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FOLIATED VECTOR FIELDS, THE GODBILLON-VEY INVARIANT AND THE INVARIANT $I(\mathcal{F})$

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Abstract

We prove that if the invariant $I(\mathcal{F})$ constructed in [1] through the Lie algebra of infinitesimal automorphisms of transversally oriented foliations $\mathcal{F}$ is trivial, then the Godbillon-Vey invariant $GV(\mathcal{F})$ of $\mathcal{F}$ is also trivial, but that the converse is not true. For codimension one foliations, the restriction $I_T(\mathcal{F})$ of $I(\mathcal{F})$ to the Lie subalgebra of vector fields tangent to the leaves is the Reeb class $\mathcal{R}(\mathcal{F})$ of $\mathcal{F}$. We also prove that if there exists a foliated vector field which is everywhere transverse to a codimension one foliation, then the Reeb class $\mathcal{R}(\mathcal{F})$ is trivial, hence so is the $GV(\mathcal{F})$ invariant.

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1 The Godbillon-Vey invariant, the invariant $I(\mathcal{F})$ and the Reeb class

1.1 The Godbillon-Vey invariant [5]

A transversally oriented codimension one foliation $\mathcal{F}$ on a smooth manifold M is tangent to the kernel of a non singular 1-form α such that

$$\alpha \wedge d\alpha = 0. \tag{1}$$

Two such 1-forms α, α' define the same foliation iff $\alpha' = \lambda\alpha$, where λ is a nowhere zero function.

The equation $\alpha \wedge d\alpha = 0$ is equivalent to the existence of a 1-form η such that

$$d\alpha = \eta \wedge \alpha. \tag{2}$$

One proves that $\eta \wedge d\eta$ is closed and that its de Rham cohomology class

$$[\eta \wedge d\eta] \in H^3_{DR}(M, R)$$

is an invariant of $\mathcal{F}$ i.e. it does not depend on the choice of α defining $\mathcal{F}$ and on the choice of η such that (2) holds. This is the Godbillon-Vey invariant $GV(\mathcal{F})$ of $\mathcal{F}$ [5].

The Godbillon-Vey invariant can also be defined for transversally oriented foliations of any codimension. If q is the codimension, the Godbillon-Vey invariant is an element of $H^{2q+1}_{DR}(M, R)$, (see for instance [13]).

Roussarie gave the first example of a foliation $\mathcal{F}$ on $PSL(2, R)$ with non trivial $GV(\mathcal{F})$, [5], [10], [13]. We will refer to this foliation as the Roussarie foliation. Thurston [11] constructed a family $\mathcal{F}^T_t$ of foliations on S^3 with non-trivial Godbillon-Vey invariant.

There is an enormous amount of literature on the Godbillon-Vey invariant. We refer the reader to the nice survey articles by E. Ghys [4] and by S. Hurder [6], who gave an up-to-date (2000) bibliography.

We will deal first with codimension one foliations, and will consider the general case in section 4.

1.2 The invariant $I(\mathcal{F})$ [1], [2]

The Lie algebra $\mathcal{L}_{\mathcal{F}}$ of infinitesimal automorphisms of $\mathcal{F}$ consists of vector fields X such that $L_X\alpha = \mu_X\alpha$, where L_X is the Lie derivative in the direction X, and μ_X is a smooth function on M.

Consider the Gelfand-Fuks cohomology $H^*(\mathcal{L}_{\mathcal{F}}, C^\infty(M))$ [3] of the Lie algebra $\mathcal{L}_{\mathcal{F}}$ with values in the space $C^\infty(M)$ of smooth functions on M, where $\mathcal{L}_{\mathcal{F}}$ operates on $C^\infty(M)$ by derivations: $(X,f) \to X.f = (df)(X)$, $\forall X \in \mathcal{L}_{\mathcal{F}}$ and $f \in C^\infty(M)$.
The p-cochains C^p are skew-symmetric p-linear functionals on $\mathcal{L}_{\mathcal{F}}$ with values in $C^\infty(M)$ and the boundary operator $\delta : C^p \longrightarrow C^{p+1}$ is given by:

$$(\delta u)(X_1, \dots X_{p+1}) = \sum (-1)^i X_i \cdot u(X_1, \dots \hat{X}_i, \dots, X_{p+1}) +$$
$$\sum_{i<j} (-1)^{i+j-1} u([X_i, X_j], X_1, \dots \hat{X}_i, \dots \hat{X}_j \dots, X_{p+1}),$$

where the hat over an argument means its deleting.

If $X \in \mathcal{L}_{\mathcal{F}}$ and $\mu_X \in C^\infty(M)$ are such that $L_X \alpha = \mu_X \alpha$, it is easy to verify [1] that the mapping

$$X \to \mu_X$$

is a 1-cocycle

$$c_\alpha : \mathcal{L}_{\mathcal{F}} \longrightarrow C^\infty(M),$$

i.e.

$$c_\alpha([X_1, X_2]) = X_1.c_\alpha(X_2) - X_2.c_\alpha(X_1) \tag{3}$$

and that if $\alpha' = \lambda \alpha$, for some nowhere zero function λ, then

$$c_{\alpha'}(X) = c_\alpha(X) + X.(ln|\lambda|) \tag{4}$$

Hence the cohomology class $[c_\alpha] \in H^1(\mathcal{L}_{\mathcal{F}}, C^\infty(M))$ is independent of the choice of α defining $\mathcal{F}$: it is then an invariant of the foliation $\mathcal{F}$, denoted $I(\mathcal{F})$.

This invariant was defined by the first named author in [1], [2] for contact structures and in general for all conformal structures: i.e. structures that are defined by a conformal class of a regular tensor field. In particular, it is defined for transversally oriented foliations of any codimension. In [1], it was proved that $I(\mathcal{F})$ is not trivial for one dimensional foliations and contact structures.

We start by pointing out the following

Proposition 1.1
If the foliation $\mathcal{F}$ can be defined by a non-closed one form α then the cocycle c_α is not identically zero.

Let $L_\alpha \subset \mathcal{L}_\mathcal{F}$ be the subalgebra of $\mathcal{L}_\mathcal{F}$ consisting of vector fields X such that $L_X\alpha = 0$. This Lie algebra is the space of strict automorphisms of α. Our proposition 1.1 says that $\mathcal{L}_\alpha$ is strictly smaller than $\mathcal{L}_\mathcal{F}$ if $d\alpha \neq 0$. We will see that we have a different situation if $d\alpha = 0$. In that case, proposition 3.1 tells us that $\mathcal{L}_\alpha = \mathcal{L}_\mathcal{F}$ iff $I(\mathcal{F})$ is trivial. This is, for instance, the case for the linear irrational flow on T^2.

1.3 The Reeb class [9], [4], [6].

The restriction c_α^L of the cocycle c_α to any Lie subalgebra L of $\mathcal{L}_\mathcal{F}$ also defines an invariant $I_L(\mathcal{F}) \in H^1(L, C^\infty(M))$. For instance, we consider L the Lie subalgebra $\tau\mathcal{F}$ of vector fields which are tangent to the leaves. The corresponding cocycle

$$c_\alpha^\tau : \tau\mathcal{F} \longrightarrow C^\infty(M)$$

is given by

$$c_\alpha^\tau(X) = \eta(X) \tag{5}$$

for all $X \in \tau\mathcal{F}$.
Indeed:

$$L_X\alpha = di(X)\alpha + i(X)(\eta \wedge \alpha) = \eta(X)\alpha.$$

We see that the Gelfand-Fuks cocycle c_α^τ is actually a differential form on each leaf. The coycle condition (3) means that differential form is closed. Hence it defines an element $[[c_\alpha^\tau]]$ in the leafwise cohomology $H^1(M, \mathcal{F})$. This is the Reeb class $\mathcal{R}(\mathcal{F})$ [9] of $\mathcal{F}$, which has been known for long time [6]: it was proved that the integral of η along curves on the leaves measures the transverse holonomy expansion (cf. [6]).

The leafwise cohomology (see [4]), $H^*(M, \mathcal{L}))$ is an elusive object (which may be infinite dimensional). It is clear however that $[[c_\alpha^\tau]] \in H^1(M, \mathcal{L})$ coincides with $[c_\alpha^\tau] \in H^1(\tau\mathcal{F}, C^\infty(M))$.

The only difference between the Gelfand-Fuks cocycles representing elements of $H^*(\tau\mathcal{F}, C^\infty(M))$ and representatives of elements in $H^*(M, \mathcal{L})$ is that the latter are linear over $C^\infty(M)$ while the Gelfand-Fuks cocycles are only R-linear. Clearly, c_α^τ is $C^\infty(M)$ linear.

Remark 1.1
Throughout this note, we fix a Riemannian metric g and let U be the vector field dual to α, i.e. $g(U, X) = \alpha(X)$ for all vector field X. The vector field U has no zeros since α does not. We normalize U to get a unit vector field ξ which verifies

$$\alpha(\xi) = 1.$$

This vector field is transverse to the foliation.

In equation (2) we may set:

$$\eta = -i_\xi(d\alpha) = -L_\xi\alpha. \tag{6}$$

and note that

$$\eta(\xi) = d\alpha(\xi, \xi) = 0. \tag{7}$$

Indeed if X, Y are tangent to the leaves, so is $[X, Y]$ by Frobenius theorem. Hence, $(d\alpha)(X,Y) = -X.\alpha(Y)+Y.\alpha(X)-\alpha([X,Y]) = 0$, and $(\eta\wedge\alpha)(X,Y) = \eta(X)\alpha(Y)-\eta(Y)\alpha(X) = 0$. Both forms $d\alpha$ and $\eta\wedge\alpha$ vanish on $\tau\mathcal{F}$ and coincide on ξ, since $i_\xi(\eta\wedge\alpha) = \eta(\xi)\alpha - \eta\wedge i_\xi\alpha = -\eta = i_\xi d\alpha$. Hence we have: $d\alpha = \eta\wedge\alpha$.

Observe that if X is a vector field tangent to the foliation, then $\eta(X) = -i_X L_\xi\alpha = -L_\xi(\alpha(X)) - \alpha([X,\xi]) = \alpha([\xi, X]) = c_\alpha(X)$. Compare with (5).

Proposition 1.2
If the Reeb invariant $\mathcal{R}(\mathcal{F})$ is trivial, so is the Godbillon-Vey invariant.

Proof.
Indeed the triviality of $\mathcal{R}(\mathcal{F})$ is the same as the existence of a function u on M such that

$$\eta = du - ((du)(\xi))\alpha.$$

Clearly this implies that

$$[\eta\wedge d\eta] = 0 \in H^3_{DR}(M, R),$$

since $\eta\wedge d\eta = d(ud(du(\xi))\wedge\alpha)$. By setting $-du(\xi) = v$, we have $\eta = du + v\alpha$ and $d\eta = d(v\alpha)$. Then $\eta\wedge d\eta = du\wedge d(v\alpha) + v\alpha\wedge d(v\alpha) = d(ud(v\alpha)) + v^2\alpha\wedge d\alpha + v\alpha\wedge dv\wedge\alpha = d(ud(v\alpha))$ since $\alpha\wedge d\alpha = 0$.

2 Thurston Formula

At this point, it is agreable to recall the well known beautiful Thurston formula for the Godbillon-Vey invariant (see for instance [13]):

The 3-form (Thurston):

$$\begin{aligned}(-L_\xi\alpha)\wedge d(-L_\xi\alpha) &= L_\xi\alpha\wedge L_\xi(d\alpha)\\ &= L_\xi\alpha\wedge L_\xi(-L_\xi\alpha\wedge\alpha)\\ &= -(L_\xi\alpha)\wedge(L_\xi)^2\alpha\wedge\alpha \end{aligned} \tag{8}$$

represents the Godbillon-Vey invariant. From this formula, it is clear that if $\xi \in \mathcal{L}_{\mathcal{F}}$, then the 3-form above vanishes identically. Hence $GV(\mathcal{F})$ is trivial. We generalize this fact in the following:

Theorem 2.1
Let $\mathcal{F}$ be a codimension one transversally oriented foliation. Suppose there exists an automorphism X everywhere transverse to $\mathcal{F}$ (i.e. $\alpha(X)$ non-zero everywhere). Then $\mathcal{R}(\mathcal{F})$ is trivial.

Proof.
Let $X \in \mathcal{L}_{\mathcal{F}}$ with $\alpha(X) \neq 0$ everywhere, and set $F = i(X)\alpha$. From $L_X\alpha = di(X)\alpha + i(X)d\alpha = \mu_X\alpha$, we get:

$$dF + i(X)(\eta \wedge \alpha) = \mu_X\alpha,$$

or

$$dF + \eta(X)\alpha - F\eta = \mu_X\alpha. \tag{9}$$

Let Y be a vector field tangent to the leaves: $dF(Y) - F\eta(Y) = 0$. Dividing by F gives

$$\frac{dF(Y)}{F} = (dln(|F|))(Y) = \eta(Y)$$

for all vector field Y tangent to $\mathcal{F}$. We see that $d(ln|F|)$ and η coincides along the leaves. Setting

$$u = (dln|F|)(\xi),$$

we get

$$\eta = d(ln|F|) - u\alpha.$$

Therefore $\mathcal{R}(\mathcal{F})$ is trivial.

Corollary 2.1 *(Lazarov-Shulman [7]).*
If there exists an automorphism everywhere transverse to $\mathcal{F}$ then $GV(\mathcal{F})$ is trivial

Remarks 2.2
1. From Thurston's formula, it is clear that if ξ is an automorphism then $GV(\mathcal{F})$ is trivial. The theorem above is not a consequence of that fact, since $\frac{X}{\|X\|}$ is an automorphism only if $d\alpha = 0$ as we will see below.

2. We are grateful to Steve Hurder who pointed out to us that theorem 2.1 is in fact due to Lazarov and Shulman [7]. They proved that if $\mathcal{F}$ is a codimension q foliation on a smooth manifold M, and if there exists a k-dimensional Lie subalgebra $\mathcal{G}$ of $\mathcal{L}_{\mathcal{F}}$ formed by vector fields everywhere transverse to $\mathcal{F}$, letting $\alpha_{\mathcal{F}} : H^*(WO_q) \rightarrow H^*(M, R)$ the characteristic map for the foliation, then $\alpha_{\mathcal{F}}(h_I C_J) = 0$ if $deg(C_J)$ is larger than $2(q-k)$. For $q = k = 1$, this reduces to the statement of corollary 2.1. Note that our proof is direct and elementary. The proof of Lazarov-Shulman (which is not complicated either) uses the existence of a special Bott connection.

Proposition 2.1
There exists an automorphism V such that $\alpha(V) = 1$ iff $d\alpha = 0$.

Proof.
Suppose $d\alpha = 0$. It is clear that if ξ is the vector field in remark 1.1, then, $L_\xi \alpha = 0$. Conversely, if V is an infinitesimal automorphism of the foliation and $\alpha(V) = 1$, then, $L_V\alpha = \mu_V\alpha = i_V(\eta \wedge \alpha) = \eta(V)\alpha - \eta$. Hence $\eta = (\eta(V) - \mu_V)\alpha$. Therefore, $d\alpha = \eta \wedge \alpha = (\eta(V) - \mu_V)\alpha \wedge \alpha = 0$.

Proof of Proposition 1.1.
The definition we gave for $\mathcal{L}_\mathcal{F}$ is equivalent to the following: "A vector field X is an automorphism of the foliation $\mathcal{F}$ iff for any vector field Y tangent to the leaves of $\mathcal{F}$, then $[X, Y]$ is tangent to the leaves [13]. This is based on the following formula

$$i_Y L_X \alpha = L_X i_Y \alpha + i_{[X,Y]}\alpha \tag{10}$$

for all vector fields X and Y.

If a vector field X is such that for any vector field Y tangent to the leaves then $[X, Y]$ is tangent to the leaves, in formula (10) $i_Y L_X \alpha = 0$ for all Y tangent to $\mathcal{F}$. Then $L_X\alpha = \mu_X\alpha$ where $\mu_X = (L_X\alpha)(\xi)$. The converse is also true: if a vector field X satisfies $L_X\alpha = \mu_X\alpha$, and Y is tangent to the leaves then, in formula (10), $i_Y L_X\alpha = \mu_X\alpha(Y) = 0$ hence $\alpha([X, Y]) = 0$. Therefore, $[X, Y]$ is tangent to the leaves of $\mathcal{F}$. The Lie algebra $\mathcal{L}_\mathcal{F}$ is also called the space of foliated vector fields. [8]

The cocycle $c_\alpha(X) = \mu_X$ can be given explicitly:

$$i_\xi L_X \alpha = \mu_X = L_X i_\xi \alpha + \alpha([\xi, X]) = \alpha([\xi, X])$$

that is:

$$c_\alpha(X) = \alpha([\xi, X]) \tag{11}$$

If c_α is identically zero, we have in particular that $\alpha([\xi, X]) = 0$ for all vector fields X tangent to $\mathcal{F}$. Therefore $\xi \in \mathcal{L}_\mathcal{F}$, i.e. $L_\xi\alpha = \mu_\xi\alpha = -\eta$, by (6).
Consequently $d\alpha = \eta \wedge \alpha = (-\mu_\xi)\alpha \wedge \alpha = 0$.

Remark 2.3
We saw that if ξ is an automorphism then $d\alpha = 0$. This in turn implies that ξ is a strict automorphism [13].

3 Relation between the invariants

We have the following:

Theorem 3.1

Let $\mathcal{F}$ be a transversally oriented codimension one foliation.
If the invariant $I(\mathcal{F})$ is trivial, so is $\mathcal{R}(\mathcal{F})$, and hence $GV(\mathcal{F})$. The converse is not true.

There are many foliations known to have non-trivial Godbillon-Vey invariant: for example the Roussarie foliation or the Thurston one parameter family.

Corollary 3.1

Let $\mathcal{F}$ be either the Roussarie foliation or any member of the family $\mathcal{F}_t^T$ of Thurston foliations, then the invariant $I(\mathcal{F})$ is not trivial.

Proof.

Assume $I(\mathcal{F})$ is trivial: this means that there exists $f \in C^\infty(M)$ such that

$$c_\alpha(X) = X.f = (df)(X), \;\; \forall X \in \mathcal{L}_{\mathcal{F}}.$$

In other words:

$$L_X\alpha = ((df)(X))\alpha$$

Let X be a vector field tangent to $\mathcal{F}$, i.e. $\alpha(X) = 0$. We have:

$$\begin{aligned} L_X\alpha &= di(X)\alpha + i(X)d\alpha \\ &= i(X)d\alpha \end{aligned}$$

Using (2), we get:

$$\begin{aligned} L_X\alpha &= i(X)(\eta \wedge \alpha) \\ &= (\eta(X))\alpha - (\alpha(X))\eta \\ &= (\eta(X))\alpha \end{aligned}$$

On the other hand

$$L_X\alpha = (df(X))\alpha$$

Hence

$$\eta(X) = (df)(X) \tag{12}$$

for all vector fields tangent to $\mathcal{F}$. This proves corollary 2.2.

Let ξ be the vector field in remark 1.1. Setting

$$u = -(df)(\xi)$$

Equation (12) implies that:

$$\eta = df + u\alpha \tag{13}$$

Hence $\mathcal{R}(\mathcal{F})$ is trivial.
To see that the converse is not true, we concentrate on the particular case where the foliation is defined by a non singular closed 1-form α. We prove the following

Lemma 3.1
Let $\mathcal{F}$ be a foliation defined by a non singular closed 1-form α. The invariant $I(\mathcal{F})$ is trivial iff every basic function is constant.

Recall that a basic function is a function which is constant along the leaves. We will denote by $BF(M) \subset C^{\infty}(M)$, the space of basic functions.

Proof.
Suppose $I(\mathcal{F})$ is trivial: there exists a smooth fucntion f such that

$$\begin{aligned} L_X\alpha &= d(i(X)\alpha) \\ &= ((df)(X))\alpha \end{aligned}$$

If X is tangent to $\mathcal{F}$: $i(X)\alpha = 0$, we have $0 = ((df)(X))\alpha$.
Hence $(df)(X) = 0$, i.e. f is constant along the leaves (i.e. belongs to $BF(M)$). It is also constant in the direction transverse to $\mathcal{F}$. Indeed, $L_\xi\alpha = d(i(\xi)\alpha) = 0 = ((df)(\xi))\alpha$.
Hence $(df)(\xi) = 0$. Therefore $df = 0$.
We conclude that

$$c_\alpha(X) = (df)(X) = 0.$$

We prove the following

Proposition 3.1
If the foliation $\mathcal{F}$ is defined by a non singular closed 1-form α, the vanishing of the cohomology class of the cocycle c_α actually implies the vanishing of the cocycle itself. Hence $I(\mathcal{F})$ is trivial iff $c_\alpha = 0$ identically.

If $X \in \mathcal{L}_{\mathcal{F}}$, then $F = i(X)\alpha \in BF(M)$: indeed if Y is tangent to the leaves $(dF)(Y) = \mu_X\alpha(Y) = 0$. We thus get a map:

$$\mathcal{L}_{\mathcal{F}} \longrightarrow BF(M) : X \mapsto \alpha(X)$$

This map is onto: if $f \in BF(M)$, the vector field $X_f = f\xi$ belongs to $\mathcal{L}_{\mathcal{F}}$.
Indeed: $L_{(X_f)}\alpha = d(i(f\xi)\alpha) = df$. But since f is basic: $df = (df)(\xi)\alpha$ i.e. $L_{(X_f)}\alpha = (df(\xi))\alpha$.

By proposition (3.1) $I(\mathcal{F})$ is trivial iff $c_\alpha = 0$, i.e. $(df)(\xi) = 0$; which means that f is constant. Conversely if every basic function is constant then $i(X)\alpha$ is constant for all $X \in \mathcal{L}_{\mathcal{F}}$. This implies that $c_\alpha = 0$.

Example 3.1.
Let $\pi : M \longrightarrow S^1$ be a fibration over the circle. If μ is the canonical 1-form on S^1 such that $\int_{S^1} \mu = 1$, then the foliation $\mathcal{F}$ by fibers of π is defined by $\alpha = \pi^*\mu$, which is closed, moreover $BF(M) \approx C^\infty(S^1)$. Therefore $I(\mathcal{F})$ is not trivial. But $GV(\mathcal{F})$ is trivial since $\mathcal{F}$ is defined by a closed form. Lemma (3.1) and example (3.1) finish the proof of theorem 3.1.

Remark 3.1
Tischler has proved that if M has a non singular closed 1-form, then M fibers overs S^1 [12].

Example 3.2 Riemannian Foliations [8].
A foliation $\mathcal{F}$ on a riemannian manifold (M, g) is called a riemannian foliation if the restriction of g to the normal bundle to $\mathcal{F}$ is invariant by all vector fields tangent to $\mathcal{F}$.
According to Molino [8], p.172, transversally oriented codimension one riemannian foliations $\mathcal{F}$ on a compact and connected manifolds M fit into two classes: either $\mathcal{F}$ are Lie R-foliations with dense leaves or $\mathcal{F}$ consist of fibers of a fibration $\pi : M \to S^1$ like in example 3.1.

These foliations are defined by closed one forms [13] (theorem 7.3). By lemma 3.1 Lie R-foliations with dense leaves have a trivial invariant $I(\mathcal{F})$. For the rest the invariant $I(\mathcal{F})$ is not trivial (see example 3.1).

Question:
Let $\mathcal{F}$ be the Reeb foliation on S^3. Is the invariant $I(\mathcal{F})$ trivial? It is shown in [3], [4],[10] that the Reeb foliation can be defined by a 1-form α such that $d\alpha = \eta \wedge \alpha$ where η is exact (proving that $GV(\mathcal{F})$ is trivial). The Reeb invariant $I_{\tau\mathcal{F}}(\mathcal{F})$ is trivial as well.

4 Higher codimension

A transversally oriented codimension q foliation $\mathcal{F}$ on a smooth manifold M, is tangent to the kernel of a transverse volume form α such that

$$d\alpha = \eta \wedge \alpha$$

where η is some 1-form. On proves that $\eta \wedge (d\eta)^q$ is closed and that its cohomology class

$$[\eta \wedge (d\eta)^q] \in H^{2q+1}_{DR}(M, R)$$

is an invariant of $\mathcal{F}$, the Godbillon-Vey invariant $GV(\mathcal{F})$.

Theorem 4.1
Let $\mathcal{F}$ be a transversally oriented codimension q foliation. If the invariant $I(\mathcal{F})$ is trivial so is $GV(\mathcal{F})$.

Proof.
Let F be the bundle of tangent vectors to $\mathcal{F}$ and $Q = TM/F$ the normal bundle. For any domain of a local chart U, there exists a local basis $\{\alpha_i, i = 1, \ldots, q\}$ of sections of Q^* such that

$$\alpha_{|U} = \alpha_1 \wedge .... \wedge \alpha_q.$$

If $I(\mathcal{F})$ is trivial, there exists a C^∞ function $f : M \to R$ such that for all vector field X,

$$L_X\alpha = ((df)(X))\alpha.$$

Let X be a vector field tangent to the leaves of the foliation, we get:

$$L_X\alpha = \eta(X)\alpha.$$

Hence $\eta = df$ on vector fields tangent to $\mathcal{F}$. Therefore

$$\eta_{|U} = df_{|U} + \sum u_i\alpha_i$$

We get:

$$(d\alpha)_{|U} = (\eta \wedge \alpha)_{|U} = (df + \sum u_i\alpha_i) \wedge \alpha_{|U} = (df \wedge \alpha)_{|U}.$$

Therefore $d\alpha = df \wedge \alpha$. Consequently, $GV(\mathcal{F})$ is trivial.

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References

[1] A. Banyaga, An Invariant of Contact Structures and Transversally Oriented Foliations, Ann. Global Analysis and Geom. 14 (1996) 427-441.

[2] A. Banyaga, The Structure of Classical Diffeomorphism Groups, Mathematics and its applications, vol 400, Kluwer Academic Publishers, 1997.

[3] D . B. Fuks, Cohomology of Infinite Dimensional Lie Algebras, Consultants Bureau. New York and London, 1986.

[4] E. Ghys, Sur l'Invariant de Godbillon-Vey. In seminaire Bourbaki, Mars 1989. Asterisque, 177-178, pages 155-181, Société Mathématique de France, 1989.

[5] C. Godbillon and J. Vey, Un Invariant pour les Feuilletages de Codimension un, C. R .Acad. Sci. Paris, 273(1971) 92-95.

[6] S. Hurder, Dynamics and the Godbillon-Vey Classes: A History and Survey, Foliations, Geometry and Dynamics (Warsaw, 2000) 23-60 World Sci. Publisher, River Edge, N.J., 2002.

[7] C. Lazarov and H. Shulman, Obstructions to Foliations Preserving Lie group Actions, Topology, Vol. 18(179) 255-256.

[8] P. Molino, Riemannian Foliations, Progress in Mathematics, Birkhauser, 1988.

[9] G. Reeb, Sur Certaines Propriétés Topologiques de Variétés Feuilletées, Actualité Sci. Indust. 1183, Hermann, Paris (1952).

[10] I. Tamura, Topology of Foliations: An Introduction, Translations of Mathematical Monographs vol 97, Amer. Math. Soc, 1992

[11] W. Thurston, Non-Cobordant Foliation on S^3, Bulletin Amer. Math.Soc., 78:511-514, 1972.

[12] D. Tischler, On Fibering certain Foliated Manifolds over S^1, Topology 9 (1970) 153-154.

[13] Ph. Tondeur, Foliations on Riemannian Manifolds, Springer Verlag, 1988.