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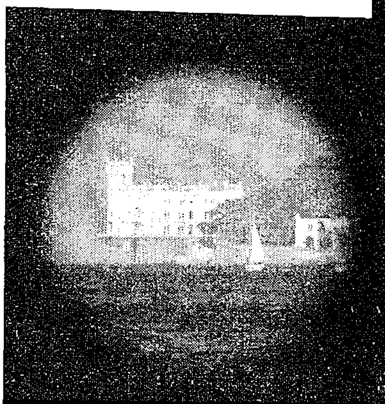

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United Nations Educational Scientific and Cultural Organization
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THE ABDUS SALAM INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

AN EXISTENCE RESULT OF ENERGY MINIMIZER MAPS BETWEEN RIEMANNIAN POLYHEDRA

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Abstract

In this paper, we prove the existence of energy minimizers in each free homotopy class of maps between polyhedra with target space without focal points. Our proof involves a careful study of some geometric properties of riemannian polhyedra without focal points. Among other things, we show that on the relevant polyhedra, there exists a convex supporting function.

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0. INTRODUCTION.

In the past decades, there has been a wide range of activity in the study of the existence of energy minimizers in various homotopy classes of maps between smooth riemannian manifolds, See [11], [15], [16], [18], [19], [20] and the references therein. In [11], Eells and Sampson obtained a fundamental theorem on the existence of harmonic maps in each free homotopy class of maps with target manifolds of non-positive sectional curvature which was generalized to the case of target manifolds without focal points by Xin [23]. In sharp contrast to the smooth case, very little is known for the case of singular spaces. In [17], Korevaar and Schoen expanded the theory of harmonic maps between smooth riemannian manifolds to the case of maps between certain singular spaces. For instance, admissible riemannian polyhedra are prototypes of the relevant singular spaces because these being both geodesic, harmonic (in the sense of Brelot cf. [10] ch 2), Dirichlet spaces and provide a wealth of examples as well. Let us mention here some examples of riemannian polyhedra (cf. [10] ch 8):

(1) -Smooth riemannian manifolds (possibly with boundary).
(2) -Triangulable riemannian lipshitz manifolds.
(3) -Riemannian join of riemannian manifolds.
(4) -Riemannian orbit spaces.
(5) -Riemannian orbifolds.
(6) -Conical singular riemannian spaces.
(7) -Normal analytic spaces with singularities.
(8) -Stratified riemannian spaces (or Tom spaces) satisfying Whitney's regularity condition. Etc...

Our goal in the present paper is to show the existence of an energy minimizer in each free homotopy class of maps between riemannian polyhedra with target spaces without focal points in the sense of [4]. This result generalizes both Xin's result [23] and the new version of the Eells-Sampson's existence theorem [11] due to Eells and Fuglede (cf.[10] ch 11) where they obtained an existence theorem in the case of target spaces polyhedra of non-positive curvature in the sense of Alexandrov [6]. This generalization seems natural but it hides several difficulties which have to be solved by different approaches. One of these difficulties arises from the fact that in our case the absence of smoothness makes the Xin's methods [23] non-valid. Another difficulty, due to the fact that a riemannian polyhedron without focal points is not necessary of non-positive curvature, leads us making things differently from Eells and Fuglede [11]. For instance, take for example a riemannian join of smooth riemannian manifold of positive sectional curvature without focal points and a riemannian manifold of non-positive curvature. This example has sense because Gulliver [14] has shown that there are manifolds without focal points of both signs of sectional curvature. In addition, Gulliver's result [14] implies that the nonexistence of focal points is weaker than non-positivity of the curvature.

In order to state and prove our results we will give in section 1 some general preliminaries on geodesic spaces, riemannian polyhedra and the energy of a map between riemannian polyhedra. In section 2, we will bring out some geometric properties of geodesic spaces which are due to the absence of focal points, we will also investigate the case of riemannian polyhedra without focal points in depth, and we will show the existence of a convex supporting function. The existence of such a function is the principal difference between our case and the Eells-Fuglede's case [10]. In their case, the square of the distance function to a geodesic is obviously convex supporting (consequence of the definition the non-positivity of the curvature). In our case the proof of such a fact is quite difficult. The geometric properties obtained in section 2, which are of self interest, are a subject of current investigations by the author. The results of such investigations will appear elsewhere. They are related to the dynamic of the generalized geodesic flow in singular space. Lastly, section 3 is devoted to the existence of minimizing maps in free homotopy classes of maps between polyhedra.

1. PRELIMINARIES.

This section is devoted to some preliminaries needed in the next sections.

1.1. Geodesic spaces [2] [6] [7] [12] [13].

Let X be a metric space with metric d. A curve $c : I \to X$ is called a *geodesic* if there is $v \geq 0$, called the speed, such that every $t \in I$ has neighborhood $U \subset I$ with $d(c(t_1), c(t_2)) = v|t_1 - t_2|$ for all $t_1, t_2 \in U$. If the above equality holds for all $t_1, t_2 \in I$, then c is called *minimal geodesic.*

The space X is called a *geodesic space* if every two points in X are connected by minimal geodesic. We assume from now on that X is a complete geodesic space.

A triangle Δ in X is a triple $(\sigma_1, \sigma_2, \sigma_3)$ of geodesic segments whose end points match in the usual way. Denote by H_k the simply connected complete surface of constant Gauss curvature k. A *comparison triangle* $\bar{\Delta}$ for a triangle $\Delta \subset X$ is a triangle in H_k with the same lengths of the sides as Δ. A comparison triangle in H_k exists and is unique up to congruence if the lengths of sides of Δ satisfy the triangle inequality and, in the case $k > 0$, if the perimeter of Δ is $< \frac{2\pi}{\sqrt{k}}$. Let $\bar{\Delta} = (\bar{\sigma}_1, \bar{\sigma}_2, \bar{\sigma}_3)$ be a comparison triangle for $\Delta = (\sigma_1, \sigma_2, \sigma_3)$, then for every point $x \in \sigma_i$, $i = 1, 2, 3$, we denote by $\bar{x}$ the unique point on $\bar{\sigma}_i$ which lies at the same distances to the ends as x.

Let d denote the distance functions in both X and H_k. A triangle Δ in X is CAT_k *triangle* if the sides satisfy the triangle inequality, the perimeter of Δ is $< \frac{2\pi}{\sqrt{k}}$ for $k > 0$, and if $d(x, y) \leq d(\bar{x}, \bar{y})$, for every two points $x, y \in X$.

We say that X has curvature at most k and write $k_X \leq k$ if every point $x \in X$ has a neighborhood U such that any triangle in X with vertices in U and minimizing sides is CAT_k. Note that we do not define k_X. If X is a riemannian manifold, then $k_X \leq k$ iff k is an upper bound for the sectional curvature of X.

A geodesic space X is called geodesically complete iff every geodesic can be stretched in two directions.

We say that a geodesic space X is without conjugate points if every two points in X are connected by unique geodesic.

1.2. Orthogonality and focal point.

For more details on the study of focal points in geodesic space, the reader can refer to [4].

Orthogonality.

(X, d) will denote a complete geodesic space. Let $\sigma : \mathbb{R} \to X$ denote a geodesic and $\sigma_1 : [a, b] \to X$ a minimal geodesic with a foot in σ (i.e. $\sigma_1(a) \in \sigma(\mathbb{R})$).

The geodesic σ_1 is *orthogonal* to σ if for all $t \in [a, b]$, the point $\sigma_1(t)$ is locally of minimal distance from σ.

In the case when for given geodesic σ and a non-belonging point p there exists an orthogonal geodesic σ' to σ and containing p, we will call the intersection point between σ and σ' the *orthogonal projection point* of p on σ.

It is shown in the paper [4] that, on one hand, if the geodesic σ is minimal then there always exists a realizing distance orthogonal geodesic to σ connecting every external point p (off σ) to σ. On the other hand, if the space (X, d) is locally compact with non-null injectivity radius and the geodesic σ is minimal on every open interval with length lower than the injectivity radius, then for every point p off σ and whose distance from σ is not greater than half of the injectivity radius, there exists a geodesic joining orthogonally the point p and the geodesic σ.

As corollaries, if the space (X, d) is a simply connected CAT_0 space then for given geodesic $\sigma : \mathbb{R} \to X$ and an off point p there always exists a realizing distance orthogonal geodesic from p to σ. When X is CAT_k for positive constant k then there always exists an orthogonal geodesic to σ from a point p whose distance from σ is not greater than $\frac{\pi}{2\sqrt{k}}$. In these last two cases the angle between two orthogonal geodesics (in the sense of the definition above) is always greater than or equal to $\frac{\pi}{2}$.

Focal points.

Let (X,d) denote a complete geodesic space, $\sigma : \mathbb{R} \to X$ a geodesic and p a point not belonging to the geodesic σ.

The point p is said to be a *focal point* of the geodesic σ or just a focal point of the space X, if there exists a minimal geodesic variation $\tilde{\sigma} :]-\epsilon,\epsilon[\times[0,l] \to X$ such that, if we note $\tilde{\sigma}(t,s) = \sigma_t(s)$, σ_0 is a minimal geodesic joining p to the point $q = \sigma(0)$ and for every $t \in]-\epsilon,\epsilon[$, σ_t is a minimal geodesic containing $\sigma(t)$, with the properties:

(1) For every $t \in]-\epsilon,\epsilon[$, each geodesic σ_t is orthogonal to σ.

(2) $\lim\limits_{t\to 0} \frac{d(p,\sigma_t(l))}{d(q,\sigma(t))} = 0$.

This definition was introduced in the article [4], us a natural generalization of the same notion in the smooth case. It is shown in the same paper that the Hadamard spaces are without a focal point.

1.3. Riemannian polyhedra.

Riemannian admissible complexes ([3] [5] [6] [9] [22]).

Let K be a locally finite simplicial complex, endowed with a piecewise smooth riemannian metric g; i.e. g is a family of smooth riemannian metrics g_Δ on simplices Δ of K such that the restriction $g_\Delta|\Delta' = g_{\Delta'}$ for any simplices Δ' and Δ with $\Delta' \subset \Delta$.

Let K be a finite dimensional simplicial complex which is connected locally finite. A map f from $[a,b]$ to K is called a broken geodesic if there is a subdivision $a = t_0 < t_1 < ... < t_{p+1} = b$ such that $f([t_i,t_{i+1}])$ is contained in some cell and the restriction of f to $[t_i,t_{i+1}]$ is a geodesic inside that cell. Then define the length of the broken geodesic map f to be:

$$L(f) = \sum_{i=0}^{i=p} d(f(t_i), f(t_{i+1})).$$

The length inside each cell being measured with respect to its metric.

Then define $\tilde{d}(x,y)$, for every two points x,y in K, to be the lower bound of the lengths of broken geodesics from x to y. $\tilde{d}$ is a pseudo-distance.

If K is connected and locally finite, then $(K,\tilde{d})$ is a complete geodesic space which is locally compact [5].

A l-simplex in K is called a *boundary simplex* if it is adjacent to exactly one $l+1$ simplex. The complex K is called *boundaryless* if there are no boundary simplices in K.

The (open) *star* of an open simplex Δ^o (i.e. the topological interior of Δ or the points of Δ not belonging to any sub-face of Δ, so if Δ is point then $\Delta^o = \Delta$) of K is defined as:

$$st(\Delta^o) = \bigcup \{\Delta_i^o : \Delta_i \text{ is simplex of } K \text{ with } \Delta_i \supset \Delta\} .$$

The star $st(p)$ of point p is defined as the star of its *carrier*, the unique open simplex Δ^o containing p. Every star is path connected and contains the star of its points. In particular K is locally path connected. The closure of any star is sub-complex.

We say that the complex K is *admissible*, if it is dimensionally homogeneous and for every connected open subset U of K, the open set $U \setminus \{U \cap \{\text{the } (k-2)-\text{skeleton}\}\}$ is connected (k is the dimension of K)(i.e. K is $(n-1)$-chainable).

Let $x \in K$ a vertex of K so that x is in the l-simplex Δ_l. We view Δ_l as an affine simplex in $\mathbb{R}^l$, that is $\Delta_l = \bigcap_{i=0}^{l} H_i$, where $H_0, H_1, ..., H_l$ are closed half spaces in general position, and we suppose that x is in the topological interior of H_0. The riemannian metric g_{Δ_l} is the restriction to Δ_l of a smooth riemannian metric defined in an open neighborhood V of Δ_l in

$\mathbb{R}^l$. The intersection $T_x\Delta_l = \bigcap_{i=1}^l H_i \subset T_xV$ is a cone with apex $0 \in T_xV$, and $g_{\Delta_l}(x)$ turns it into an euclidean cone. Let $\Delta_m \subset \Delta_l$ $(m < l)$ be another simplex adjacent to x. Then, the face of $T_x\Delta_l$ corresponding to Δ_m is isomorphic to $T_x\Delta_m$ and we view $T_x\Delta_m$ as a subset of $T_x\Delta_l$.

Set $T_xK = \bigcup_{\Delta_i \ni x} T_x\Delta_i$, we call it the *tangent cone* of K at x. Let $S_x\Delta_l$ denote the subset of all unit vectors in $T_x\Delta_l$ and set $S_x = S_xK = \bigcup_{\Delta_i \ni x} S_x\Delta_i$. The set S_x is called the *link* of x in K. If Δ_l is a simplex adjacent to x, then $g_{\Delta_l}(x)$ defines a riemannian metric on the $(l-1)$-simplex $S_x\Delta_l$. The family g_x of riemannian metrics $g_{\Delta_l}(x)$ turns $S_x\Delta_l$ into a simplicial complex with a piecewise smooth riemannian metric such that the simplices are spherical.

We call an admissible connected locally finite simplicial complex, endowed with a piecewise smooth riemannian metric, an *admissible riemannian complex.*

Riemannian polyhedron [10].

We mean by *polyhedron* a connected locally compact separable Hausdorff space X for which there exists a simplicial complex K and homeomorphism $\theta : K \to X$. Any such pair (K, θ) is called a *triangulation* of X. The complex K is necessarily countable and locally finite (cf. [21] page 120) and the space X is path connected and locally contractible. The *dimension* of X is by definition the dimension of K and it is independent of the triangulation.

A *sub-polyhedron* of a polyhedron X with given triangulation (K, θ), is polyhedron $X' \subset X$ having as a triangulation $(K', \theta|_{K'})$ where K' is a subcomplex of K (i.e. K' is complex whose vertices and simplexes are some of those of K).

If X is a polyhedron with specified triangulation (K, θ), we shall speak of vertices, simplexes, $i-$skeletons or stars of X respectively of a space of links or tangent cones of X as the image under θ of vertices, simplexes, $i-$skeletons or stars of K respectively the image of space of links or tangent cones of K. Thus our simplexes become compact subsets of X and the $i-$skeletons and stars become sub-polyhedrons of X.

If for given triangulation (K, θ) of the polyhedron X, the homeomorphism θ is locally bi-lipschitz then X is said *Lip polyhedron* and θ *Lip homeomorphism.*

A *null set* in a Lip polyhedron X is a set $Z \subset X$ such that Z meets every maximal simplex Δ, relative to a triangulation (K, θ) (hence any) in set whose pre-image under θ has $n-$dimensional Lebesgue measure 0, $n = dim\Delta$. Note that 'almost everywhere' (a.e.) means everywhere except in some null set.

A *Riemannian polyhedron* $X = (X, g)$ is defined as a Lip polyhedron X with a specified triangulation (K, θ) such that K is a simplicial complex endowed with a covariant bounded measurable riemannian metric tensor g, satisfying the ellipticity condition below. In fact, suppose that X has homogeneous dimension n and choose a measurable riemannian metric g_Δ on the open euclidean $n-$simplex $\theta^{-1}(\Delta^o)$ of K. In terms of euclidean coordinates $\{x_1, ..., x_n\}$ of points $x = \theta^{-1}(p)$, g_Δ thus assigns to almost every point $p \in \Delta^o$ (or x), an $n \times n$ symmetric positive definite matrix $g_\Delta = (g_{ij}^\Delta(x))_{i,j=1,\ldots,n}$ with measurable real entries and there is a constant $\Lambda_\Delta > 0$ such that (ellipticity condition):

$$\Lambda_\Delta^{-2} \sum_{i=0}^{i=n} (\xi^i)^2 \leq \sum_{i,j} g_{ij}^\Delta(x) \xi^i \xi^j \leq \Lambda_\Delta^2 \sum_{i=0}^{i=n} (\xi^i)^2$$

for *a.e.* $x \in \theta^{-1}(\Delta^o)$ and every $\xi = (\xi^1, ..., \xi^n) \in \mathbb{R}^n$. This condition amounts to the components of g_Δ being bounded and it is independent not only of the choice of the euclidean frame on $\theta^{-1}(\Delta^o)$ but also of the chosen triangulation.

For simplicity of statements we shall sometimes require that, relative to a fixed triangulation (K, θ) of riemannian polyhedron X (uniform ellipticity condition),

$$\Lambda := \sup\{\Lambda_\Delta : \Delta \text{ is simplex of } X\} < \infty \ .$$

A riemannian polyhedron X is said to be admissible if for a fixed triangulation (K, θ) (hence any) the riemannian simplicial complex K is admissible.

There is a natural question we can ask about riemannian polyhedra: Is the theorem of Gromov-Nash still true in the case of riemannian polyhedra? In general, if we don't put more conditions on the polyhedron, the answer to the question is no. In fact a non-differentiable triangulable riemannian Lipschitz manifold is an admissible riemannian polyhedron and, De Cecco and Palmieri [8] showed that certain of these polyhedra are not isometrically embeddable in any euclidean space (and therefore not in any smooth riemannian manifold). But we know that finite dimensional Lip polyhedron is affinely Lip embedded in some finite dimensional euclidean space.

We underline that (for simplicity) the given definition of a riemannian polyhedron (X, g) contains already the fact (because of the definition above of the riemannian admissible complex) that the metric g is *continuous* relative to some (hence any) triangulation (i.e. for every maximal simplex Δ the metric g_Δ is continuous up to the boundary). This fact is sometimes in the literature omitted. The polyhedron is said to be simplexwise smooth if relative to some triangulation (K, θ) (and hence any), the complex K is simplexwise smooth. Both continuity and simplexwise smoothness are preserved under subdivision.

In the case of a general bounded measurable riemannian metric g on X, we often consider, in addition to g, the *euclidean riemannian metric* g^e on the Lip polyhedron X with a specified triangulation (K, θ). For each simplex Δ, g^e_Δ is defined in terms of euclidean frame on $\theta^{-1}(\Delta^o)$ as above by unitmatrix (δ_{ij}). Thus g^e is by no means covariantly defined and should be regarded as a mere reference metric on the triangulated polyhedron X.

Relative to a given triangulation (K, θ) of an $n-$dimensional riemannian polyhedron (X, g) (not necessarily admissible), we have on X the distance function e induced by the euclidean distance on the euclidean space V in which K is affinely Lip embedded. This distance e is not intrinsic but it will play an auxiliary role in defining an equivalent distance d_X as follows:

Let $\mathfrak{Z}$ denote the collection of all null sets of X. For given triangulation (K, θ) consider the set $Z_K \subset \mathfrak{Z}$ obtained from X by removing from each maximal simplex Δ in X those points of Δ^o which are Lebesgue points for g_Δ. For $x, y \in X$ and any $Z \in \mathfrak{Z}$ such that $Z \subset Z_K$ we set:

$$d_X(x,y) = \sup_{\substack{Z\in\mathfrak{Z} \\ Z\supset Z_K}} \inf_{\substack{\gamma \\ \gamma(a)=x,\gamma(b)=y}} \{L_K(\gamma)\colon \gamma \text{ is Lip continuous path and transversal to } Z\},$$

where $L_K(\gamma)$ is the length of the path γ defined as:

$$L_K(\gamma) = \sum_{\Delta\subset X} \int_{\gamma^{-1}(\Delta^o)} \sqrt{(g^\Delta_{ij} \circ \theta^{-1} \circ \gamma)\dot{\gamma}^i\dot{\gamma}^j}, \text{ the sum is over all simplexes meeting } \gamma.$$

It is shown in [10] that the distance d_X is intrinsic, in particular it is independent of the chosen triangulation and it is equivalent to the euclidean distance e (due to the Lip affinely and homeomorphically embedding of X in some euclidean space V).

1.4. Energy of maps.

The concept of energy in the case of a map of riemannian domain into an arbitrary metric space Y was defined and investigated by Korevaar and Shoen [17]. Later this concept was extended by Eells and Fuglede [10] to the case of map from an admissible riemannian polyhedron X with simplexwise smooth riemannian metric. Thus, the energy $E(\varphi)$ of a map φ from X to the space Y is defined as the limit of suitable approximate energy expressed in terms of the distance function d_Y of Y.

It is shown in [10] that the maps $\varphi : X \longrightarrow Y$ of finite energy are precisely those quasicontinuous (i.e. has a continuous restriction to closed sets, whose complements have arbitrarily small capacity, (cf. [10] page 153) whose restriction to each top dimensional simplex of X has finite energy in the sense of Korevaar-Schoen, and $E(\varphi)$ is the sum of the energies of these restrictions.

Now, let (X,g) be an admissible $m-$dimensional riemannian polyhedron with simplexwise smooth riemannian metric. It is not required that g is continuous across lower dimensional simplexes. The target (Y,d_Y) is an arbitrary metric space.

Denote $L^2_{loc}(X,Y)$ the space of all μ_g-measurable (μ_g the volume measure of g) maps $\varphi: X \longrightarrow Y$ having separable essential range and for which the map $d_Y(\varphi(.),q) \in L^2_{loc}(X,\mu_g)$ (i.e. locally μ_g-squared integrable) for some point q (hence by triangle inequality for any point). For $\varphi,\psi \in L^2_{loc}(X,Y)$ define their distance $D(\varphi,\psi)$ by:

$$D^2(\varphi,\psi) = \int_X d^2_Y(\varphi(x),\psi(y))d\mu_g(x).$$

Two maps $\varphi,\psi \in L^2_{loc}(X,Y)$ are said to be *equivalent* if $D(\varphi,\psi)=0$, i.e. $\varphi(x)=\psi(x)$ μ_g-a.e. If the space X is compact then $D(\varphi,\psi)<\infty$ and D is a metric on $L^2_{loc}(X,Y)=L^2(X,Y)$ and complete if the space Y is complete [17].

The *approximate energy density* of the map $\varphi \in L^2_{loc}(X,Y)$ is defined for $\epsilon>0$ by:

$$e_\epsilon(\varphi)(x) = \int_{B_X(x,\epsilon)} \frac{d^2_Y(\varphi(x),\varphi(x'))}{\epsilon^{m+2}} d\mu_g(x').$$

The function $e_\epsilon(\varphi) \geq 0$ is locally μ_g-integrable.

The *energy* $E(\varphi)$ of a map φ of class $L^2_{loc}(X,Y)$ is:

$$E(\varphi) = \sup_{f\in C_c(X,[0,1])} (\limsup_{\epsilon\longrightarrow 0} \int_X fe_\epsilon(\varphi)d\mu_g),$$

where $C_c(X,[0,1])$ denotes the space of continuous functions from X to the interval $[0,1]$ with compact support.

A map $\varphi: X \longrightarrow Y$ is said to be *locally of finite energy*, and we write $\varphi \in W^{1,2}_{loc}(X,Y)$, if $E(\varphi|U)<\infty$ for every relatively compact domain $U \subset X$, or equivalently if X can be covered by domains $U \subset X$ such that $E(\varphi|U)<\infty$.

For example (cf. [10] lemma 4.4), every Lip continuous map $\varphi: X \longrightarrow Y$ is of class $W^{1,2}_{loc}(X,Y)$. In the case when X is compact $W^{1,2}_{loc}(X,Y)$ is denoted $W^{1,2}(X,Y)$ the space of all maps of finite energy.

We can show (cf. [10] theorem 9.1) that a map $\varphi \in L^2_{loc}(X)$ is locally of finite energy iff there is a function $e(\varphi) \in L^1_{loc}(X)$, named *energy density* of φ, such that (weak convergence):

$$\lim_{\epsilon\longrightarrow 0} \int_X fe_\epsilon(\varphi)d\mu_g = \int_X fe(\varphi)d\mu_g, \text{ for each } f \in C_c(X).$$

2. Geodesic spaces without focal points.

The aim of this section is to bring out some geometric properties of a geodesic space, which are due to the absence of the focal points. In particular, we will investigate in depth the example of the riemannian polyhedra.

2.1. Complete geodesic space without focal points.

In this paragraph (X,d) is a complete geodesic space. Our first result is the following.

Theorem 2.1.

Suppose that the geodesic space X is without focal points and it is simply connected. Then (X,d) is without conjugate points, in the sense that for every pair of points $(p,q) \in X \times X$ there exists a unique minimal geodesic σ_{pq} connecting the point p to q.

For the purpose of proving Theorem 2.1 we begin with the following lemma:

Lemma 2.2.

Under the same hypothesis of Theorem 2.1, let $\sigma : I \subseteq \mathbb{R} \to X$ be a geodesic and p a point not belonging to σ. If there exists a point $q \in \sigma$ which is an orthogonal (see the paragraph 1.2) projection point of p on σ, then the point q is unique.

Proof.

In fact, the conclusion of the lemma means that the function $t \mapsto L(t) = d^2(p, \sigma(t))$ reaches its minimum at most one time.

Arguing by contradiction, then suppose that there exist $t_1 \neq t_2 \in I$ such that $\sigma(t_1) \neq \sigma(t_2)$ and both these two points are orthogonal projection points of p on σ. Thus $d(\sigma(t_1), \sigma(t_2)) > 0$ (because σ is a geodesic).

Now, let σ_i, $i = 1, 2$, be a minimal geodesic connecting the point p to $\sigma(t_i)$. According to the definition of orthogonality, σ_i, for $i = 1, 2$, is orthogonal to the geodesic σ.

We have supposed that our space X is without focal points. This fact is expressed by the following:

$$\forall s_1, s_2 \geq 0,\ \exists K_{s_1 s_2} > 0, \text{ such that, } d(\sigma_1(s_1), \sigma_2(s_2)) \geq K_{s_1 s_2} d(\sigma(t_1), \sigma(t_2)) > 0.$$

But this last assertion contradicts the fact that the geodesic σ_1 meets the geodesic σ_2 at the point p.

This completes the proof of our result.

□

Now, we are ready to prove Theorem 2.1.

Proof of Theorem 2.1.

As in the theorem, let (X, d) be a simply connected complete geodesic space without focal points. Suppose that there exists (at least) two points $p, q \in X$ such that, there is at least two distinct minimal geodesics σ_1 and σ_2, connecting them. Suppose that both σ_1 and σ_2 are parameterized by the arc-length. Accordingly, there exists $s \in]0, d(p,q)[$ such that $\sigma_1(s) \neq \sigma_2(s)$.

Let γ be a minimal geodesic connecting $\sigma_1(s)$ to $\sigma_2(s)$. Thus, by hypothesis we have $d(p, \sigma_1(s)) = d(p, \sigma_2(s))$. By Lemma 2.2, there exists $t_0 \in]0, d(\sigma_1(s), \sigma_2(s))[$, such that $\gamma(t_0)$ is the unique orthogonal projection point of p on the geodesic γ. So we have for every $t \neq t_0$, $d(p, \gamma(t_0)) < d(p, \gamma(t))$. It follows from this last inequality the following:

$$\begin{cases} d(q, \gamma(t_0)) > d(q, \sigma_1(s)) = d(q, \gamma(0)) \\ d(q, \gamma(t_0)) > d(q, \sigma_2(s)) = d(q, \gamma(d(\sigma_1(s), \sigma_2(s)))) \end{cases}.$$

Just now, look at the restrictions $\gamma|_{[0,t_0]}$ and $\gamma|_{[t_0, d(\sigma_1(s), \sigma_2(s))]}$, both are minimal geodesics. Therefore, using Lemma 2.2, we obtain two distinct orthogonal projection points of q, one of them is on the geodesic $\gamma([0, t_0])$ the other is on the geodesic $\gamma([t_0, d(\sigma_1(s), \sigma_2(s))])$. This last conclusion is in contradiction with Lemma 2.2 because the concatenation of the two geodesics is exactly γ which is also a minimal geodesic. Our theorem is thereby proved.

□

2.2. Riemannian polyhedra without focal points.

This paragraph is devoted to a deep investigation of the geometry of riemannian polyhedra without focal points. For simplicity of statements we shall require that, our riemannian polyhedra are simplexwise smooth. But the results of this paragraph are also valid with mostly the same proofs if the riemannian polyhedra are just Lip. First we will begin with some definitions.

Let (X, d_X, g) be a riemannian polyhedron endowed with simplexwise riemannian metric g and (K, θ) a fixed triangulation.

Recall that for each point $p \in X$ (or $\theta(p) \in K$), there are well defined notions, the tangent cone over p denoted T_pX and the link over p noted S_pX which generalizes respectively the tangent space and the unit tangent space if X is smooth manifold (see the paragraph 1.3.).

Just now, we will suggest a generalization of the concepts of the normal bundle and the unit normal bundle of a geodesic in some riemannian manifold, in the case of riemannian polyhedra.

Definitions 2.3.

Let $\sigma : I \subseteq \mathbb{R} \to X$ be a geodesic, p a point belonging to σ and $v \in T_pX$ a tangent vector.

(1) *v is said orthogonal to the geodesic σ iff, there exists a geodesic γ issuing from p tangent to v and orthogonal (see the definition above) to σ.*
(2) *We name the set of all orthogonal vectors to σ at the point p (could be empty), the normal cone of σ over p and we denote it $\perp_p \sigma$.*
(3) *We name the set of all unitary orthogonal vectors $u \in \perp_p \sigma$, the normal link of σ over p.*

Remark 2.4.

As an immediate consequence we can derive from Theorem 2.1 that in the case of riemannian polyhedra without focal points it make sense to talk about a generalized exponential map $E_p : S_pX \to X$ which is an homeomorphism between the link over each point p and the space of all minimal geodesics deriving from this point (because X is locally without conjugate points).

Next we prove a crucial geometric lemma which will play an important role in all the following.

Lemma 2.5.

Let (X, d_X, g) be a riemannian polyhedron without focal points and let σ be a geodesic of X. Then for every point p belonging to σ, the spherical distance in the link S_pX between the two directions corresponding to the ingoing and the outgoing of σ at p, is greater or equal than π.

Proof.

(X, d_X, g) denotes a riemannian polyhedron without focal points and $\sigma \subset X$ a geodesic. Suppose that there exists a point $p = \sigma(0)$ where the conclusion of the lemma is not valid. Let $\bigcup_{i=1}^n \Delta_i$ be a locale triangulation (we omit in the notation the homeomorphism of the triangulation) which contains $\sigma(0)$ in its (topological) interior. So there exists $\epsilon > 0$ such that $\sigma(]-\epsilon, \epsilon[) \subset \bigcup_{i=1}^n \Delta_i$.

As a consequence of what we suppose on p, the point p necessarily belongs to the boundary of some simplex of the triangulation. Otherwise, p will be in the (topological) interior of some simplex, but every open simplex is endowed with smooth metric and in this case, the distance between the two directions defined by σ is equal to π which is in contradiction with the hypothesis on the point p.

Now suppose, that $p \in \partial\Delta_1 \cap \partial\Delta_2$ and note v_1, v_2 the two unitary tangent vectors to σ which are pointing respectively inside Δ_1 and Δ_2 (these vectors are completely determined by the fact that the space is locally without focal points). So we have supposed that $dist(v_1, v_2) < \pi$ ($dist(v_1, v_2)$ means the spherical distance between v_1 and v_2). Thus, it should exist $\epsilon_0 > 0$ very close to 0 with $\sigma(\epsilon_0) \in \Delta_2$ and a neighborhood U of p satisfying the radial uniqueness property such that:

$$U \cap E(\perp_{\sigma(\epsilon_0)} \sigma) \cap E(\perp_p^{v_1} \sigma) \neq \emptyset,$$

with E denoting the generalized exponential map and $\perp_p^{v_1} \sigma$ the set of vectors $v \in \perp_{\sigma(p)} \sigma$ which are orthogonal to v_1. This contradicts the fact that the space X is without focal points.

In fact, if such ϵ_0 didn't exist, we will have for each $t > 0$:

$$U \cap E(\perp_{\sigma(t)} \sigma) \cap E(\perp_p^{v_1} \sigma) = \emptyset,$$

and by continuity (because the space is locally without focal points) we will have:

$$U \cap E(\perp_p^{v_2} \sigma) \cap E(\perp_p^{v_1} \sigma) = \emptyset,$$

or,

$$U \cap E(\perp_p^{v_2} \sigma) \cap E(\perp_p^{v_1} \sigma) \supset \gamma, \text{ where } \gamma \text{ is a minimal geodesic segment.}$$

In fact, the last intersection cannot be a discrete set of points because that implies that U contains conjugate points which is in contradiction with the radial uniqueness property.

Both of the two last intersections lead to the fact that the distance $dist(\perp_p^{v_2} \sigma, \perp_p^{v_1} \sigma) = dist(v_1, v_2) - \pi$ is greater or equal to 0, which leads to $dist(v_1, v_2 \geq \pi$. This ends the proof of the lemma.

□

Now, we are ready to prove the following theorem.

Theorem 2.6.
Let (X, g, d) denote a simple connected riemannian polyhedron without focal points, $\sigma : I \subseteq \mathbb{R} \to X$ a geodesic and $p \in X$. Then the function $L : t \mapsto d^2(p, \sigma(t))$ is continuous and it is convex.

Proof.
Firstly, the continuity of the function L is a consequence of the fact that the geodesic space (X, d) is without conjugate points (see Theorem 2.1).

Secondly, following the same argument used by Alexander and Bishop in [1], where they show that a simply connected complete locally convex geodesic space is globally convex, it is sufficient to show that every point $x \in X$ admits an open convex neighborhood U_x. In other terms, we just have to show the following: for every $x \in X$ there is an open neighborhood U_x such that, every geodesic σ with end points in U_x belongs to U_x and the function $L : t \mapsto d^2(x, \sigma(t))$ is convex.

Let p be a point of the polyhedron (X, g, d) and (K, θ) be a fixed triangulation of X. In the following we will omit the homeomorphism of the triangulation in our notations and so we will do any distinction between the simplexes of X and the simplexes of K.

At first, we remark that for every $p \in X$, every real $r > 0$ and every geodesic σ with ends in the open ball $B(p, r)$, with center p and ray r, is entirely contained in the ball $B(p, r)$. In fact, the geodesic space X is without focal points so by the lemma of the last section (2.1.) we have:

$$\text{For every } t,\ d^2(p, \sigma(t)) \leq \sup\{d^2(p, \text{first end of } \sigma), d^2(p, \text{second end of } \sigma)\}\ .$$

Second, there are two cases to investigate, the first one is when the point p is in the topological interior of some maximal simplex and the second one is when the point p is vertex (to the triangulation (K, θ)).

Suppose that p is in the interior of the maximal simplex Δ. Then there exists a positive real $r_p > 0$ such that the open ball $B(p, r_p)$ with center p and ray r_p is contained in Δ. Thanks to the riemannian metric g_Δ, the open ball $B(p, r_p)$ can be thought of as sub-manifold of some smooth riemannian manifold endowed with the riemannian metric g_Δ. Take now a geodesic σ with end points in $B(p, r_p)$ then it is contained in the ball $B(p, r_p)$.

The polyhedron X is without focal points so the neighborhood (sub-manifold) $B(p, r_p)$ is without focal points too. Thus, by a result of Xin [23], the function L is convex for every geodesic σ contained in $B(p, r_p)$.

Now, look at the case where p is vertex of X. Let r_p be a positive real such that the open ball $B(p, r_p)$ is included in the open star $st(p)$ of p. Let $\sigma : [a, b] \to X$ be a geodesic of $B(p, r_p)$ and let $\bigcup_i \Delta_i^\circ$ (finite union) denote the star of p. We know that there is a subdivision $t_0 = a, t_1, ..., t_n = b$ such that each restriction $\sigma|_{[t_i, t_{i+1}]}$ is a geodesic in the sense of smooth riemannian geometry. So thanks to the result of Xin [23], the question about the convexity of the function $L(t) = d^2(p, \sigma(t))$ is asked when σ transits from a simplex Δ_i to a simplex Δ_{i+1} i.e. at the points t_i.

Suppose that for fixed t_j the function L is not convex at t_j. This hypothesis implies on one hand that t_j is not the minimum of the function L, because there is $\epsilon > 0$ such that both the restrictions $\sigma|_{[t_j - \epsilon, t_j]}$ and $\sigma|_{[t_j, t_j + \epsilon]}$ are convex. So by taking an ϵ smaller, the trace $\sigma|_{[t_j - \epsilon, t_j + \epsilon]}$ is strictly monotone and suppose it increasing. On the other hand, the non-convexity of L at t_j implies that the left derivative of L at t_j is strictly greater than its right derivative. Let us now traduce this last fact in terms of angles ; so let $\tau : [0,1] \times \mathbb{R} \rightarrow X$ be a map such that, for every $s \in \mathbb{R}$, $\tau(.,s) = \tau_s(.)$ is the unique geodesic relating p to $\sigma(s)$ (because X is globally without conjugate points). Note θ_s^- and θ_s^+ the left angle (to s) respectively the right angle (to s) between the two geodesics σ and τ_s. In a nutshell, if we traduce the non-convexity of L in term of angles we will have $\pi - \theta_{t_j}^- > \theta_{t_j}^+$ otherwise $\theta_{t_j}^- + \theta_{t_j}^+ < \pi$ which is in contradiction with the last lemma (because σ is a geodesic). So L is still convex at t_j. This ends the proof.

□

3. The existence theorem

This section is devoted to the existence of minimizing maps in the free homotopy classes of maps between polyhedra. Henceforth all polyhedra considered are supposed simplexwise smooth.

Theorem 3.1.
Let X and Y be compact riemannian polyhedra. Suppose that X is admissible and Y is without focal points.

Then every homotopy class $[u]$ of each continuous map u between the polyhedra X and Y has an energy (see the definition above) minimizer relative to $[u]$.

To prove this theorem we will adapt an original proof due to Eells and Fuglede that we can find in [10], where they prove an equivalent theorem in the case when the target polyhedron is supposed of nonpositive curvature (in the sense of Alexandrov). In fact we will adjust the first step of their proof (because there are two steps in the proof of Eells and Fuglede) to our case, the second step remains the same. But for the sake of completeness we will give all the proof and just before we do some remarks.

Remarks 3.2.
An immediate consequence of Theorem 2.1, is that the universal covering of a complete geodesic space without focal points is contractible (because it is simple connected and without focal points, see [1]).

Let X and Y be two locally finite polyhedra. If Y has contractible universal covering space, then the homotopy classes of maps $u : X \rightarrow Y$ are in natural bijective correspondence $u \mapsto u_*$ with the conjugacy classes of homomorphisms $u_* : \pi_1(X) \rightarrow \pi_1(Y)$ of their fundamental groups (cf. [21] ch 8.1, theorem 9).

Let X be an admissible compact riemannian polyhedron and (Y, d_y) be a complete geodesic space. Let a sequence $(u_i)_i \subset W^{1,2}(X,Y)$ such that:

$$E(u_i) + \int_X d_Y^2(u_i(x), q) d\mu_g(x) \leq c,$$

for some constant c and some fixed point $p \in Y$. Then $(u_i)_i$ has a subsequence which converges in $L^2(X,Y)$ to a map $u \in W^{1,2}(X,Y)$ (cf. [10] ch 9 Lemma 9.2).

After these remarks, we are actually ready to prove Theorem 3.1.

Proof of Theorem 3.1.
Let X and Y be two riemannian polyhedra such that X is admissible and Y is without focal points.

In the second section, and through the proof of Theorem 2.6, we have shown that a riemannian polyhedron without focal points is locally convex. Thus, in the case of the compact polyhedron Y there exists a uniform $r > 0$ such that for every point $y \in Y$, the ball $B_Y(y, 2r)$ is convex (i.e. $\forall\sigma : [a,b] \to Y$ with end points in $B_Y(y,2r)$, is completely contained in $B_Y(y,2r)$ and the distance function $t \mapsto d^2(y,\sigma(t))$ is convex). So every geodesic contained in a ball $B_Y(y,2r)$ is uniquely determined up to its end points.

So now, for given triangulations of both X and Y which are compact, every continuous map $u : X \to Y$ can be approximated uniformly (because X and Y are compact) by a simplicial map u^S which is Lipschitz and hence of finite energy (see 1.4). In addition, if we assume that $d_Y(u(x) - u^S(x)) < r$ for all $x \in X$, the simplicial map u^S becomes homotopic to u. In fact, finitely many balls $B_Y(y_i,2r)$, $i = 1,...,l$ cover the polyhedron Y, so for given i and $x \in U_i := u^{-1}(B_Y(y_i,2r))$ there is a unique minimal geodesic $\sigma_x : I \subset \mathbb{R} \to B_Y(y_i,2r)$ joining $u(x)$ to $u^S(x)$ within the ball $B_Y(y_i,2r)$. Thus, the map $\tau : I \times U_i \to B_Y(y_i,2r)$ such that $\tau(t,x) = \sigma_x(t)$, is continuous because geodesic segments in the convex balls $B_Y(y_i,2r)$ vary continuously with their endpoints [1].

In summary, we have shown that for every continuous map $u : X \to Y$, there is a representative element of the homotopy class $[u]$ which is of finite energy.

Now, take an energy minimizing sequence $(u_i)_i$ of continuous maps of finite energy in a homotopy class $[u]$. By the last remark (see above) and the fact that, $d_Y(u_i(x),q)$ is bounded by the diameter of Y (because Y is compact) and the polyhedron X is of finite volume (X is compact), there is a subsequence, noted always $(u_i)_i$, which converges in $L^2(X,Y)$ to an element $u \in W^{1,2}(X,Y)$. Every element u_i from this convergent subsequence $(u_i)_i$ lifts to a continuous map $\tilde{u}_i \in W^{1,2}_{loc}(\tilde{X},\tilde{Y})$ where $\tilde{X}$ and $\tilde{Y}$ are respectively the universal cover of X and the universal cover of Y. In addition, every such map $\tilde{u}_i$ is equivariant with respect to their fundamental groups $\pi_1(X)$ and $\pi_1(Y)$ in the sense that, if $(u_i)_* : \pi_1(X) \to \pi_1(Y)$ denote the homomorphism induced by the map u_i, then:

$$\tilde{u}_i \circ \gamma = (u_i)_*(\gamma) \circ \tilde{u}_i \qquad \text{for all } \gamma \in \pi_1(X) \qquad (\star_i).$$

To normalize these lifted map we fix a point $x_0 \in X_0$ and choose an image point noted $\tilde{u}_i(\tilde{x}_0) \in \tilde{Y}$ from the inverse image of the point $u_i(x_0)$ by the covering map (to the universal covering) of Y. The second remark above insures that the class $[u]$ is identified with the conjugacy class of homomorphism $u_* : \pi_1(X) \to \pi_1(Y)$ so independently of the choice of $\tilde{u}_i(\tilde{x}_0)$, in our case of the minimizing subsequence $(u_i)_i$ we can choose for all i the same representative of the class of the homeomorphism and let this choice be $(\tilde{u}_i)_* = (\tilde{u}_1)_*$. Thus, the pointwise limit $\tilde{u} = \lim_{i\to\infty} \tilde{u}_i$ satisfies the following:

$$\tilde{u} \circ \gamma = (u_1)_*(\gamma) \circ \tilde{u} \qquad \text{for all } \gamma \in \pi_1(X) \qquad (\star).$$

Now, we will use some algebraic topological arguments. Remember that the fundamental group $\pi_1(X)$ acts isometrically and simplicially on $\tilde{X}$ thus there exists a compact set $\tilde{F} \subset \tilde{X}$ called a *fundamental domain* of $\pi_1(X)$ whose boundary $\partial\tilde{F}$ has measure 0 and each point of $\tilde{X}$ is $\pi_1(X)$-equivalent either to exactly one point of the interior of $\tilde{F}$ or to at least one point of $\partial\tilde{F}$. The fact that X is compact implies that the compact $\tilde{F}$ can be obtained as a suitable union of maximal simplexes of $\tilde{X}$. Furthermore, $\tilde{F}$ is contained in the interior $\tilde{U}$ of suitable union of finitely, say N, many $\pi_1(X)$-translates of $\tilde{F}$.

Let $\mathfrak{E}$ denote the class of all maps in $W^{1,2}_{loc}(\tilde{X},\tilde{Y})$ which are equivariant as in $(\star)$. So as we saw, the limit map $\tilde{u} = \lim_{i\to\infty} \tilde{u}_i$ belongs to $\mathfrak{E}$.

Actually, modify the above construction. Let $(\tilde{u}_i)_i$ denote a minimizing sequence for $\int_{\tilde{F}} e(\tilde{u})$ in the class $\mathfrak{E}$, $e(\tilde{u})$ denoting the energy density of the map $\tilde{u} \in W^{1,2}_{loc}(\tilde{X},\tilde{Y})$. Thanks to the equivariance equality $(\star)$, the sequence $(\tilde{u}_i)_i$ is likewise minimizing for $E(\tilde{u}_{|\tilde{U}}) = N\int_{\tilde{F}} e(\tilde{u})$. Now, by the third remark of 3.2. applied to compact subsets of $\tilde{U}$, thus the sequence of traces $(\tilde{u}_{i_{|\tilde{U}}})$ converges in $L^2_{loc}(\tilde{U},\tilde{Y})$ and pointwise *a.e.* in $\tilde{U}$ to some map $\tilde{u}_{\tilde{U}} \in W^{1,2}(\tilde{U},\tilde{Y})$ which minimizes the energy of restrictions to $\tilde{U}$ of all maps belonging to the class $\mathfrak{E}$. Consequently, and by $(\star)$ again the sequence $(\tilde{u}_i)_i$ converges pointwise *a.e.* in $\tilde{X}$ to an extension $\tilde{u}$ of the map $\tilde{u}_{\tilde{U}}$. Of course this new limit $\tilde{u}$ satisfies $(\star)$ but likewise it minimizes the integral $\int_{\tilde{F}} e(\tilde{u}) = N^{-1}E(\tilde{u}_{|\tilde{U}})$ among all the restrictions to $\tilde{F}$ of maps of class $\mathfrak{C}$. Furthermore, such minimizer is also locally E-minimizing on $\tilde{X}$; indeed, every point of $\tilde{X}$ has a relatively compact neighborhood $\tilde{V}$ such that $\tilde{V} \cap \gamma(\tilde{V}) = \emptyset$ for all $\gamma \in \pi_1(X) \setminus \{id\}$. So if an element $\tilde{v} \in W^{1,2}_{loc}(\tilde{X}, \tilde{Y})$ satisfies $\tilde{v} = \tilde{u}$ in $\tilde{X} \setminus \tilde{V}$, then the map $\tilde{v}^* : \tilde{X} \to \tilde{Y}$ defined by $\tilde{v}^*(\gamma\tilde{x}) = \tilde{v}(\tilde{x})$ for every $\tilde{x} \in \tilde{V}$ and $\gamma \in \pi_1(X)$, while $\tilde{v}^* = \tilde{v}$ elsewhere, belongs to the class $\mathfrak{C}$, and satisfies $E(\tilde{v}_{|\tilde{V}}) \geq E((\tilde{v}^*)_{|\tilde{V}}) \geq E(\tilde{u}_{|\tilde{V}})$.

Furthermore the map $u : X \to Y$ covered by $\tilde{u}$ is in the class $[u]$ and minimizes the energy in $[u]$; indeed, any map $v \in [u]$ lifts to a map $\tilde{v}$ belonging to the class $\mathfrak{C}$, and $E(v) = \int_{\tilde{F}} e(\tilde{v}) \geq \int_{\tilde{F}} e(\tilde{u}) = E(u)$ because $\tilde{u}$ is minimizing relative to the class $\mathfrak{C}$. This ends the proof of the theorem.

□

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