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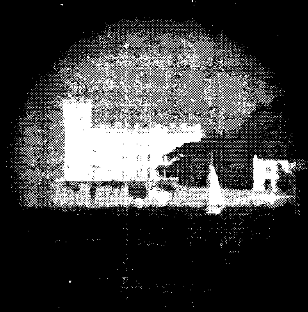

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HIGHER CLASS GROUPS OF EICHLER ORDERS

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United Nations Educational Scientific and Cultural Organization
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THE ABDUS SALAM INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

HIGHER CLASS GROUPS OF EICHLER ORDERS

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Abstract

In this paper, we prove that if A is a quaternion algebra and Λ an Eichler order in A, then the only p-torsion possible in even dimensional higher class groups $Cl_{2n}(\Lambda)$ $(n \geq 1)$ are for those rational primes p which lie under prime ideals of $\mathcal{O}_F$ at which Λ are not maximal.

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1. Introduction

Let F be a number field and $\mathcal{O}_F$ the ring of integers in F. Let A be a semi-simple algebra over F and Λ an $\mathcal{O}_F$-order in A. The higher class group of Λ is defined as

$$Cl_n(\Lambda) = ker(SK_n(\Lambda) \longrightarrow \bigoplus_{\wp} SK_n(\Lambda_\wp)),$$

where $\wp$ runs through all maximal ideals of $\mathcal{O}_F$ and

$$SK_n(\Lambda) := ker(K_n(\Lambda) \longrightarrow K_n(A)),$$

for all integers $n \geq 0$. In [4], it is proved that the only p-torsion possible in $Cl_{2n+1}(\Lambda)$ is for those rational primes p which lie under a prime of $\mathcal{O}_F$ at which Λ is not maximal. In this paper, we prove that if A is a quaternion algebra and Λ an Eichler order, then the only p-torsion possible in even dimensional higher class groups $Cl_{2n}(\Lambda)$ $(n \geq 1)$ are for those rational primes p which lie under prime ideals of $\mathcal{O}_F$ at which Λ are not maximal.

In this paper, we use the same notations as in [4].

2. Main Results

Let F be a number field and $\mathcal{O}_F$ the ring of integers in F. Let A be a quaternion algebra over F and Λ an Eichler order in A. For any maximal ideals $\wp$ of $\mathcal{O}_F$, let $F_\wp$, $\mathcal{O}_{F_\wp}$, $A_\wp$, $\Lambda_\wp$ be the $\wp$-completions of F, $\mathcal{O}_F$, A, Λ respectively.

The local order $\Lambda_\wp$ is maximal or isomorphic to some

$$\begin{pmatrix} \mathcal{O}_{F_\mathcal{P}} & \mathcal{O}_{F_\mathcal{P}} \\ \pi_\mathcal{P}^{k_\mathcal{P}} \mathcal{O}_{F_\mathcal{P}} & \mathcal{O}_{F_\mathcal{P}} \end{pmatrix},$$

where $\pi_\mathcal{P}$ is a uniformizer of $\mathcal{O}_{F_\mathcal{P}}$ and $k_\mathcal{P}$ is a rational integer (cf. Ch II, §2 of [6]). Hence if $\Lambda_\wp$ is not maximal, then there is an element $a_\wp \in A_\wp$ such that

$$\Lambda_\wp = a_\wp \begin{pmatrix} \mathcal{O}_{F_\mathcal{P}} & \mathcal{O}_{F_\mathcal{P}} \\ \pi_\mathcal{P}^{k_\mathcal{P}} \mathcal{O}_{F_\mathcal{P}} & \mathcal{O}_{F_\mathcal{P}} \end{pmatrix} a_\wp^{-1}.$$

We choose

$$\Gamma'_\wp = \begin{pmatrix} \mathcal{O}_{F_\mathcal{P}} & \mathcal{O}_{F_\mathcal{P}} \\ \mathcal{O}_{F_\mathcal{P}} & \mathcal{O}_{F_\mathcal{P}} \end{pmatrix}$$

if $\Lambda_\wp$ is not maximal, and $\Gamma'_\wp = \Lambda_\wp$ if $\Lambda_\wp$ is maximal. By Proposition 5.1 in ch3 of [6], there is a global maximal order Γ such that the $\wp$-completion $\Gamma_\wp = \Gamma'_\wp$ for any $\wp$. Let

$$I = \prod_{\wp} \wp^{k_\wp}$$

throughout this paper, where $\wp$ runs through all $\wp$ at which $\Lambda_\wp$ is not maximal.

Lemma 2.1. *For all $n \geq 1$, the natural homomorphism*

$$K_n(\Lambda_\wp / I\Gamma_\wp) \longrightarrow K_n(\Gamma_\wp / I\Gamma_\wp)$$

is surjective.

Proof. If $\Lambda_\wp$ is maximal, then the lemma obviously holds. So we suppose $\Lambda_\wp$ is not maximal. Without loss of generality, we assume

$$\Lambda_\wp = \begin{pmatrix} \mathcal{O}_{F_\mathcal{P}} & \mathcal{O}_{F_\mathcal{P}} \\ \pi_\mathcal{P}^{k_\mathcal{P}} \mathcal{O}_{F_\mathcal{P}} & \mathcal{O}_{F_\mathcal{P}} \end{pmatrix} \qquad \text{and} \qquad \Gamma_\wp = \begin{pmatrix} \mathcal{O}_{F_\mathcal{P}} & \mathcal{O}_{F_\mathcal{P}} \\ \mathcal{O}_{F_\mathcal{P}} & \mathcal{O}_{F_\mathcal{P}} \end{pmatrix}.$$

Then

$$\Lambda_\wp / I\Gamma_\wp = \begin{pmatrix} R & R \\ 0 & R \end{pmatrix} \qquad \text{and} \qquad \Gamma_\wp / I\Gamma_\wp = \begin{pmatrix} R & R \\ R & R \end{pmatrix},$$

where

$$R = \mathcal{O}_{F_\mathcal{P}} / \pi_\mathcal{P}^{k_\mathcal{P}} \mathcal{O}_{F_\mathcal{P}}.$$

The natural inclusion

$$i: \quad \begin{pmatrix} R & R \\ 0 & R \end{pmatrix} \longrightarrow \begin{pmatrix} R & R \\ R & R \end{pmatrix}$$

induces a homomorphism of K_n groups

$$i_*: \quad K_n \begin{pmatrix} R & R \\ 0 & R \end{pmatrix} \longrightarrow K_n \begin{pmatrix} R & R \\ R & R \end{pmatrix}.$$

By Theorem A of [3],

$$K_n \begin{pmatrix} R & R \\ 0 & R \end{pmatrix} \simeq K_n R \bigoplus K_n R.$$

By Proposition B of [3], the kernel of i_* is isomorphic to $K_n R$. So the image of i_* is isomorphic to $K_n(R)$. However we have

$$K_n \begin{pmatrix} R & R \\ R & R \end{pmatrix} \simeq K_n R.$$

Since R is a finite ring, $K_n R$ is a finite group (see [5], 1.1). So i_n must be surjective. □

For any abelian group G, let $G(\frac{1}{s})$ be the group $G \bigotimes \mathbb{Z}[\frac{1}{s}]$.

Lemma 2.2. *For all $n \geq 1$, the natural homomorphism*

$$f_{1*}: \qquad K_n(\Lambda_\wp)(\frac{1}{s}) \longrightarrow K_n(\Gamma_\wp)(\frac{1}{s})$$

is surjective, where s is the generator of $I \bigcap \mathbb{Z}$.

Proof. For any ring homomorphism

$$f: A \longrightarrow B,$$

we shall write f_* for the induced homomorphism

$$K_n(A)(\frac{1}{s}) \longrightarrow K_n(B)(\frac{1}{s})$$

The square

$$\text{(I).} \qquad \begin{array}{ccc} \Lambda_\wp & \xrightarrow{\quad f_1 \quad} & \Gamma_\wp \\ {\scriptstyle f_2} \downarrow & & \downarrow {\scriptstyle g_1} \\ \Lambda_\wp / I\Gamma_\wp & \xrightarrow[\quad g_2 \quad]{} & \Gamma_\wp / I\Gamma_\wp \end{array}$$

has an associated $K_*(\frac{1}{s})$ Mayer-Vietoris sequence

$$\cdots \longrightarrow K_n(\Lambda_\wp)(\frac{1}{s}) \stackrel{(f_{1*},\, f_{2*})}{\longrightarrow} K_n(\Gamma_\wp)(\frac{1}{s}) \bigoplus K_n(\Lambda_\wp/I\Gamma_\wp)(\frac{1}{s}) \stackrel{(g_{1*},\, g_{2*})}{\longrightarrow} K_n(\Gamma_\wp/I\Gamma_\wp)(\frac{1}{s}) \longrightarrow \cdots$$

by [1] or [7], where

$$(f_{1*},\ f_{2*})(x) = (f_{1*}(x),\ f_{2*}(x))$$

for $x \in K_n(\Lambda_\wp)(\frac{1}{s})$ and

$$(g_{1*},\ g_{2*})(a,\ b) = g_{1*}(a) - g_{2*}(b)$$

for $a \in K_n(\Gamma_\wp)(\frac{1}{s})$ and $b \in K_n(\Lambda_\wp/I\Gamma_\wp)(\frac{1}{s})$.

For any element $x \in K_n(\Gamma_\wp)(\frac{1}{s})$, we can find $y \in K_n(\Lambda_\wp/I\Gamma_\wp)$ such that

$$(g_{1*},\ g_{2*})(x,\ y) = g_{1*}(x) - g_{2*}(y) = 0$$

by Lemma 2.1. So

$$(x,\ y) \in ker(g_{1*},\ g_{2*}) = im(f_{1*},\ f_{2*}).$$

Hence $x \in im(f_{1*})$ which implies f_{1*} is surjective. □

Corollary 2.3. *For all $n \geq 1$, the cokernel of*

$$K_n(\Lambda_\wp) \longrightarrow K_n(\Gamma_\wp)$$

has no nontrivial p-torsion elements, where p is an arbitrary rational prime which does not divide s.

Proof. By Lemma 2.2, the cokernel of

$$K_n(\Lambda_\wp) \longrightarrow K_n(\Gamma_\wp)$$

is s-torsion. Hence the result follows. □

Corollary 2.4. *For all $n \geq 0$, the map*

$$f:\ K_{2n+1}(\Lambda_\wp)(\frac{1}{s}) \longrightarrow K_{2n+1}(A_\wp)(\frac{1}{s})$$

is surjective.

Proof. The map

$$f:\ K_{2n+1}(\Lambda_\wp)(\frac{1}{s}) \longrightarrow K_{2n+1}(A_\wp)(\frac{1}{s})$$

is the composition of

$$K_{2n+1}(\Lambda_\wp)(\frac{1}{s}) \stackrel{f_1}{\longrightarrow} K_{2n+1}(\Gamma_\wp)(\frac{1}{s}) \stackrel{f_1'}{\longrightarrow} K_{2n+1}(A_\wp)(\frac{1}{s}).$$

By Theorem 1 of [2], f_1' is surjective. Since f_1 and f_1' are all surjective, f is surjective. □

If R is any ring, we will denote the quotient of $K_m(R)$ by its divisible subgroup by $K_m^c(R)$. Let S be the set of primes at which $\Lambda_\wp$ is not maximal. As in [4], we define $Cl_m(\Gamma, S)$ to be the cokernel of the map

$$K_{m+1}^c(A) \longrightarrow \bigoplus_{\wp \in S} K_{m+1}^c(A_\wp) \oplus \bigoplus_{\wp \notin S} K_{m+1}^c(A_\wp)/im(K_{m+1}^c(\Gamma_\wp)),$$

where Γ is a maximal order containing Λ. Let P_S denote the set of rational primes lying under the prime ideals in S.

Theorem 2.5. *For all $n \geq 1$, the q-primary part of $Cl_{2n}(\Lambda)$ is trivial for $q \notin P_S$.*

Proof. By Lemma 1.2 of [4],

$$Cl_{2n}(\Lambda) \simeq coker(\bigoplus_{\wp \in S} K_{2n+1}^c(\Lambda_\wp) \longrightarrow Cl_{2n}(\Gamma, S)).$$

By Theorem 1 of [2],

$$\bigoplus_{\wp \notin S} K_{2n+1}^c(D_\wp)/im(K_{2n+1}^c(\Gamma_\wp)) = 0.$$

So

$$Cl_{2n}(\Gamma, S) = coker(K_{2n+1}^c(A) \longrightarrow \bigoplus_{\wp \in S} K_{2n+1}^c(A_\wp)).$$

Hence $Cl_{2n}(\Lambda)$ is a homomorphic image of

$$coker(\bigoplus_{\wp \in S} K_{2n+1}^c(\Lambda_\wp) \longrightarrow K_{2n+1}^c(A_\wp)).$$

We have proved in Corollary 2.4 that

$$coker(K_{2n+1}^c(\Lambda_\wp) \longrightarrow K_{2n+1}^c(A_\wp))(q) = 0$$

for $q \notin P_S$. So the q-primary part of $Cl_{2n}(\Lambda)$ is trivial for $q \notin P_S$. □

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