{\Gamma}$ and the angular point C_a is a conic point on $\hat{\Gamma}$. In Ω_a some circular sector neighborhood G_a of the angular points E_a and C_a, respectively, is defined by (see Figure 2.1 and Figure 2.2)

$$\bar{G}_a := \{(r,z) \in \bar{\Omega}_a : 0 \le R \le R_0, 0 \le \theta \le \theta_0\}, \quad G_a := \bar{G}_a \setminus \partial G_a, \quad \partial_0 G_a := \partial G_a \setminus \bar{\Gamma}_0, \tag{2.4}$$

where ∂G_a is the boundary of G_a. Let $\hat{G}$ denote the 3D domain generated by rotation of G_a about the x_3-axis and $\partial\hat{G}$ its boundary. Then $G = G_a \times (-\pi, \pi]$ and $\partial_0 G := \partial_0 G_a \times (-\pi, \pi]$ are the images of $\hat{G}$ and $\partial\hat{G}$ in the (r, φ, z)-system.

By (2.2) and (2.3) mappings $W_2^k(\hat{\Omega}\setminus\Gamma_0) \to X_{1/2}^k(\Omega)$ ($k = 0, 1, 2,\ W_2^0 = L_2$), $(W_2^1(\hat{\Omega}\setminus\Gamma_0))^3 \to V(\Omega)$ and $(W_2^2(\hat{\Omega}\setminus\Gamma_0))^3 \to Z(\Omega)$ are defined (see e.g. [22, 23, 24] for details), where the norms $\|\cdot\|_{X_{1/2}^k(\Omega)}$, $\|\cdot\|_{V(\Omega)}$ and $\|\cdot\|_{Z(\Omega)}$ coincide with the usual ones in Cartesian coordinates. The variational formulation of the 3D BVP (2.1) in cylindrical coordinates is given by: Find $\mathbf{u} \in V_0(\Omega) := \{\mathbf{v} \in V(\Omega) : v|_{\partial_0\Omega} = 0\}$:

$$b(\mathbf{u}, \mathbf{v}) := \int_\Omega (\sigma(\mathbf{u}))^T \varepsilon(\mathbf{v}) r dr d\varphi dz = \int_\Omega \mathbf{f}^T \mathbf{v} r dr d\varphi dz =: f(\mathbf{v}) \quad \text{for} \quad \mathbf{v} \in V_0(\Omega), \tag{2.5}$$

where $\sigma(\mathbf{u})$ and $\varepsilon(\mathbf{u})$ denote the stress and the strain tensors, respectively, in the (r, φ, z)-system and $\mathbf{f} \in (X_{1/2}^0(\Omega))^3$. The bilinear for $b(\cdot,\cdot)$ from (2.5) is coercive in $V_0(\Omega)$. Thus, the Lax/Milgram lemma infers that problem (2.5) is well posed (cf. [13, 14, 22]).

2.2 Partial Fourier decomposition

For functions $\mathbf{v} \in V(\Omega)$ partial Fourier series expansion with respect to the rotational angle φ is employed using the orthogonal and complete system $\{1, \sin\varphi, \cos\varphi, \cdots, \sin n\varphi, \cos n\varphi, \cdots\}$ in $L_2(-\pi, \pi)$. Thus (cf. [5, 23, 24])

$$\begin{aligned}
v_r(r,\varphi,z) &= \sum_{n=0}^{\infty} (v_{rn}^s(r,z)\cos n\varphi + v_{rn}^a(r,z)\sin n\varphi), \\
v_\varphi(r,\varphi,z) &= \sum_{n=0}^{\infty} (v_{\varphi n}^s(r,z)(-\sin n\varphi) + v_{\varphi n}^a(r,z)\cos n\varphi), \\
v_z(r,\varphi,z) &= \sum_{n=0}^{\infty} (v_{zn}^s(r,z)\cos n\varphi + v_{zn}^a(r,z)\sin n\varphi),
\end{aligned} \tag{2.6}$$

with the Fourier coefficients defined as usual (cf. [5, 23, 24]). For the analysis of function defined on Ω_a we introduce the function spaces (see also [21, 15, 23, 24]):

$$
\begin{aligned}
L_2(\Omega_a) &:= \{w = w(r,z) : \int_{\Omega_a} |w|^2 dr dz < \infty\}, \\
L_{2,\alpha}(\Omega_a) &:= \{w = w(r,z) : r^\alpha w \in L_2(\Omega_a)\}, \quad X(\Omega_a) := L_{2,1/2}(\Omega_a), \\
W_\alpha^{1,2}(\Omega_a) &:= \{w \in L_{2,\alpha}(\Omega_a) : \frac{\partial w}{\partial r}, \frac{\partial w}{\partial z} \in L_{2,\alpha}(\Omega_a)\}, \quad \alpha \text{ a real number} \\
W_{1/2}^{2,2}(\Omega_a) &:= \{w \in W_{1/2}^{1,2}(\Omega_a) : \frac{\partial^2 w}{\partial r^2}, \frac{\partial^2 w}{\partial z^2}, \frac{\partial^2 w}{\partial r \partial z} \in X(\Omega_a)\}, \qquad (2.7) \\
V^a(\Omega_a) &:= \{\mathbf{w} = (w_1, w_2, w_3)^T \in \left(W_{1/2}^{1,2}(\Omega_a)\right)^3 : r^{-1} w_1,\ r^{-1} w_2 \in X(\Omega_a)\}, \\
W^a(\Omega_a) &:= \{\mathbf{w} = (w_1, w_2, w_3)^T \in \left(W_{1/2}^{1,2}(\Omega_a)\right)^3 : r^{-1}(w_1 - w_2),\ r^{-1} w_3 \in X(\Omega_a)\}, \\
V_0^a(\Omega_a) &:= \{\mathbf{w} \in V^a(\Omega_a) : \mathbf{w} = \mathbf{0} \quad \text{auf} \quad \Gamma_a\}, \\
W_0^a(\Omega_a) &:= \{\mathbf{w} \in W^a(\Omega_a) : \mathbf{w} = \mathbf{0} \quad \text{auf} \quad \Gamma_a\}.
\end{aligned}
$$

These spaces are endowed with the norms:

$$
\begin{aligned}
\|w\|_{L_{2,\alpha}(\Omega_a)} &:= \left\{ \int_{\Omega_a} |r^\alpha w|^2 dr dz \right\}^{1/2}, \\
\|w\|_{W_\alpha^{1,2}(\Omega_a)} &:= \left\{ \|w\|^2_{L_{2,\alpha}(\Omega_a)} + \left\| \frac{\partial w}{\partial r} \right\|^2_{L_{2,\alpha}(\Omega_a)} + \left\| \frac{\partial w}{\partial z} \right\|^2_{L_{2,\alpha}(\Omega_a)} \right\}^{1/2}, \\
|w|_{W_{1/2}^{2,2}(\Omega_a)} &:= \left\{ \left\| \frac{\partial^2 w}{\partial r^2} \right\|^2_{X(\Omega_a)} + \left\| \frac{\partial^2 w}{\partial z^2} \right\|^2_{X(\Omega_a)} + 2 \left\| \frac{\partial^2 w}{\partial r \partial z} \right\|^2_{X(\Omega_a)} \right\}^{1/2}, \\
\|w\|_{W_{1/2}^{2,2}(\Omega_a)} &:= \left\{ |w|^2_{W_{1/2}^{2,2}(\Omega_a)} + \|w\|^2_{W_{1/2}^{1,2}(\Omega_a)} \right\}^{1/2}, \qquad (2.8) \\
\|\mathbf{w}\|_{V_0^a(\Omega_a)} &:= \{ \|\frac{w_1}{r}\|^2_{X(\Omega_a)} + \|\frac{w_2}{r}\|^2_{X(\Omega_a)} + \|\mathbf{w}\|^2_{(W_{1/2}^{1,2}(\Omega_a))^3} \}^{1/2}, \\
\|\mathbf{w}\|_{W_0^a(\Omega_a)} &:= \left\{ \|\mathbf{w}\|^2_{(W_{1/2}^{1,2}(\Omega_a))^3} + \left\| \frac{1}{r}(w_1 - w_2) \right\|^2_{X(\Omega_a)} + \left\| \frac{w_3}{r} \right\|^2_{X(\Omega_a)} \right\}^{1/2}.
\end{aligned}
$$

Using generalized completeness relationships norms of functions defined on $\hat{\Omega}$ can be represented by norms of their Fourier coefficients on Ω_a.

Lemma 2.1 ([23, 24]). *Let $\mathbf{u} \in V(\Omega)$ and let $\mathbf{u}_n^s = (u_{rn}^s, u_{\varphi n}^s, u_{zn}^s)^T$ and $\mathbf{u}_n^a = (u_{rn}^a, u_{\varphi n}^a, u_{zn}^a)^T$ ($n \in \mathbf{N}_0 := \{0, 1, 2, \cdots\}$) denote its Fourier coefficients. Then (use $t = r, \varphi, z$)*

$$
\|u_t\|^2_{X_{1/2}^0(\Omega)} = 2\pi \{ \|u_{t0}^s\|^2_{X(\Omega_a)} + \|u_{t0}^a\|^2_{X(\Omega_a)} \} + \pi \sum_{n=1}^{\infty} \left(\|u_{tn}^s\|^2_{X(\Omega_a)} + \|u_{tn}^a\|^2_{X(\Omega_a)} \right) < \infty, \qquad (2.9)
$$

$$
\begin{aligned}
\|\mathbf{u}\|^2_{V(\Omega)} &= 2\pi \Big\{ \|u_{r0}^s\|^2_{W_{1/2}^{1,2}(\Omega_a)} + \|u_{z0}^s\|^2_{W_{1/2}^{1,2}(\Omega_a)} + \|u_{\varphi 0}^a\|^2_{W_{1/2}^{1,2}(\Omega_a)} + \left\| \frac{1}{r} u_{r0}^s \right\|^2_{X(\Omega_a)} \\
&+ \left\| \frac{1}{r} u_{\varphi 0}^a \right\|^2_{X(\Omega_a)} \Big\} + \pi \sum_{n=1}^{\infty} \sum_{e \in \{s,a\}} \Big\{ \|\mathbf{u}_\mathbf{n}^\mathbf{e}\|^2_{(W_{1/2}^{1,2}(\Omega_a))^3} + \left\| \frac{1}{r}(u_{rn}^e - n u_{\varphi n}^e) \right\|^2_{X(\Omega_a)} \\
&+ \left\| \frac{1}{r}(n u_{rn}^e - u_{\varphi n}^e) \right\|^2_{X(\Omega_a)} + n^2 \left\| \frac{1}{r} u_{zn}^e \|^2_{X(\Omega_a)} \right. \Big\} < \infty, \qquad (2.10)
\end{aligned}
$$

where $\sum_{e \in \{s,a\}}$ means summation over s and a.

Remark 2.1 (cf. [16]). *If the function $v(r,\varphi,z)$ and only some of its derivatives belong to $X^0_{1/2}(\Omega)$, then corresponding completeness relations of the type* (2.9) *hold. For example, let $\frac{\partial^l v}{\partial\varphi^l} \in X^0_{1/2}(\Omega)$ $(l=0,1)$, then*

$$\|v\|^2_{X^0_{1/2}(\Omega)} + \left\|\frac{\partial v}{\partial\varphi}\right\|^2_{X^0_{1/2}(\Omega)} = 2\pi\|v^s_0\|^2_{X(\Omega_a)} + \pi\sum_{n=1}^{\infty}(1+n^2)\{\|v^s_n\|^2_{X(\Omega_a)} + \|v^a_n\|^2_{X(\Omega_a)}\} < \infty. \quad (2.11)$$

Hereafter the notation u^e will always mean that some relation holds for u^s as well as for u^a. The solution $\mathbf{u} \in V_0(\Omega)$ of the three-dimensional variational problem (2.5) can be represented by partial Fourier series according to relation (2.6) with Fourier coefficients $\mathbf{u}^e_n$ being solutions of an infinite sequence of decoupled two-dimensional variational problems on Ω_a.

Lemma 2.2 (cf. [23, 24]). *The functionals $b(\mathbf{u},\mathbf{v})$ and $f(\mathbf{v})$ from* (2.5) *admit the representation*

$$b(\mathbf{u},\mathbf{v}) = 2\pi\{b_0(\mathbf{u}^s_0,\mathbf{v}^s_0) + b_0(\mathbf{u}^a_0,\mathbf{v}^a_0)\} + \pi\sum_{n=1}^{\infty}\{b_n(\mathbf{u}^s_n,\mathbf{v}^s_n) + b_n(\mathbf{u}^a_n,\mathbf{v}^a_n)\}, \quad (2.12)$$

$$f(\mathbf{v}) = 2\pi\{\mathbf{f}^s_0(\mathbf{v}^s_0) + \mathbf{f}^a_0(\mathbf{v}^a_0)\} + \pi\sum_{n=1}^{\infty}\{\mathbf{f}^s_n(\mathbf{v}^s_n) + \mathbf{f}^a_n(\mathbf{v}^a_n)\},$$

$$b_n(\mathbf{u}^e_n,\mathbf{v}^e_n) = \int_{\Omega_a}(\varepsilon^e_n(\mathbf{u}^e_n))^T\sigma^e_n(\mathbf{v}^e_n)\,r\,dr\,dz, \quad (2.13)$$

$$\mathbf{f}^e_n(\mathbf{v}^e_n) = \int_{\Omega_a}\mathbf{f}^{e^T}_n\mathbf{v}^e_n\,r\,dr\,dz, \quad (2.14)$$

where $\varepsilon^e_n(\mathbf{u}^e_n)$ and $\sigma^e_n(\mathbf{u}^e_n)$ denote the Fourier coefficients of the linearized strain tensor $\varepsilon(\mathbf{u})$ and stress tensor $\sigma(\mathbf{u})$, respectively.

Theorem 2.1 (cf. [23, 24]). *Let $\mathbf{u} \in V_0(\Omega)$ be the unique solution of the 3D BVP* (2.5) *with $\mathbf{f} \in (X^0_{1/2}(\Omega))^3$. If $\mathbf{u}^e_n$ and $\mathbf{f}^e_n$, $n \in \mathbf{N}_0$, are the Fourier coefficients of $\mathbf{u}$ and $\mathbf{f}$ according to* (2.6), *then $\mathbf{u}^e_n$ are the unique solutions of the following variational equations in Ω_a:*

$$n=0: \quad \text{find } \mathbf{u}^e_0 \in V^a_0(\Omega_a): \quad b_0(\mathbf{u}^e_0,\mathbf{w}) = f^e_0(\mathbf{w}) \quad \text{for} \quad \mathbf{w} \in V^a_0(\Omega_a), \quad (2.15)$$

$$n \in \mathbf{N}: \quad \text{find } \mathbf{u}^e_n \in W^a_0(\Omega_a): \quad b_n(\mathbf{u}^e_n,\mathbf{w}) = f^e_n(\mathbf{w}) \quad \text{for} \quad \mathbf{w} \in W^a_0(\Omega_a). \quad (2.16)$$

Moreover, the solutions $\mathbf{u}^e_n$ satisfy the a priori estimates

$$\|\mathbf{u}^e_0\|^2_{V^a_0(\Omega_a)} \le C_1\|\mathbf{f}^e_0\|^2_{(X(\Omega_a))^3}, \quad \|\mathbf{u}^e_n\|^2_{W^a_0(\Omega_a)} \le C_2\|\mathbf{f}^e_n\|^2_{(X(\Omega_a))^3}, \quad n \in \mathbf{N}, \quad (2.17)$$

$$\|\mathbf{u}^e_n\|^2_{W^a_0(\Omega_a)} \le C_3\left\{\|\mathbf{u}^e_n\|^2_{(W^{1,2}_{1/2}(\Omega_a))^3} + n^2\left\|\frac{1}{r}\mathbf{u}^e_n\right\|^2_{(X(\Omega_a))^3}\right\} \le \frac{C_4}{n^2}\|\mathbf{f}^e_n\|^2_{(X(\Omega_a))^3}, \ n \ge 2. \quad (2.18)$$

2.3 Domains with reentrant edges

It is well known that the solution $\mathbf{u} \in V_0(\Omega)$ of the BVP (2.5) exhibits singularities near reentrant edges of the domain Ω (cf. [12, 13, 14, 27]) and that the knowledge of the singularity functions can be used to improved the rate of convergence of numerical methods which approximate $\mathbf{u}$ (cf. [4, 12, 16, 17]). In this section we consider axisymmetric domains $\hat{\Omega}$ with reentrant edges and describe the asymptotic behavior of the Fourier coefficients $\mathbf{u}_n^e$ near the angular points of the meridian domain Ω_a that generate on rotation the edges on $\hat{\Omega}$. Since the regularity problem is a local one, we suppose for simplicity that the axisymmetric domain $\hat{\Omega}$ has only one reentrant edge, that is the meridian domain Ω_a has only one reentrant corner with vertex $E_a \in \Gamma_a$ and angle $\pi < \theta_0 < 2\pi$. In [23] the asymptotic behavior of the Fourier coefficients $\mathbf{u}_n^e$, solutions of (2.15), (2.16) near the corner E_a is described by means of singular vector-functions which involve a smooth cut-off function $\eta = \eta(r,z) = \tilde{\eta}(R)$ (R from (2.4)),

$$\tilde{\eta}(R) = \begin{cases} 1 & \text{if } 0 \le R \le R_0/3 \\ 0 & \text{if } R \ge 2R_0/3 \end{cases}, \qquad \tilde{\eta} \in \mathbf{C}^\infty[0,\infty). \tag{2.19}$$

For any real number α, let $\mathbf{\Psi}_\alpha(\theta)$ ($0 \le \theta \le \theta_0$ from (2.4)) denote the vector field, see e.g. [13, 14]

$$\begin{aligned} \mathbf{\Psi}_\alpha(\theta) &= (\Psi_{\alpha r}(\theta), \Psi_{\alpha z}(\theta))^T \quad \text{with} \\ \Psi_{\alpha r}(\theta) &= A\sin\alpha\theta + B\cos(\alpha-2)\theta - C\sin(\alpha-2)\theta - B\cos\alpha\theta, \\ \Psi_{\alpha z}(\theta) &= C\cos\alpha\theta - D\sin\alpha\theta - C\cos(\alpha-2)\theta - B\sin(\alpha-2)\theta. \end{aligned} \tag{2.20}$$

The constants A, B, C and D depend only on the number α and the angle θ_0. Then the regularity of the solutions $\mathbf{u}_n^e$ of (2.15), (2.16) can be described as follows.

Theorem 2.2 (cf. [23]). *For $\mathbf{f} \in (X^0_{1/2}(\Omega))^3$ let $\mathbf{u} \in V_0(\Omega)$ be the solution of the BVP (2.5). Let $\mathbf{u}_n^e$ and $\mathbf{f}_n^e$ ($n \in \mathbf{N}_0$) denote the Fourier coefficients of $\mathbf{u}$ and $\mathbf{f}$, respectively. Suppose that the corner $E_a \in \Gamma_a$ with interior angle θ_0 is reentrant, i.e. $\beta := \frac{\pi}{\theta_0}$ satisfies $\frac{1}{2} < \beta < 1$. Then there exist real constants $c_{n\alpha}^e$ and c_n^e such that the solutions $\mathbf{u}_n^e$ can be split in the form*

$$\begin{aligned} u_{rn}^e &= w_{rn}^e + s_{rn}^e, \quad s_{rn}^e = \eta \sum_{\alpha<1} c_{n\alpha}^e R^\alpha \Psi_{\alpha r}(\theta), \\ u_{zn}^e &= w_{zn}^e + s_{zn}^e, \quad s_{zn}^e = \eta \sum_{\alpha<1} c_{n\alpha}^e R^\alpha \Psi_{\alpha z}(\theta), \end{aligned} \tag{2.21}$$

$$u_{\varphi n}^e = w_{\varphi n}^e + s_{\varphi n}^e, \quad s_{\varphi n}^e = \eta c_n^e R^\beta \sin(\beta\theta), \tag{2.22}$$

$$\mathbf{w}_n^e \in (W^{2,2}_{1/2}(\Omega_a))^3, \quad \|\mathbf{w}_n^e\|_{(W^{2,2}_{1/2}(\Omega_a))^3} + \sum_{\alpha<1} |c_{n\alpha}^e| + |c_n^e| \le C\|\mathbf{f}_n^e\|_{(X(\Omega_a))^3}. \tag{2.23}$$

The functions $\Psi_{\alpha r}$ and $\Psi_{\alpha z}$ are taken from (2.20) and the sum over α is for those values of real $\alpha < 1$ solutions of

$$\sin^2\alpha\theta_0 = \frac{(\lambda+\mu)^2}{(\lambda+3\mu)^2}\alpha^2\sin^2\theta_0. \tag{2.24}$$

Remark 2.2. *(i) Equation (2.24) has no real roots in the strip $0 < Re.\alpha \le 1$ if $\theta_0 < \pi$ and has two distinct real roots in the same strip if $\theta_0 > \pi$ (cf. [13, 14]).*
(ii) The solutions $\mathbf{u}_n^e$ ($n \in \mathbf{N}_0$) have the property $\mathbf{u}_n^e \in (W^{2,2}_{1/2}(\Omega_a))^3$ if Ω_a is convex.
(iii) If n_e reentrant corners are present on Γ_a, then the singular parts in (2.21) – (2.23) are to be replaced by sums of singular parts corresponding to the n_e corners.

2.4 Domains with conical points

It has been shown that the solution $\mathbf{u} \in V_0(\Omega)$ of the BVP (2.5) entails singularities near conical points of the domain Ω and that the space of the singularity functions is finite dimensional (cf. [6, 18, 19]). This fact implies that the regularity of only a finite number of the Fourier coefficients $\mathbf{u}_n^e$ of $\mathbf{u}$ can be affected by the presence of a conical point on the boundary of Ω. For an efficient implementation of the Fourier-finite-element method for the numerical approximation of $\mathbf{u}$, it is important to describe explicitly the behavior of the Fourier coefficients near conical points of the boundary. In the paper [26] it is shown that for $\mathbf{f} \in (X_{1/2}^0(\Omega))^3$, only the Fourier coefficients $\mathbf{u}_0^e$ and $\mathbf{u}_1^e$ can exhibit singularities near conical points. In order to describe this behavior we consider a corner with vertex $C_a \in \bar{\Gamma}_a \cap \bar{\Gamma}_0$ and angle $0 < \theta_0 < \pi$, $\theta_0 \neq \pi/2$. For any real number α_n and angle $\theta \in (0, \theta_0)$, let the 3×3 matrix $\mathbf{M}_n(\alpha_n, \theta)$ be defined such that the columns are given by

$$\begin{pmatrix} (\alpha_n + 1)\sin^{-1}\theta P_{\alpha_n+1}^{-n} - (\alpha_n + 1 - n)\cot\theta P_{\alpha_n}^{-n} \\ n\sin^{-1}\theta P_{\alpha_n+1}^{n} \\ -(\alpha_n + 1 - n)P_{\alpha_n}^{-n} \end{pmatrix};$$

$$\begin{pmatrix} -2n\sin^{-1}\theta P_{\alpha_n+1}^{-n} \\ -2(\alpha_n + 1)\sin^{-1}\theta P_{\alpha_n+1}^{-n} + 2(\alpha_n + 1 - n)\cot\theta P_{\alpha_n}^{-n} \\ 0 \end{pmatrix};$$

$$\begin{pmatrix} ((\alpha_n + 1) - (2\alpha_n + 1)\sin^2\theta)\cot\theta P_{\alpha_n}^{-n} - (\alpha_n + n + 1)\cos\theta\cot\theta P_{\alpha_n+1}^{-n} \\ -n\cot\theta P_{\alpha_n}^{-n} \\ ((2\alpha_n + 1)\cos^2\theta - (3 - 4\nu))P_{\alpha_n}^{-n} - (\alpha_n + n + 1)\cos\theta P_{\alpha_n+1}^{-n} \end{pmatrix},$$

where $P_{\alpha_n}^{-n}(\theta)$ denote the Legendre functions of the first kind (cf. [1]). Suppose for simplicity that the domain Ω has only one conical point C_a and no reentrant edges. Then the form of the Fourier coefficients $\mathbf{u}_n^e$ can be described as follows:

Theorem 2.3 (cf. [26]). *Let $\mathbf{u} \in V_0(\Omega)$ be the solution of the three-dimensional BVP (2.5) with $\mathbf{f} \in (X_{1/2}^0(\Omega))^3$. Let $\mathbf{u}_n^e$ and $\mathbf{f}_n^e$ ($n \in \mathbf{N}_0$) denote the Fourier coefficients of $\mathbf{u}$ and $\mathbf{f}$, respectively. Suppose that the angle θ_0 at the vertex $C_a \in \bar{\Gamma}_0$ satisfies $\pi/2 < \theta_0 < \pi$. For each $n \in \mathbf{N}_0$, let α_{n_l} ($l = 1, 2, \cdots$) denote the positive real roots of the equation*

$$det\,\mathbf{M}_n(\alpha_n, \theta_0) = 0 \tag{2.25}$$

and let the function $\mathbf{U}_n(\alpha_{n_l}, \theta)$ be defined by

$$\mathbf{U}_n(\alpha_n, \theta) = \mathbf{M}_n(\alpha_n, \theta)\begin{pmatrix} a \\ b \\ c \end{pmatrix}. \tag{2.26}$$

where the vector $(a, b, c)^T$ is a non-trivial solution of the equation

$$\mathbf{M}_n(\alpha_n, \theta_0)\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \mathbf{0}. \tag{2.27}$$

(a) *For any* $\theta_0 \in (0, \pi)$ *the Fourier coefficients* $\mathbf{u}_n^e$, $n \geq 2$, *satisfy the relations*

$$\mathbf{u}_n^e \in (W^{2,2}_{1/2}(\Omega_a))^3, \quad \|\mathbf{u}_n^e\|_{(W^{2,2}_{1/2}(\Omega_a))^3} \leq M_0 \|\mathbf{f}_n^e\|_{(X(\Omega_a))^3}, \quad n = 2, 3, \cdots . \tag{2.28}$$

(b) *If* $\alpha_n > 1/2$, $n = 0, 1$, *then the coefficients* $\mathbf{u}_n^e$, $n = 0, 1$, *also satisfy relation* (2.28). *If* $\alpha_n \leq 1/2$, $n = 0, 1$, *then there exist real constants* γ_n^e $(n = 0, 1)$ *such that for the coefficients* $\mathbf{u}_0^e$ *and* $\mathbf{u}_1^e$ *the following splitting holds:*

$$\mathbf{u}_n^e = \mathbf{w}_n^e + \mathbf{s}_n^e, \quad \mathbf{s}_n^e = \eta(R)\gamma_n^e R^{\alpha_n} \mathbf{U}_n(\alpha_n, \theta), \quad \mathbf{w}_n^e \in (W^{2,2}_{1/2}(\Omega_a))^3, \tag{2.29}$$

$$\sum_{l=1}^{L_n} |\gamma_{n_l}^e| + \|\mathbf{w}_n^e\|_{(W^{2,2}_{1/2}(\Omega_a))^3} \leq M_1 \|\mathbf{f}_n^e\|_{(X(\Omega_a))^3} \quad \textit{for} \quad n = 0, 1. \tag{2.30}$$

3 The Fourier-finite-element method and error estimates

3.1 Graded refined meshes

The plane meridian domain $\bar{\Omega}_a$ is partitioned into a set of shape regular and admissible triangular elements $\mathcal{T}_h = \{T\}$, $h \in (0, h_0]$ $(h_0 < 1)$ in such a way that the usual assumptions are satisfied (cf. [5, 9, 20, 30]). Furthermore, we introduce near singular points E_a of the domain Ω_a which are associated with singularities of the Fourier coefficients $\mathbf{u}_n^e$ local mesh grading by means of a grading parameter $0 < \kappa \leq 1$, a grading function R_j and a step size h_j as follows:

$$R_j = 2/3 R_0 (jh)^{\frac{1}{\kappa}} \; (j = 1, \cdots, J), \quad h_1 := R_1, \quad h_j = R_j - R_{j-1} \; (j = 2, \cdots, J), \tag{3.1}$$

where $J := [h^{-1}]$ is the integer part of h^{-1} and R_0 is from (2.4). Thus the mesh grading is located within the neighborhood $\bar{G}_a$ of the vertex E_a. The grading parameter κ would be chosen in relation to the singular exponents, see Theorems 2.2 and 2.3. Using the relationships in (3.1) the following assertions can easily be verified.

Lemma 3.1. *For* h, h_j, R_j *and* $0 < \kappa \leq 1$ *the following relations hold:*

$$c_1 h R_j^{1-\kappa} \leq h_j \leq c_2 h R_j^{1-\kappa}, \quad c_3 R_j^{1-\kappa} \frac{1}{j} \leq h_j \leq c_4 R_j^{1-\kappa} \frac{1}{j}, \quad j = 1, \cdot, J, \tag{3.2}$$

$$h_{j-1} \leq h_j \leq c_5 h_{j-1}, \quad R_{j-1} \leq R_j \leq c_6 R_{j-1}. \tag{3.3}$$

Let R_T denote the distance of T from E_a, i.e. $R_T := \operatorname{dist}(T, E_a) := \inf_{P \in T} |E_a - P|$. Then near the vertex E_a of Ω_a the triangulation $\mathcal{T}_h$ is further refined such that $h_T := \operatorname{diam} T$ depends on R_T according to the following assumption.

Assumption 3.1. *The triangulation* $\mathcal{T}_h$ *is refined locally by means of the grading parameter* $0 < \kappa \leq 1$ *such that the following assumptions hold:*

$$\begin{aligned} &\rho_1 h^{1/\kappa} \leq h_T \leq \rho_1^{-1} h^{1/\kappa} \quad \textit{for} \quad T \in \mathcal{T}_h : R_T = 0, \\ &\rho_2 h R_T^{1-\kappa} \leq h_T \leq \rho_2^{-1} h R_T^{1-\kappa} \quad \textit{for} \quad T \in \mathcal{T}_h : 0 < R_T < R_J, \\ &\rho_3 h \leq h_T \leq \rho_3^{-1} h \quad \textit{for} \quad T \in \mathcal{T}_h : R_J \leq R_T, \end{aligned} \tag{3.4}$$

with some constants ρ_j, $0 < \rho_j \leq 1$ $(j = 1, 2, 3)$ *and* R_J *from* (3.1).

We note that if $\kappa = 1$, then there is no actual refinement and the mesh is shape regular. Owing to Assumption 3.1 and Lemma 3.1 the asymptotic behavior of the parameter h_T near the vertex E_a is determined by the relations

$$\begin{aligned} c_1 h_j \le h_T &\le c_1^{-1} h_j \quad \text{for} \quad T \in \mathcal{T}_h : R_{j-1} \le R_T < R_j \ (j = 1, \cdots, J), \\ c_2 h \le h_T &\le c_2^{-1} h \quad \text{for} \quad T \in \mathcal{T}_h : R_J \le R_T, \quad 0 < c_i \le 1 \ (i = 1, 2). \end{aligned} \tag{3.5}$$

For error analysis, we distinguish the following subsets of the triangulation $\mathcal{T}_h$ contained in G_a.

$$\begin{aligned} D_{ih} &:= \{T \in \mathcal{T}_h : R_{i-1} \le R_T < R_i\}, \quad i = 2, \cdots, J, \\ D_{1h} &:= \{T \in \mathcal{T}_h : 0 \le R_T < R_1\}, \quad D_h := \cup_{i=1}^{J} D_{ih}. \end{aligned} \tag{3.6}$$

Clearly $D_h \subset \bar{G}_a$ and the sets D_{ih} $(i = 1, \cdots J)$ are pairwise disjoint. Let n_j denote the total number of triangles T with the property $T \in D_{jh}$, $j \in \{1, 2, \cdots, J\}$, then the relation

$$n_j \le Cj, \quad j = 1, \cdots, J \tag{3.7}$$

can be verified (cf. [17]), where the constant C does not dependent on h.

3.2 The Fourier-Galerkin approximation

Let $\mathbf{R}_n^s$ and $\mathbf{R}_n^a$ denote the 3×3-diagonal matrices given by

$$\mathbf{R}_n^s := \operatorname{diag}[\cos n\varphi, -\sin n\varphi, \cos n\varphi]; \ \mathbf{R}_n^a := \operatorname{diag}[\sin n\varphi, \cos n\varphi, \sin n\varphi]. \tag{3.8}$$

Let $\mathbf{u} \in V_0(\Omega)$ be the solution of the three-dimensional BVP (2.5). Then the function $\mathbf{u}_N$ defined by the truncated Fourier series

$$\mathbf{u}_N = \sum_{n=0}^{N} (\mathbf{R}_n^s \mathbf{u}_n^s + \mathbf{R}_n^a \mathbf{u}_n^a) \quad \text{for} \quad N > 0 \tag{3.9}$$

is called the Fourier approximation of $\mathbf{u}$.

The undetermined $2N + 2$ Fourier coefficients $\mathbf{u}_n^s$ and $\mathbf{u}_n^a$ from (3.9) are solutions of the two-dimensional BVPs (2.15), (2.16) for $n = 0, 1, \cdots, N$ and will be approximated by means of the finite-element method. We introduce the finite-element subspaces by

$$\begin{aligned} R_h &:= \{v_h(r,z) : v_h \in C(\bar{\Omega}_a) : v_h|_T \in P_1(T) \text{ for any } T \in \mathcal{T}_h, v_h = 0 \text{ on } \bar{\Gamma}_a\}, \\ V_h^a &:= \{\mathbf{v}_h = (u_h(r,z), v_h(r,z), w_h(r,z)) : u_h, v_h, w_h \in R_h, u_h = v_h = 0 \text{ on } \Gamma_0\}, \quad (3.10) \\ U_h^a &:= \{\mathbf{v}_h = (u_h(r,z), v_h(r,z), w_h(r,z)) : u_h, v_h, w_h \in R_h, u_h = v_h, \ w_h = 0 \text{ on } \Gamma_0\}, \\ W_h^a &:= \{\mathbf{v}_h = (u_h(r,z), v_h(r,z), w_h(r,z)) : u_h, v_h, w_h \in R_h, u_h = v_h = w_h = 0 \text{ on } \Gamma_0\}. \end{aligned}$$

The Fourier-finite-element subspace V_{hN} is defined by

$$V_{hN} := \{\mathbf{v}_{hN} : \mathbf{v}_{hN} = \sum_{n=0}^{N} (\mathbf{R}_n^s \mathbf{v}_{nh}^s + \mathbf{R}_n^a \mathbf{v}_{nh}^a), \ \mathbf{v}_{0h}^e \in V_h^a, \ \mathbf{v}_{1h}^e \in U_h^a, \ \mathbf{v}_{nh}^e \in W_h^a, \ 2 \le n \le N\},$$

where $\mathbf{R}_n^s$ and $\mathbf{R}_n^a$ are from (3.8). The symbol $P_1(T)$ in (3.10) stands for the space of all polynomials of degree ≤ 1 on T. It is easy to see that the relations $V_h^a \subset V_0^a(\Omega_a)$, $U_h^a \subset W_0^a(\Omega_a)$, $W_h^a \subset W_0^a(\Omega_a)$ and $V_{hN} \subset V_0(\Omega)$ hold.

The Fourier-Galerkin approximation of the solution $\mathbf{u} \in V_0(\Omega)$ of the 3D BVP (2.5) is obtained as follows: Find $\mathbf{u}_{hN} = (u_{rhN}(r,\varphi,z), u_{\varphi hN}(r,\varphi,z), u_{zhN}(r,\varphi,z)) \in V_{hN}$ such that

$$b(\mathbf{u}_{hN}, \mathbf{v}) = f(\mathbf{v}) \quad \text{for all} \quad \mathbf{v} \in V_{hN}, \tag{3.11}$$

where the functionals $b(\cdot,\cdot)$ and $f(\cdot)$ are from (2.5). The Lax/Milgram and Cea's lemmas infer the existence of a unique solution $\mathbf{u}_{hn} \in V_{hN}$ that satisfies the a priori estimate

$$\|\mathbf{u} - \mathbf{u}_{hN}\|_{V_0(\Omega)} \leq \inf_{\mathbf{w}_{hN} \in V_{hN}} \|\mathbf{u} - \mathbf{w}_{hN}\|_{V(\Omega)}. \tag{3.12}$$

The solution $\mathbf{u}_{hN}$ of (3.11) admits the decomposition

$$\mathbf{u}_{hN} = \sum_{n=0}^{N} (R_n^s \mathbf{u}_{nh}^s + R_n^a \mathbf{u}_{nh}^a) \tag{3.13}$$

with the coefficients $\mathbf{u}_{nh}^s$ and $\mathbf{u}_{nh}^a$ being the unique solutions of the variational (Galerkin) equations:

$$\begin{aligned} n = 0: \quad & \text{find} \quad \mathbf{u}_{0h}^e \in V_h^a : b_0(\mathbf{u}_{0h}^e, \mathbf{w}) &=& f_0^e(\mathbf{w}) \quad \text{for} \quad \mathbf{w} \in V_h^a, \\ n = 1: \quad & \text{find} \quad \mathbf{u}_{1h}^e \in U_h^a : b_1(\mathbf{u}_{1h}^e, \mathbf{w}) &=& f_1^e(\mathbf{w}) \quad \text{for} \quad \mathbf{w} \in U_h^a, \\ 2 \leq n \leq N: \quad & \text{find} \quad \mathbf{u}_{nh}^e \in W_h^a : b_n(\mathbf{u}_{nh}^e, \mathbf{w}) &=& f_n^e(\mathbf{w}) \quad \text{for} \quad \mathbf{w} \in W_h^a, \end{aligned} \tag{3.14}$$

where the bilinear and linear forms $b_n(\cdot,\cdot)$ and $f_n^e(\cdot)$, respectively, are from (2.14).

For proving convergence $\mathbf{u}_{hN} \to \mathbf{u}$ ($h \to 0$, $N \to \infty$) of the Fourier-finite-element approximation we need to define an appropriate projection operator of the space $V_0(\Omega)$ into the Fourier-finite-element subspace V_{hN}.

3.3 Recursion formula for the element stiffness matrix

The computational advantage of this method over the full 3D problem lies in the fact that the stiffness matrices for the approximation of the Fourier coefficients $\mathbf{u}_n^e$ ($0 \leq n \leq N$) can be computed recursively. Let $T \in \mathcal{T}_h$, then for any n, $0 \leq n \leq N$, the element stiffness matrix $\mathbf{A}_T^{(n)}$ corresponding to the Galerkin equation (cf. (3.14))

$$b_n(\mathbf{u}_{nh}^e, \mathbf{w}) = f_n^e(\mathbf{w})$$

can be written in the form

$$\mathbf{A}_T^{(n)} = \int_T (\mathbf{D}_n \boldsymbol{\omega})^T \mathbf{E} (\mathbf{D}_n \boldsymbol{\omega}) r dr dz,$$

where the differential operator $\mathbf{D}_n$, the elasticity matrix $\mathbf{E}$ and the matrix of shape functions $\boldsymbol{\omega}$ are given by (we will use block matrices with the dimensions of the various blocks indicated below it)

$$\mathbf{D}_n = \begin{bmatrix} \frac{\partial}{\partial r} & 0 & 0 \\ \frac{1}{r} & -\frac{n}{r} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial r} \\ \frac{n}{r} & \frac{\partial}{\partial r} - \frac{1}{r} & 0 \\ 0 & \frac{\partial}{\partial z} & \frac{n}{r} \end{bmatrix} = \begin{bmatrix} \underset{[4\times1]}{\mathbf{D}_{11}} & \underset{[4\times1]}{n\mathbf{D}_{12}} & \underset{[4\times1]}{\mathbf{D}_{13}} \\ \underset{[2\times1]}{n\mathbf{D}_{21}} & \underset{[2\times1]}{\mathbf{D}_{22}} & \underset{[2\times1]}{n\mathbf{D}_{23}} \end{bmatrix},$$

$$\mathbf{E} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}-\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}-\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}-\nu \end{bmatrix} = \begin{bmatrix} \underset{[4\times4]}{\mathbf{E}_{11}} & \underset{[4\times2]}{\mathbf{0}} \\ \underset{[2\times4]}{\mathbf{0}} & \underset{[2\times2]}{\mathbf{E}_{22}} \end{bmatrix},$$

$$\boldsymbol{\omega} = \begin{bmatrix} \omega_1 & \omega_2 & \omega_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega_1 & \omega_2 & \omega_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \omega_1 & \omega_2 & \omega_3 \end{bmatrix} = \begin{bmatrix} \underset{[3\times1]}{\tilde{\boldsymbol{\omega}}^T} & \underset{[3\times1]}{\mathbf{0}^T} & \underset{[3\times1]}{\mathbf{0}^T} \\ \underset{[3\times1]}{\mathbf{0}^T} & \underset{[3\times1]}{\tilde{\boldsymbol{\omega}}^T} & \underset{[3\times1]}{\mathbf{0}^T} \\ \underset{[3\times1]}{\mathbf{0}^T} & \underset{[3\times1]}{\mathbf{0}^T} & \underset{[3\times1]}{\tilde{\boldsymbol{\omega}}^T} \end{bmatrix}$$

with $\tilde{\boldsymbol{\omega}}^T = (\omega_1, \omega_2, \omega_3)^T$. Now the product $\mathbf{D}_n\boldsymbol{\omega}$ can be represented in the form

$$\boldsymbol{\alpha}_n = \mathbf{D}_n\boldsymbol{\omega} = \begin{bmatrix} \underset{[4\times3]}{\boldsymbol{\alpha}_{11}} & \underset{[4\times3]}{n\boldsymbol{\alpha}_{12}} & \underset{[4\times3]}{\boldsymbol{\alpha}_{13}} \\ \underset{[2\times3]}{n\boldsymbol{\alpha}_{21}} & \underset{[2\times3]}{\boldsymbol{\alpha}_{22}} & \underset{[2\times3]}{n\boldsymbol{\alpha}_{23}} \end{bmatrix},$$

where $\boldsymbol{\alpha}_{ij} = \mathbf{D}_{ij}\tilde{\boldsymbol{\omega}}^T$. We observe that the following relationship holds.

$$\begin{aligned} \mathbf{A}_T^{(n)} &= \int_T \boldsymbol{\alpha}^T \mathbf{E} \boldsymbol{\alpha} r dr dz \\ &= \begin{bmatrix} \mathbf{A}_1 + n^2\mathbf{A}_2 & n\mathbf{A}_3 & \mathbf{A}_4 + n^2\mathbf{A}_5 \\ n\mathbf{A}_3^T & \mathbf{A}_6 + n^2\mathbf{A}_7 & n\mathbf{A}_8 \\ \mathbf{A}_4^T + n^2\mathbf{A}_5^T & n\mathbf{A}_8^T & \mathbf{A}_9 + n^2\mathbf{A}_{10} \end{bmatrix} = \mathbf{S}_0 + n\mathbf{S}_1 + n^2\mathbf{S}_2, \end{aligned}$$

where the 3×3-matrices $\mathbf{A}_i$ $i = 1, 2, \cdots, 10$, are defined as follows:

$$\begin{aligned} \mathbf{A}_1 &= \int_T \boldsymbol{\alpha}_{11}^T \mathbf{E}_{11} \boldsymbol{\alpha}_{11} r dr dz, \qquad \mathbf{A}_2 = \int_T \boldsymbol{\alpha}_{21}^T \mathbf{E}_{22} \boldsymbol{\alpha}_{21} r dr dz \\ \mathbf{A}_3 &= \int_T (\boldsymbol{\alpha}_{11}^T \mathbf{E}_{11} \boldsymbol{\alpha}_{12} r dr dz + \boldsymbol{\alpha}_{21}^T \mathbf{E}_{22} \boldsymbol{\alpha}_{22}) r dr dz, \qquad \mathbf{A}_4 = \int_T \boldsymbol{\alpha}_{11}^T \mathbf{E}_{11} \boldsymbol{\alpha}_{13} r dr dz \\ \mathbf{A}_5 &= \int_T \boldsymbol{\alpha}_{21}^T \mathbf{E}_{22} \boldsymbol{\alpha}_{23} r dr dz, \qquad \mathbf{A}_6 = \int_T \boldsymbol{\alpha}_{22}^T \mathbf{E}_{22} \boldsymbol{\alpha}_{22} r dr dz \\ \mathbf{A}_7 &= \int_T \boldsymbol{\alpha}_{12}^T \mathbf{E}_{11} \boldsymbol{\alpha}_{12} r dr dz, \qquad \mathbf{A}_8 = \int_T (\boldsymbol{\alpha}_{12}^T \mathbf{E}_{11} \boldsymbol{\alpha}_{13} r dr dz + \boldsymbol{\alpha}_{22}^T \mathbf{E}_{22} \boldsymbol{\alpha}_{23}) r dr dz \\ \mathbf{A}_9 &= \int_T \boldsymbol{\alpha}_{13}^T \mathbf{E}_{11} \boldsymbol{\alpha}_{13} r dr dz, \qquad \mathbf{A}_{10} = \int_T \boldsymbol{\alpha}_{23}^T \mathbf{E}_{22} \boldsymbol{\alpha}_{23} r dr dz \end{aligned}$$

Thus, the element stiffness matrix $\mathbf{A}_T^{(n)}$ for any arbitrary Fourier coefficient $\mathbf{u}_n^e$ is determined from the fixed matrices $\mathbf{A}_i$, $i = 1, 2, \cdots, 10$, which need to computed only once.

3.4 Estimates of the interpolation error

We define a projection $\mathbf{r}_{hN} : V_0(\Omega) \to V_{hN}$ such that for $\mathbf{u} \in V_0(\Omega)$

$$\mathbf{r}_{hN}\mathbf{u} = \sum_{n=0}^{N}(R_n^s \mathbf{r}_h \mathbf{u}_n^s + R_n^a \mathbf{r}_h \mathbf{u}_n^a) \quad \text{with} \quad \mathbf{r}_h \mathbf{u}_n^e = \Pi_h \mathbf{s}_n^e + \Pi_h \mathbf{w}_n^e, \quad 0 \le n \le N. \tag{3.15}$$

In (3.15) the symbol Π_h denotes the usual Lagrange interpolation operator (cf. [5, 9, 20, 21, 30]) and $\mathbf{u}_n^e = \mathbf{s}_n^e + \mathbf{w}_n^e$ corresponds to the splitting of the Fourier coefficients in singular and regular parts according to Theorem 2.2 and Theorem 2.3.

The Lagrange interpolation operator $\Pi_h : C(\bar{\Omega}_a) \to R_h$ (R_h from (3.10) and $h \in (0, h_0]$, $h_0 < 1$) is defined (see also [9, 20]) such that for each $\psi \in C(\bar{\Omega}_a)$, $\Pi_h\psi \in R_h$ and $\Pi_h\psi(Q) = \psi(Q)$ for each node Q of the triangulation $\mathcal{T}_h$. The local interpolation operator $\Pi_T : C(T) \to P_1(T)$ is given by $\Pi_T\psi = \Pi_h\psi|_T$.

Lemma 3.2. *Let the triangulation $\mathcal{T}_h$ satisfy Assumption 3.1 with $0 < \kappa \le 1$. Then there exist constants $C > 0$ independent of h and v such that for any $v \in R_h$ the following estimates hold:*

$$\|v - \Pi_h v\|_{W^{1,2}_{1/2}(\Omega_a)} \le Ch\|v\|_{W^{2,2}_{1/2}(\Omega_a)} \quad \text{for} \quad v \in W^{1,2}_{1/2}(\Omega_a) \cap W^{2,2}_{1/2}(\Omega_a), \tag{3.16}$$

$$\|r^{-1}(v - \Pi_h v)\|_{X(\Omega_a)} \le Ch\|v\|_{W^{2,2}_{1/2}(\Omega_a)} \quad \text{for} \quad v \in L_{2,-1/2}(\Omega_a) \cap W^{2,2}_{1/2}(\Omega_a). \tag{3.17}$$

Proof: The proof of this lemma can be taken from *Mercier/Raugel* [21, *Proposition*6.1]. ■

Let us consider the interpolation error $\mathbf{s}_n^e - \Pi_h \mathbf{s}_n^e$ of the singular vector-functions $\mathbf{s}_n^e = (s_{rn}^e(r,z), s_{\varphi n}^e(r,z), s_{zn}^e(r,z))^T$ (cf. Theorem 2.2 and Theorem 2.3). Obviously, these functions are continuous and their interpolates are well defined. Since $\mathbf{s}_n^e(r,z)$ vanishes for $R \ge 2/3R_0$, it suffices to study the error $\mathbf{s}_n^e - \Pi_h \mathbf{s}_n^e$ within $\bar{G}_a$.

Lemma 3.3. *Let the triangulation $\mathcal{T}_h$ ($0 < h \le h_0 < 1$) satisfy Assumption 3.1 with $0 < \kappa \le 1$. Then there exist constants $C > 0$ independent of h and n such that*

$$\|r^{-1}(\mathbf{s}_n^e - \Pi_h \mathbf{s}_n^e)\|^2_{(X(G_a))^3} \le C\|\mathbf{f}_n^e\|^2_{(X(\Omega_a))^3} \begin{cases} h^{2\beta_0/\kappa} & \text{if } \beta_0 < \kappa \le 1 \\ h^2 & \text{if } 0 < \kappa < \beta_0 \end{cases} \tag{3.18}$$

$$\|\mathbf{s}_n^e - \Pi_h \mathbf{s}_n^e\|^2_{(W^{1,2}_{1/2}(G_a))^3} \le C\|\mathbf{f}_n^e\|^2_{(X(\Omega_a))^3} \begin{cases} h^{2\beta_0/\kappa} & \text{if } \beta_0 < \kappa \le 1 \\ h^2 & \text{if } 0 < \kappa < \beta_0 \end{cases} \tag{3.19}$$

where β_0 is the minimum value of the singular exponents (cf. Theorem 2.2 and Theorem 2.3) and $0 < \kappa \le 1$ is the grading parameter (cf. (3.1))

Proof. We have

$$\|\mathbf{s}_n^e - \Pi_h \mathbf{s}_n^e\|^2_{(W^{1,2}_{1/2}(G_a))^3} = \|\mathbf{s}_n^e - \Pi_h \mathbf{s}_n^e\|^2_{(X(G_a))^3} + |\mathbf{s}_n^e - \Pi_h \mathbf{s}_n^e|^2_{(W^{1,2}_{1/2}(G_a))^3}, \tag{3.20}$$

$$\begin{aligned} |\mathbf{s}_n^e - \Pi_h \mathbf{s}_n^e|^2_{(W^{1,2}_{1/2}(G_a))^3} &= |s_{rn}^e - \Pi_h s_{rn}^e|^2_{W^{1,2}_{1/2}(G_a)} + |s_{\varphi n}^e - \Pi_h s_{\varphi n}^e|^2_{W^{1,2}_{1/2}(G_a)} \\ &+ |s_{zn}^e - \Pi_h s_{zn}^e|^2_{W^{1,2}_{1/2}(G_a)}. \end{aligned} \tag{3.21}$$

We consider the error $|s^e_{rn} - \Pi_h s^e_{rn}|^2_{W^{l,2}_{1/2}(G_a)}$ and suppose that G_a is a neighborhood of a corner with vertex $E_a \in \Gamma_a$ with $\mathrm{dist}(E_a, \Gamma_0) > 0$, that is the singular function s^e_n has the form

$$s^e_n(r,z) = \eta \sum_{\alpha<1} c^e_{\alpha n} R^\alpha \Psi_{\alpha r}(\theta).$$

According to (3.6) the global error $|s^e_{rn} - \Pi_h s^e_{rn}|^2_{W^{1,2}_{1/2}(G_a)}$ can be represented as follows:

$$|s^e_{rn} - \Pi_h s^e_{rn}|^2_{W^{1,2}_{1/2}(G_a)} = \sum_{T\in D_{1h}} |s^e_{rn} - \Pi_T s^e_{rn}|^2_{W^{1,2}_{1/2}(T)} + \sum_{i=2}^{J} \sum_{T\in D_{ih}} |s^e_{rn} - \Pi_T s^e_{rn}|^2_{W^{1,2}_{1/2}(T)}. \quad (3.22)$$

In local polar coordinates R, θ (cf. (2.4)) the semi-norms $|v|_{W^{l,2}_{1/2}(T)}$ $(l = 1, 2)$ are given by

$$|v|^2_{W^{1,2}_{1/2}(T)} = \int_T \left\{ \left|\frac{\partial v}{\partial R}\right|^2 + \frac{1}{R^2}\left|\frac{\partial v}{\partial \theta}\right|^2 \right\} R^2 \sin\theta dR d\theta \quad (3.23)$$

$$|v|^2_{W^{2,2}_{1/2}(T)} = \int_T \left\{ \left|\frac{\partial^2 v}{\partial R^2}\right|^2 + 2\left|\frac{1}{R}\frac{\partial^2 v}{\partial R\partial\theta} - \frac{1}{R^2}\frac{\partial v}{\partial\theta}\right|^2 + \left|\frac{1}{R^2}\frac{\partial^2 v}{\partial\theta^2} + \frac{1}{R}\frac{\partial v}{\partial R}\right|^2 \right\} R^2 \sin\theta dR d\theta.$$

We consider from (3.22) the set D_{1h} which contains also triangles which have the point E_a as a vertex, and thus $s^e_{rn} \notin W^{2,2}_{1/2}(T)$. For $T \in D_{1h}$ we employ the estimate

$$|s^e_{rn} - \Pi_h s^e_{rn}|^2_{W^{1,2}_{1/2}(T)} \le |s^e_{rn}|^2_{W^{1,2}_{1/2}(T)} + |\Pi_h s^e_{rn}|^2_{W^{1,2}_{1/2}(T)}. \quad (3.24)$$

Taking account of (3.23) and the explicit representation of s^e_{rn} and $\Pi_T s^e_{rn}$ the norm on the right-hand side of (3.24) can be estimated by (see [23] for more details)

$$|s^e_{rn}|^2_{W^{1,2}_{1/2}(T)} + |\Pi_T s^e_{rn}|^2_{W^{1,2}_{1/2}(T)} \le C \sum_{\alpha<1} |c^e_{\alpha n}|^2 h^{\frac{2\alpha+1}{\kappa}}. \quad (3.25)$$

For triangles $T \in D_h \setminus D_{1h}$ which do not have E_a as a vertex, the semi-norm $|s^e_{rn}|_{W^{2,2}_{1/2}(T)}$ is bounded and the classical local interpolation error estimate holds, i.e.

$$|s^e_{rn} - \Pi_T s^e_{rn}|^2_{W^{1,2}_{1/2}(T)} \le C h_T^2 |s^e_{rn}|^2_{W^{2,2}_{1/2}(T)} \quad \text{for} \quad s^e_{rn} \in W^{2,2}_{1/2}(T). \quad (3.26)$$

The norm $|s^e_{rn}|^2_{W^{2,2}_{1/2}(T)}$ with $T \in D_{jh}$, $j = 2, \cdots, J$, can be estimated by

$$\left|\eta \sum_{\alpha<1} c^e_{\alpha n} R^\alpha \Psi_{\alpha r}(\theta)\right|^2_{W^{2,2}_{1/2}(T)} \le C \sum_{\alpha<1} |c^e_{\alpha n}|^2 h_T^2 \Big(\inf_{(r,z)\in T} R\Big)^{2\alpha-3} \le C \sum_{\alpha<1} |c^e_{\alpha n}|^2 h_T^2 R_{j-1}^{2\alpha-3}. \quad (3.27)$$

Now taking into account relations (3.25), (3.26), (3.27) and (3.1) – (3.7) we get by summing up all triangles $T \in D_{jh}$ $(j = 1, \cdots .J)$ the estimates

$$\begin{aligned} |s^e_{rn} - \Pi_T s^e_{rn}|^2_{W^{1,2}_{1/2}(G_a)} &\le C \sum_{\alpha<1} |c^e_{\alpha n}|^2 h^{\frac{2\alpha+1}{\kappa}} \Big(1 + \sum_{j=2}^{J} j^{\frac{4}{\kappa}-3} (j-1)^{\frac{2\alpha-3}{\kappa}}\Big) \\ &\le C \sum_{\alpha<1} |c^e_{\alpha n}|^2 \begin{cases} h^{2\alpha/\kappa} & \text{for } \alpha_0 < \kappa \le 1 \\ h^2 & \text{for } 0 < \kappa < \alpha \end{cases}. \end{aligned} \quad (3.28)$$

For getting the inequality (3.28) we made used of the estimates $J \leq Ch^{-1}$ and

$$\sum_{j=1}^{J} j^{\frac{2\alpha+1}{\kappa}-3} \leq C \begin{cases} J^{\frac{1}{\kappa}} & \text{for} \quad \alpha < \kappa \leq 1, \\ J^{\frac{2\alpha+1}{\kappa}-2} & \text{for} \quad 0 < \kappa < \alpha \end{cases}.$$

Similar estimates as above can be established for the other terms in the right-hand side of relations (3.20) and (3.21). Using relation (2.23) we get finally the assertion (3.19). Relation (3.18) can be proved by analogy. ■

3.5 Error estimates in $W_2^1(\hat{\Omega})$

We investigate subsequently estimates of the error $\mathbf{u} - \mathbf{u}_{hN}$ generated by replacing the exact solution $\mathbf{u}$ of the three-dimensional BVP (2.5) by its Fourier-finite-element approximation $\mathbf{u}_{hN}$ given by (3.11). Let us consider first the error $\mathbf{u} - \mathbf{u}_N$ due to the Fourier approximation.

Theorem 3.1. *Let $\mathbf{u} \in V_0(\Omega)$ be the solution of the BVP (2.5) with $\mathbf{f} \in (X^0_{1/2}(\Omega))^3$ and let $\mathbf{u}_N$ be its Fourier approximation defined by (3.9). Then there exists a constant $C > 0$ independent of $N > 0$, $\mathbf{u}$ and $\mathbf{f}$ such that*

$$\|\mathbf{u} - \mathbf{u}_N\|_{V_0(\Omega)} \leq CN^{-1}\|\mathbf{f}\|_{(X^0_{1/2}(\Omega))^3}. \tag{3.29}$$

Proof: It follows immediately from the completeness relation (2.10) and the a priori estimate (2.18) the relations

$$\begin{aligned} \|\mathbf{u} - \mathbf{u}_N\|^2_{V_0(\Omega)} &\leq C \sum_{n=N+1}^{\infty} \sum_{e\in\{s,a\}} \Big\{\|\mathbf{u}^e_n\|^2_{(W^{1,2}_{1/2}(\Omega_a))^3} + n^2\|\frac{1}{r}\mathbf{u}^e_n\|^2_{(X(\Omega_a))^3}\Big\} \\ &\leq CN^{-2} \sum_{n=N+1}^{\infty} \sum_{e\in\{s,a\}} n^2\Big\{\|\mathbf{u}^e_n\|^2_{(W^{1,2}_{1/2}(\Omega_a))^3} + n^2\|\frac{1}{r}\mathbf{u}^e_n\|^2_{(X(\Omega_a))^3}\Big\} \leq CN^{-2}\|\mathbf{f}\|_{(X^0_{1/2}(\Omega))^3}. \end{aligned}$$

Theorem 3.2. *Let $\mathbf{u} \in V_0(\Omega)$ be the solution of the BVP (2.5) with $\mathbf{f} \in (X^0_{1/2}(\Omega))^3$ and let $\mathbf{u}_{hN} \in V_{hN}$ be its Fourier-finite-element approximation given by (3.11). Suppose that $\partial\mathbf{f}/\partial\varphi \in (X^0_{1/2}(\Omega))^3$ and that the triangulation $\mathcal{T}_h$ satisfies Assumption 3.1. Then there exists a constant $C > 0$ independent of h, $N > 0$ and $\mathbf{f}$ such that*

$$\|\mathbf{u} - \mathbf{u}_{hN}\|_{V_0(\Omega)} \leq C(N^{-1} + \beta(h,\kappa))\Big(\|\mathbf{f}\|_{(X^0_{1/2}(\Omega))^3} + \Big\|\frac{\partial\mathbf{f}}{\partial\varphi}\Big\|_{(X^0_{1/2}(\Omega))^3}\Big) \tag{3.30}$$

with

$$\beta(h,\kappa) = \begin{cases} h^{\beta_0/\kappa} & \text{for} \quad \beta_0 < \kappa \leq 1 \\ h & \text{for} \quad 0 < \kappa < \beta_0 \end{cases}.$$

Proof: The a priori estimate (3.12) and the triangle inequality give

$$\|\mathbf{u} - \mathbf{u}_{hN}\|_{V_0(\Omega)} \leq \|\mathbf{u} - \mathbf{u}_N\|_{V_0(\Omega)} + \|\mathbf{u}_N - \mathbf{r}_{hN}\mathbf{u}\|_{V_0(\Omega)}. \tag{3.31}$$

Using the completeness relation (2.10), Lemma 3.2, Lemma 3.3 and Remark 2.1 we get

$$
\begin{aligned}
\|\mathbf{u}_N - \mathbf{r}_{hN}\mathbf{u}\|^2_{V_0(\Omega)} \le C\Big(&\Big\|\frac{1}{r}(u^s_{r0} - \Pi_h u^s_{r0})\Big\|^2_{X(\Omega_a)} + \Big\|\frac{1}{r}(u^a_{\varphi 0} - \Pi_h u^a_{\varphi 0})\Big\|^2_{X(\Omega_a)} \\
+ &\sum_{n=0}^{N}\sum_{e\in\{s,a\}}\Big\{\|\mathbf{u}^e_n - \Pi_h \mathbf{u}^e_n\|^2_{(W^{1,2}_{1/2}(\Omega_a))^3} + n^2\Big\|\frac{1}{r}(\mathbf{u}^e_n - \Pi_h \mathbf{u}^e_n)\Big\|^2_{(X(\Omega_a))^3}\Big\}\Big) \\
\le\ & C\Big(\Big\|\frac{1}{r}(w^s_{r0} - \Pi_h w^s_{r0})\Big\|^2_{X(\Omega_a)} + \Big\|\frac{1}{r}(w^a_{\varphi 0} - \Pi_h w^a_{\varphi 0})\Big\|^2_{X(\Omega_a)} \\
+ &\sum_{n=0}^{N}\sum_{e\in\{s,a\}}\Big\{\|\mathbf{w}^e_n - \Pi_h \mathbf{w}^e_n\|^2_{(W^{1,2}_{1/2}(\Omega_a))^3} + n^2\Big\|\frac{1}{r}(\mathbf{w}^e_n - \Pi_h \mathbf{w}^e_n)\Big\|^2_{(X(\Omega_a))^3}\Big\}\Big) \\
+ & C\Big(\Big\|\frac{1}{r}(s^s_{r0} - \Pi_h s^s_{r0})\Big\|^2_{X(\Omega_a)} + \Big\|\frac{1}{r}(s^a_{\varphi 0} - \Pi_h s^a_{\varphi 0})\Big\|^2_{X(\Omega_a)} \\
+ &\sum_{n=0}^{N}\sum_{e\in\{s,a\}}\Big\{\|\mathbf{s}^e_n - \Pi_h \mathbf{s}^e_n\|^2_{(W^{1,2}_{1/2}(\Omega_a))^3} + n^2\Big\|\frac{1}{r}(\mathbf{s}^e_n - \Pi_h \mathbf{s}^e_n)\Big\|^2_{(X(\Omega_a))^3}\Big\}\Big) \\
\le\ & C\beta^2(h,\kappa)\sum_{n=0}^{\infty}\sum_{e\in\{s,a\}}(1+n^2)\|f^e_n\|^2_{(X(\Omega_a))^3} \\
\le\ & C\beta^2(h,\kappa)\Big(\|\mathbf{f}\|^2_{(X^0_{1/2}(\Omega))^3} + \Big\|\frac{\partial \mathbf{f}}{\partial\varphi}\Big\|^2_{(X^0_{1/2}(\Omega))^3}\Big).
\end{aligned}
\tag{3.32}
$$

Theorem 3.2 follows finally from Theorem 3.1 and relations (3.31), (3.32). ■

Remark 3.1. *The additional smoothness assumption on the function* $\mathbf{f}$ *in Theorem 3.2, i.e. the condition* $\partial\mathbf{f}/\partial\varphi \in (X^0_{1/2}(\Omega))^3$, *is only needed to uncouple the discretization parameters* h *and* N *as in* (3.30). *One could also employ the method introduce by Mercier/Raugel [21], in which a mixed projection combining the usual Lagrange interpolation operator* Π_h *and Clement's [10]* L_2*-projection operator is used for the approximation of the Fourier Coefficients* $\mathbf{u}^e_n$. *In this case the additional smoothness requirement on* $\mathbf{f}$ *is not necessary to achieve the same order of convergence.*

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