Global Low-Frequency Modes in Weakly Ionized Magnetized Plasmas: Effects of Equilibrium Plasma Rotation

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Abstract

The linear and non-linear properties of global low-frequency oscillations in cylindrical weakly ionized magnetized plasmas are investigated analytically for the conditions of equilibrium plasma rotation. The theoretical results are compared with the experimental observations of rotating plasmas in laboratory devices, such as Mistral and Mirabelle in France, and KIWI in Germany.

Key words: magnetized plasma; rotation; reduced model; global structures

Classification: **52.25.-b; 52.30.-q; 52.35.-g; 52.72.+v**

Comment: 12th International Congress on Plasma Physics,

25-29 October 2004, Nice (France)

Introduction

The important role of low-frequency oscillations is well known, and they continue to be the subject of intense investigation in fusion and astrophysical situations, ionosphere and laboratory applications. Electrostatic low-frequency turbulence and related anomalous phenomena (turbulent transport, transport barriers, zonal flows) attract a special attention in the edge tokamak region, where collisions are essential. Cylindrical laboratory devices containing weakly ionized plasmas can have special importance for controlled fusion studies, or other complicated natural conditions. They allow to test our understanding of basic plasma physics including particle collisions, turbulence, and transport, as well as to develop new reliable methods of plasma diagnostics and control.

In this paper the linear and non-linear properties of global low-frequency oscillations in cylindrical weakly ionized magnetized plasmas are investigated analytically for the conditions of equilibrium plasma rotation. A new reduced non-linear model for the global oscillations in rotating plasmas is derived and analyzed. The various instabilities in rotating plasmas (current-dissipative and rotational centrifugal instabilities) are carefully revisited and identified in the linear theory. The effect of rigid plasma rotation on the flute and drift modes is investigated, and the relative importance of these modes in the plasma is established. Integral constraints are derived for the general case of arbitrary density and temperature profiles, and eigenfrequencies and instability rates are analyzed on their bases.

Our investigation allows for a detailed comparison between the theoretical models and experimental results for eigenfrequencies and instability rates in rotating cylindrical plasmas. The theoretical results are compared with the experimental observations of rotating plasmas in laboratory devices, such as Mistral [Th. Pierre et al. 2004] and Mirabelle [Th. Pierre, G. Leclert, and F. Braun 1987] in France, and KIWI [A. Latten, T. Klinger, A. Piel, Th. Pierre 1995] in Germany. The centrifugal effects can be used as a basis for developing effective methods for turbulence control. Recently, they have been incorporated in new fusion confinement concepts [R. Ellis et al. 2001].

Ideal Ion Equilibrium

The continuity and momentum balance equations for the ions are

(1) $\partial_t n + \vec{\nabla}\cdot(n\vec{V}) = 0$,

(2) $(\partial_t + \vec{V}\cdot\vec{\nabla})\,\vec{V} = \Omega\vec{V}\times\vec{b} - \dfrac{q}{m}\vec{\nabla}\phi - \nu\vec{V}$,

where $\Omega = qB_0/mc$, and the ion index is omitted. In the case of stationary equilibrium, the equations of motion in cylindrical coordinates are

(3) $(V_r\partial_r + \Omega_z\partial_\vartheta)V_r - \dfrac{1}{r}V_\vartheta^2 = \dfrac{q}{m}E_r + \Omega V_\vartheta$, $(V_r\partial_r + \Omega_z\partial_\vartheta)V_\vartheta + \dfrac{1}{r}V_\vartheta V_r = \dfrac{q}{m}E_\vartheta - \Omega V_r$,

where the frequency of charge-neutral collisions is neglected in comparison with the cyclotron frequency, and $\Omega_z = V_\vartheta/r$ is a rotation frequency.

Let us assume a rigid equilibrium rotation:

(4) $\vec{V}_0 = V(r)\hat{\vartheta}$, $\Omega_z = V/r = const$,

with axial symmetry $\partial_\vartheta = 0$. Then

(5) $V_r = 0$, $n = n_0(r)$,

and the quadratic equation for the rotation frequency follows from the equations of motion

(6) $\Omega_z^2 + \Omega\Omega_z - \Omega\Omega_E = 0$,

where $\Omega_E \equiv -cE_r / Br$ is an electric drift rotation frequency. There are two solutions,

(7) $\Omega_z / \Omega = \frac{1}{2}\left(-1 \pm \sqrt{1 + 4\Omega_E / \Omega}\right)$,

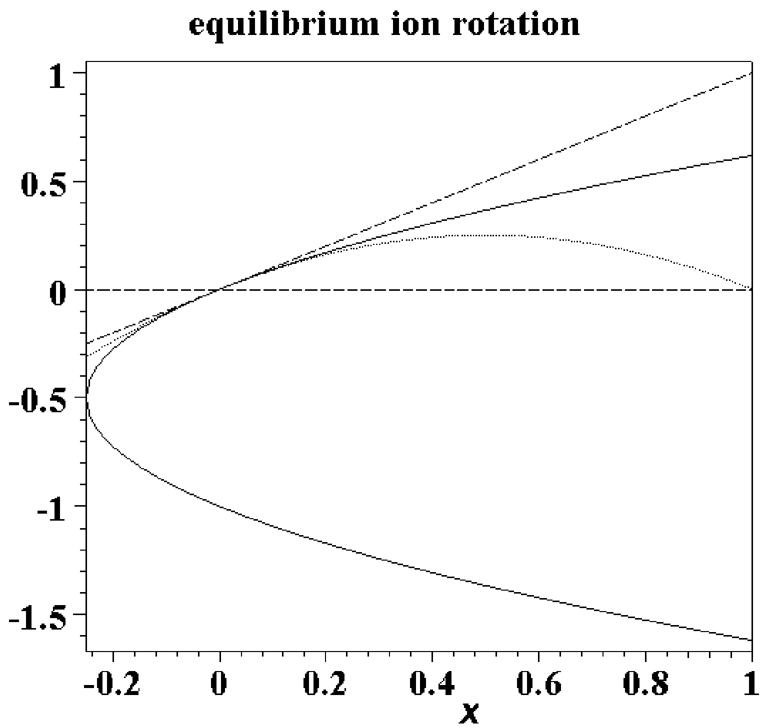
which can correspond to fast and slow rotation in the clock or counter-clock direction depending on the magnitude and the sign of Ω_E / Ω. When $\Omega_E \Omega < 0$, the physical solutions exist if only

(8) $4\left|\Omega_E / \Omega\right| < 1$.

For the very slow rotation, with $4\left|\Omega_E / \Omega\right| << 1$,

(9) $\Omega_z = \Omega_E (1 - \Omega_E / \Omega)$.

The dependence of Ω_z / Ω on $x \equiv \Omega_E / \Omega$ is shown below together with approximations for slow rotation, $\Omega_z = \Omega_E$ (straight dashed line – (1)) and $\Omega_z = \Omega_E (1 - \Omega_E / \Omega)$ (curve (2)).



Ideal Electron Equilibrium

In the same way for the slow electron rotation:

(10) $\Omega_{Ze} = \Omega_E + \omega_d$, $\omega_d = -\dfrac{cT_e \kappa_{n0}}{eB_0 r}(1 + \eta)$, where

$\eta \equiv d \ln T_e / d \ln n_0$ is the temperature-density gradient ratio, $\kappa_{n0} = d \ln n_0 / dr$.

Reduced Wave Model

In the presence of equilibrium flows, the linearized ion continuity equation and the linearized equation of ion motion govern the fluctuations δn_i and $\delta \vec{V}_i = \vec{V}_i - \vec{V}_{i0}$:

(11) $$\left[\partial_t + \vec{V}_{i0}\cdot\vec{\nabla} + (\vec{\nabla}\cdot\vec{V}_{i0})\right]\delta n_i = -n_{i0}\vec{\nabla}\cdot\delta\vec{V}_i - \delta\vec{V}_i\cdot\vec{\nabla} n_{i0}$$

(12) $$(\partial_t + \vec{V}_{i0}\cdot\vec{\nabla} + \nu_i)\delta\vec{V}_i = \Omega_i\delta\vec{V}_i\times\vec{b} - \frac{q_i}{m_i}\vec{\nabla}\delta\Phi - \delta\vec{V}_i\cdot\vec{\nabla}\vec{V}_{i0} \ .$$

In cylindrical geometry, with the magnetic field along z and the macroscopic gradients in the radial direction, the potential perturbation is represented as follows

(13) $$\delta\phi \equiv \sum_{\omega}\sum_{k_{//}}\sum_{l=-\infty}^{\infty} \exp(il\theta + ik_{//}z - i\omega t)\delta\hat{\phi}(l, k_{//}, \omega, r) \ ,$$

where $\hat{\phi}$ is a cylindrical wave amplitude, l is an azimuthal mode number, and the frequency summation is over all the eigen-frequencies $\omega(l, k_{//})$. Other wave quantities are represented in a similar manner. The reality constraints must be imposed. For example, in the cylindrical case

(14) $$\omega(-l,-k_{//}) = -\omega(l,k_{//}) \ , \ \hat{\phi}(-l,-k_{//}) = \hat{\phi}^*(l,k_{//}) .$$

In the case of axially symmetric equilibrium with rigid rotation, the following identity is valid for an azimuthal mode of mode number l:

(15) $$\vec{V}_{i0}\cdot\vec{\nabla}\delta\vec{V}_i + \delta\vec{V}_i\cdot\vec{\nabla}\vec{V}_{i0} = il\Omega_{zi}\delta\vec{V}_i - 2\Omega_{zi}\delta\vec{V}_i\times\vec{b} \ .$$

Therefore one can introduce a renormalized frequency scale $\Omega_R \equiv \Omega + 2\,\Omega_{zi}$, taking into account the Coriolis effect. Thus increasing the rotation frequency is equivalent to increasing the magnetic field.

It is convenient to use the dimensionless variables, – with the scales $\Omega_R \equiv \Omega + 2\,\Omega_{zi}$ for the frequency, $\rho_R \equiv c_s/\Omega_R$ for the space coordinates ($c_s^2 = Z_i\overline{T}/m_i$), $\overline{n}$ for the density, $\overline{T}$ for the temperature, – and the dimensionless electric field potential is $\phi = e\Phi/\overline{T}$. $q_i = Z_i e$ and m_i are the ion charge and mass, e is the absolute value of the electron charge, m_e is the electron mass.

Then the linearized ion continuity equation and the linearized equation of ion motion take the same form as without rotation except $\partial_t + il\Omega_{zi}$ replacing ∂_t in the equation of motion :

(16) $$(\partial_t + il\Omega_{zi} + \nu_i)\delta\vec{V}_i = \delta\vec{V}_i\times\vec{b} - \vec{\nabla}\delta\hat{\phi}$$

In the low-frequency approximation, the latter equation yields the velocity perturbation as the sum of electric and polarization drift terms:

(17) $$\boxed{\delta\vec{V}_i = \vec{b}\times\vec{\nabla}\delta\hat{\phi} + i\tilde{\omega}\vec{\nabla}_\perp\delta\hat{\phi} - \frac{i}{\tilde{\omega}}\vec{b}\nabla_{//}\delta\hat{\phi}}$$

which is the same as in the case without rotation but with $\tilde{\omega} = \omega - l\Omega_{zi} + i\nu_i$. Then the ion continuity equation takes the same form as in the case without rotation:

(18) $$(\Delta_\perp + \vec{\kappa}_{n0} \cdot \vec{\nabla}_\perp)\,\delta\hat{\phi} = \frac{\tilde{\omega} - i\nu_i}{\tilde{\omega}} \frac{\delta\, n_i}{n_{i0}} - \left[\frac{\omega_*}{\tilde{\omega}} + \frac{k_{//}^2}{\tilde{\omega}^2} \right] \delta\hat{\phi}$$

but with $\tilde{\omega} = \omega - l\Omega_{zi} + i\nu_i$. Here $\kappa_{n0} \equiv d\ln n_{e0}/dr$, $\omega_* = -l\kappa_{n0}/r$ is a density-gradient-driven drift frequency, and $\Delta_\perp + \vec{\kappa}_{n0} \cdot \vec{\nabla}_\perp = \frac{1}{r}\frac{d}{dr} r \frac{d}{dr} - \frac{l^2}{r^2} + \kappa_{n0}\frac{d}{dr}$.

The Poisson equation can be used to eliminate the ion density n_i in favor of the electron one n_e:

(19) $$Z_i n_i = n_e - \frac{\Delta\phi}{\overline{\alpha}_R} ,$$

where the dimensionless variables are introduced, $\overline{\alpha}_R \equiv \frac{4\pi\overline{n} m_i c^2}{Z_i B_0^2} \frac{\Omega_i^2}{\Omega_R^2} = \overline{\alpha}_i \frac{\Omega_i^2}{\Omega_R^2}$, and $\overline{\alpha}_i$ is the static electric permittivity constant in the case without rotation.

One can imagine some relationship between the self-consistent electric field potential and the induced electron density perturbation $n_e \equiv n_{e0} F(\phi - \phi_0)$, where $F(x)$ is in general some non-linear function of the potential oscillation around its equilibrium value ϕ_0. This function can be normalized according to $F(0) = 1$. Then n_{e0} is the value of a non-perturbed electron density, without potential perturbations ($\phi = \phi_0$). In this manner the ion continuity equation becomes a closed equation for the electric field potential. In the general case, one cannot find such a relationship; therefore some useful models of electron motion are usually introduced. For small potential perturbations, we can develop $F(\phi - \phi_0)$ in Taylor series. Then the linearized electron models are introduced by a relationship

(20) $$n_e / n_{e0} = 1 + \hat{\chi}_e(\phi - \phi_0) \;\rightarrow\; \delta\,\hat{n}_e / n_{e0} = \tilde{\chi}_e\, \delta\,\hat{\phi} ,$$

where $\hat{\chi}_e$ is an electron-density response operator, which determines the linear electron susceptibility introduced in plasma electrodynamics, while $\tilde{\chi}_e$ relates wave amplitudes.

Then the ion continuity equation becomes a closed wave equation :

(21) $$(\Delta_\perp + \vec{\kappa}_{n0} \cdot \vec{\nabla}_\perp)\,\delta\,\hat{\phi} = -\left[\frac{\omega_*}{\tilde{\omega}} + \frac{k_{//}^2}{\tilde{\omega}^2} - \frac{\tilde{\omega} - i\nu_i}{\tilde{\omega}} \left(\tilde{\chi}_e - \frac{\Delta_\perp - k_{//}^2}{\overline{\alpha} n_{e0}} \right) \right] \delta\,\hat{\phi} ,$$

and one arrives at the eigen-value problem. In the case of very weak deviations from the quasi-neutrality, when $\overline{\alpha}_i >> 1 \rightarrow Z_i n_i = n_e$, the eigen-value equation simplifies:

(22) $$(\Delta_\perp + \vec{\kappa}_{n0}\cdot\vec{\nabla}_\perp)\,\delta\hat{\phi} = -\left[\frac{\omega_*}{\tilde{\omega}} + \frac{k_{//}^2}{\tilde{\omega}^2} - \tilde{\chi}_e\,\frac{\tilde{\omega} - i\nu_i}{\tilde{\omega}}\right]\delta\hat{\phi}\;.$$

Electron Fluid Motion

In the drift approximation, the electron continuity equation is

(23) $$\left[\partial_t - \nu_p + \vec{V}_E\cdot\vec{\nabla}\right]n_e + \nabla_{//}(n_e V_{//e}) = 0 \quad .$$

The electron velocity can be expressed in terms of the current density $J_{//} = e(Z_i n_i V_{//\mathrm{i}} - n_e V_{//\mathrm{e}}) \approx -en_e V_{//\mathrm{e}}$.

The parallel velocity $V_{//\mathrm{e}}$ is governed by the equation of electron motion

(24) $$(d_{te} + \vec{V}_{de}\cdot\vec{\nabla} + V_{//e}\nabla_{//} + \nu)V_{//e} = \frac{e}{m_e}\nabla_{//}\Phi - \frac{1}{m_e n_e}\nabla_{//}(n_e T_e)\ ,$$

with the convective derivative $d_t \equiv \partial_t + \vec{V}_E\cdot\vec{\nabla}$. The term $\vec{V}_{de}\cdot\vec{\nabla}V_{//e}$ in this equation must be omitted, since a more consistent treatment reveals the well-known cancellation of this term with the contribution from $\vec{\nabla}\cdot\pi\cdot\vec{b}$, the collisionless part of the stress tensor. For sufficiently large collision frequency ν_e, the electron inertia can be disregarded, and the last equation yields $V_{//e}:\ V_{//e} = \dfrac{1}{m_e\nu_\mathrm{e}}\left(e\nabla_{//}\Phi - T_e\nabla_{//}\ln n_e\right),\quad \nabla_{//}T_e = 0$. Thus, we can eliminate $V_{//e}$ from the continuity equation:

(25) $$d_t\, n_e = \frac{1}{m_e\nu_\mathrm{e}}\left(T_e\nabla_{//}n_e - en_e\nabla_{//}\Phi\right).$$

If there is an equilibrium electron drift along the magnetic field produced by a constant electric field $\vec{E}_0$, with the velocity $U = -eE_0/m_e\nu_e$, then the potential in the basic equations must be formally replaced according to $\nabla_{//}\Phi \to \nabla_{//}\Phi - E_0$ and the parallel velocity is $V_{//e} = U + \dfrac{1}{m_e\nu_\mathrm{e}}\left(e\nabla_{//}\Phi - T_e\nabla_{//}\ln n_e\right)$.

In this case, the continuity equation becomes

(26) $$(d_t + U\,\nabla_{//} - \nu_p)\,n_e = \frac{1}{m_e\nu_\mathrm{e}}\left(T_e\nabla_{//}n_e - en_e\nabla_{//}\Phi\right)\ .$$

The reduced continuity equation for electrons in the drift approximation can be expressed in dimensionless variables:

(27) $$\left(\partial_t + U\,\nabla_{//} + R\vec{\mathrm{v}}_E\cdot\vec{\nabla}\right)\frac{\delta n_e}{n_{e0}} + R\vec{\mathrm{v}}_*\cdot\vec{\nabla}\delta\phi = D_e\nabla_{//}^2\left(\frac{\delta n_e}{n_{e0}} - \frac{\delta\phi}{\tau}\right),\quad R \equiv \Omega_R/\Omega_i = 1 + 2\Omega_{zi}/\Omega_i$$

where $\nabla_{//} n_{e0} = 0$, $\tau = T_e / \overline{T}$, $D_e = \tau M / Z_i \nu_e$ is a dimensionless diffusion coefficient ($D_e = S_e^2 / \nu'$, in dimensional units), $M = m_i / m_e$, $\vec{v}_* \equiv \vec{\nabla} \ln n_{e0} \times \vec{b}$ is the density-gradient-driven drift velocity, $R \equiv \Omega_R / \Omega_i = 1 + 2\Omega_{zi} / \Omega_i$ is an ion rotation parameter. This is a non-linear equation. From here the density perturbation follows in the linear approximation: $\delta \hat{n}_e / n_{e0} = \widetilde{\chi}_e \, \delta \hat{\phi}$.

In cylindrical geometry,

$$(28) \quad \widetilde{\chi}_e = \frac{R\omega_* + i\Gamma_{//}/\tau}{\omega - \omega_U - \omega_E + i\,\Gamma_{//}}$$

Here $\omega_U = \overline{U} k_{//} \sqrt{M / Z_i}$ is the frequency shift in the presence of axial electron flow, with $\overline{U} = U / S_e$ the ratio of parallel drift velocity and parallel thermal velocity for electrons, $\Gamma_{//} = D_e k_{//}^2$ is the rate of electron diffusion along the magnetic field. $\omega_E = l\Omega_E$, Ω_E is the angular frequency, which corresponds to the electric drift rotation with the velocity $\vec{V}_E \equiv V_E \hat{\vartheta}$, $\Omega_E = V_E / r = const$. In dimensionless variables $\vec{V}_E \equiv R\vec{b} \times \vec{\nabla}\phi_0 = V_E \hat{\vartheta}$, $V_E = R d\phi_0 / dr$, $\Omega_E = V_E / r$.

If the electron inertia is important, then ν_e in $\Gamma_{//}$ is replaced with $\nu_e - i(\omega - \omega_U - R\omega_E)$. The electron electric susceptibility in the local theory is related to $\widetilde{\chi}_e$: $\chi_e(\vec{k},\omega) = \widetilde{\chi}_e / k^2 \lambda_e^2$ in dimensional units, with $\lambda_e^2 = T_e / 4\pi e^2 \overline{n}_{e0}$. A misprint in [R.F. Ellis, E. Marden-Marshall, 1979] is to be noticed in the expression for the electron density response, given for $\tau = 1$, $\omega_E = 1$: ω_U is missing in the denominator.

When $\Gamma_{//}$ is much larger then all other frequencies in $\widetilde{\chi}_e$, then the electron density perturbation is governed approximately by the linearized Boltzmann's law ("δ-models"),

$$(29) \quad \widetilde{\chi}_e = \frac{1}{\tau}(1 + i\delta), \quad \delta = \frac{1}{\Gamma_{//}}(\omega - \omega_U - \omega_E - \tau R\omega_*).$$

Small deviations from the Boltzmann's law, described by δ, determine possible mechanisms of weak instability.

Dispersion Equation

Let us consider the approximation of constant electron temperature, $\tau = 1$. The Gaussian density profile $n_0 \sim \exp(-r^2 / L_N^2)$ is a good approximation for many experimental situations. Then the wave equation simplifies and reduces to the eigen-value problem :

$$(1) \quad \boxed{(\Delta_\perp + \vec{\kappa}_{n0} \cdot \vec{\nabla}_\perp)\, \delta\hat{\phi} = -K^2 \delta\hat{\phi}} \, .$$

A real positive constant K^2 governs the eigen-frequencies via the dispersion equation. In cylindrical geometry, the eigen-functions are related to the confluent hypergeometric function $M(a,b,z)$:

(2) $$\delta\hat{\phi} \sim r^l M\left(\frac{2l - K^2 L_N^2}{4}, l+1, \frac{r^2}{L_N^2}\right).$$

The eigen-values come from the boundary condition $\delta\phi(R_0) = 0$,

(3) $$K^2 L_N^2 = 2l - 4a_{l\,n}(R_0^2/L_N^2), \quad \mathrm{n} \geq 0,$$

where $a_{l\,n}(z) < 0$ is the real root number $n+1$ of the equation $\mathbf{M}(a, |l|+1, z) = 0$.

Thus in the case of low-frequency global modes in the rotating plasma, the dispersion equation is

(4) $$K^2 = \frac{\omega_*}{\tilde{\omega}} + \frac{k_{//}^2}{\tilde{\omega}^2} - \tilde{\chi}_e \frac{\tilde{\omega} - i\nu_i}{\tilde{\omega}},$$

where $\tilde{\chi}_e = \dfrac{R\omega_* + i\Gamma_{//}/\tau}{\omega - \omega_U - \omega_E + i\,\Gamma_{//}}$ for the simple electron fluid model introduced above, $\omega_* = 2l/L_N^2$, $K^2 = \omega_*(1+\Delta_{\ln})$, and $\Delta_{\ln} \equiv -2a_{l\,n}(R_0^2/L_N^2)/l$ is a size parameter. In the large-radius approximation, when $R_0^2/L_N^2 >> 1$, and when l is not very large $\Rightarrow a_{l\,n} \approx -n$, $K_0^2 \approx 2(l+2n)/L_N^2$. Then $K^2 = \omega_*$ for the modes with $n = 0$.

Centrifugal Flute Instability

The dispersion equation is especially simple for flute modes, $k_{//} \to 0$, when

$$\tilde{\chi}_e = \frac{R\omega_*}{\omega - \omega_U - \omega_E}.$$

In what follows the ideal limit is considered, and the frequency is counted from $l\Omega_{zi}$. Then the dispersion equation takes the form

(30) $$K^2 = \frac{\omega_*}{\omega} - \frac{R\omega_*}{\omega - \omega_U + l(\Omega_{zi} - \Omega_E)}.$$

One can clearly see the basic effects responsible for charge separation in wave motion : different equilibrium rotation frequencies for ions and electrons, $\Omega_{zi} \neq \Omega_E$, the Coriolis effect $R \neq 1$, and axial electron current, $\omega_U \neq 0$. The latter factor is omitted for simplicity.

This dispersion equation demonstrates the limitations of the effective gravity model for the instabilities in rotating plasmas. Such a model will miss the Coriolis effect.

This equation is quadratic in frequency : $fx^2 + 2x + 1 = 0$, where $x \equiv \omega / l\Omega_{zi}$, $f \equiv K_0^2 l/\omega_* = l - 2a_{l\,n}(R_0^2/L_N^2) > 1$,

(31) $$x = \frac{1}{f}\left(-1 \pm \sqrt{1-f}\right).$$

Thus all the modes are instable. The eigenfrequency and the instability rate are

(32) $$\omega_r = l\Omega_{zi}\frac{f-1}{f}\ ,\quad \omega_i = l\Omega_{zi}\frac{\sqrt{f-1}}{f}\ .$$

The rate is maximal for the modes with $n = 0$. In the large-radius approximation, $\omega_r = \Omega_{zi}(l-1)$, $\omega_i = \Omega_{zi}\sqrt{l-1}$.

Drift Wave Instability

In the non-ideal case, and for non-adiabatic electrons, when $\delta \neq 0$, the dispersion relation yields the growth/damping rate ω_i , $\omega \equiv \omega_r + i\omega_i = \frac{\omega_* - \nu_i K^2}{1+K^2+i\delta}$. The necessary instability condition is

(5) $$\boxed{\omega_r \mathrm{Re}\delta < 0}.$$

For weakly non-adiabatic electrons, with $|\delta| << 1$, one can expand in powers of δ:

(6) $$\boxed{\omega_r = \frac{\omega_*}{1+K^2}\ ,\ \omega_i = -\frac{\omega_r \mathrm{Re}\delta + \nu_i K^2}{1+K^2}}\ ,\ |\delta| << 1\ .$$

The necessary and sufficient instability condition is

(7) $$\boxed{\omega_r \mathrm{Re}\delta + \nu_i K^2 < 0}\ .$$

Evidently, the ion-neutral collisions is a stabilizing factor of wave dissipation.

If the frequency is counted from $l\Omega_{zi}$, and for the very slow electron rotation, the instability factor for the electron drift waves is the following

(33) $$\delta = \frac{1}{\Gamma_{//}}\left[\omega - \omega_U + l(\Omega_{zi} - \Omega_E) - \tau R\omega_*\right] = \frac{1}{\Gamma_{//}}\left[\omega - \omega_U - R(\tau\omega_* + l\Omega_{zi}^2)\right]\ .$$

This expression reveals explicitly the mechanisms of centrifugal instability for the electron drift waves.

Thus in the case of constant electron temperature and Gaussian density profile,

(34) $$\omega_r = \frac{\omega_*}{1+K^2}$$ (the frequency is counted from $l\Omega_{zi}$),

(35) $$\omega_i = \frac{1}{\Gamma_{//}}\left[\omega_r^2 + \frac{\omega_r}{K^2}\left(\omega_U + (R-1)\omega_* + Rl\Omega_{zi}^2\right) - \nu_i\Gamma_{//}\right]\frac{K^2}{1+K^2}\ .$$

The instability condition is

(36) $$\omega_r^2 + \frac{\omega_r}{K^2}\left(\omega_U + (R-1)\omega_* + Rl\Omega_{zi}^2\right) > \nu_i\Gamma_{//}\ .$$

The latter inequality shows explicitly the modification of the dissipative instability in the current carrying and rotating plasma. For the very slow ion rotation $\Omega_{zi} \cong \Omega_E(1-\Omega_E/\Omega_i+...)$. In the opposite limite, for large Ω_E/Ω_i, $\Omega_i < \Omega_{zi} \cong \sqrt{\Omega_E\Omega_i} < \Omega_E$.

Let us analyze the dependence of the eigen-frequency on the rotation parameter
$R \equiv \Omega_R/\Omega_i = 1+2\Omega_{zi}/\Omega_i = \pm\sqrt{1+4\Omega_E/\Omega_i}$,
with different signes for two equilibrium states :

(37) $$\omega_r/\Omega_i = \omega_{*0}\frac{R}{R^2+K_0^2},$$

where ω_{*0} and $K_0^2 = \omega_{*0}(1+\Delta_{\ln})$ are the dimensionless drift frequency and the eigen-value without rotation, and the eigen-frequency is always counted from
$l\Omega_{zi} = \frac{1}{2}l\Omega_i(R-1)$.
Larger values of R correspond to stronger rotation if $\Omega_E\Omega_i > 0$, and to weaker rotation if $\Omega_E\Omega_i < 0$. When $R^2 << \omega_{*0}(1+\Delta_{\ln})$ (this is possible if $\Omega_E\Omega_i < 0$), then the eigen-frequency grows with R : $\omega_r/\Omega_i = \frac{R}{1+\Delta_{\ln}}$. In the opposite limit, when $R^2 >> \omega_{*0}(1+\Delta_{\ln})$, the eigen-frequency decreases when R grows : $\omega_r/\Omega_i = \omega_{*0}/R$.

The instability rate is represented as a sum $\omega_i \equiv \gamma_{dis}+\gamma_U+\gamma_{rot}+\gamma_{in}$,

(38) $$\gamma_{dis} = \frac{1}{\Gamma_{//}}\frac{\omega_r^2K^2}{1+K^2}, \quad \gamma_U = \frac{\omega_U}{\Gamma_{//}}\frac{\omega_r}{1+K^2}, \quad \gamma_{rot} = \frac{\omega_r}{\Gamma_{//}}\frac{(R-1)\omega_* + Rl\Omega_{zi}^2}{1+K^2}, \quad \gamma_{in} = -\frac{\nu_iK^2}{1+K^2}.$$

The part γ_{dis} coincides with the instability rate in the ideal case without axial current and rotation. This part has the following dependence on R :

(39) $$\frac{\gamma_{dis}}{\Omega_i} = \frac{1+\Delta_{\ln}}{\Gamma_{//}\Omega_i^2}\frac{\omega_r^3}{R}.$$

It is maximal when $R^2 = \frac{1}{2}\omega_{*0}(1+\Delta_{\ln})$. Thus it grows with R, when $R^2 < \frac{1}{2}\omega_{*0}(1+\Delta_{\ln})$: $\frac{\gamma_{dis}}{\Omega_i} = \frac{\Omega_i}{\Gamma_{//}}\frac{R^2}{(1+\Delta_{\ln})^2}$. In the opposite limit, $\frac{\gamma_{dis}}{\Omega_i} = (1+\Delta_{\ln})\omega_{*0}^3\frac{\Omega_i}{\Gamma_{//}}\frac{1}{R^4}$, and it decreases rapidly as R grows.

Let us analyze the part γ_U :

(40) $$\frac{\gamma_U}{\Omega_i} = \frac{\omega_U}{\omega_{*0}\Gamma_{//}}\frac{R\omega_r^2}{\Omega_i^2}.$$

It is maximal when $R^2 = 3\omega_{*0}(1+\Delta_{\ln})$. In the limit $R^2 < 3\omega_{*0}(1+\Delta_{\ln})$, this part grows more rapidly with R then γ_{dis} : $\frac{\gamma_U}{\Omega_i} = \frac{\omega_U}{\omega_{*0}\Gamma_{//}}\frac{R^3}{(1+\Delta_{\ln})^2}$. In the opposite limit, it decreases much slower then γ_{dis} :

$\frac{\gamma_U}{\Omega_i} = \frac{\omega_U}{\omega_{*0}\Gamma_{//}} \frac{\omega_{*0}^2}{R}$. Thus the current modification of the instability rate becomes more important in the rotating plasma.

In a similar manner, one finds

(41) $$\frac{\gamma_{rot}}{\Omega_i} = \frac{\omega_r^2}{\Gamma_{//}\Omega_i}(R-1)(1+l\frac{R-1}{\omega_{*0}}) .$$

The maximum of this part as a function of mode number corresponds to the eigen-frequency maximum. The Coriolis effect is not important when $\omega_{*0} << |R-1|$, then the effective gravity model can be also applied to describe plasma rotation. In the limit $R^2 << \omega_{*0}(1+\Delta_{\ln})$, this part has the same dependence on R as γ_{dis} : $\frac{\gamma_{rot}}{\Omega_i} = \frac{l\omega_r^2}{\omega_{*0}\Gamma_{//}\Omega_i}$. In the opposite limit, $\frac{\gamma_{rot}}{\Omega_i} = \frac{\omega_{*0}\Omega_i}{\Gamma_{//}R^2}(R-1)(\omega_{*0} + l(R-1))$. Therefore the part γ_{rot} can become dominant.

Finally, for ion-neutral collisions,

(42) $$\frac{\gamma_{in}}{\Omega_i} = -\frac{1+\Delta_{\ln}}{R}\frac{\nu_i\omega_r}{\Omega_i^2} .$$

This part is smaller for larger R. In the limit $R^2 << \omega_{*0}(1+\Delta_{\ln})$, there is no dependence on R , while in the opposite limit $\frac{\gamma_{in}}{\Omega_i} = -\frac{1+\Delta_{\ln}}{R^2}\frac{\nu_i\omega_{*0}}{\Omega_i}$. Thus the effect of ion-neutral collisions can become less important in the rotating plasma, which means that the instability threshold can become lower.

Integral Constraints

For arbitrary density and temperature profiles, one can use integral constraints in order to evaluate the eigen-frequency and the instability rate. The eigen-value equation yields the following integral constraint

(43) $$\int n_0 \left|\vec{\nabla}\hat{\phi}\right|^2 d\vec{r} = \int n_0 \left|\hat{\phi}\right|^2 \left(\frac{\omega_* - \omega\tilde{\chi}_e}{\omega + i\nu_i} + \frac{k_{//}^2}{(\omega + i\nu_i)^2} \right) d\vec{r} .$$

It is derived after multiplying the eigen-value equation by $n_0\hat{\phi}^*$ and integrating by parts for zero boundary conditions. Again, the frequency is counted from $l\Omega_{zi}$. When the parallel ion motion is disregarded,

(44) $$\omega = \frac{\int n_0(\omega_* \left|\hat{\phi}\right|^2 - i\nu_i \left|\vec{\nabla}\hat{\phi}\right|^2) d\vec{r}}{\int n_0(\left|\vec{\nabla}\hat{\phi}\right|^2 + \tilde{\chi}_e \left|\hat{\phi}\right|^2) d\vec{r}} .$$

In the interesting case of weak ion-neutral collisions, when $\nu_i << \omega$, and weak deviations from the adiabatic electron-density response, when $\tilde{\chi}_e = \frac{1}{\tau}(1 + i\delta)$, with $|\delta| << 1$, the imaginary part ω_i of the frequency $\omega \equiv \omega_r + i\omega_i$ is small (transparency domain),

(45) $$|\omega_i / \omega_r| << 1 \ , \quad \omega_r = \frac{1}{W}\int n_0 \omega_* |\hat{\phi}|^2 d\vec{r} \ ,$$

where the following notation is introduced: $W \equiv W_K + W_P = \int n_0 |\vec{\nabla}\hat{\phi}|^2 d\vec{r} + \int (n_0/\tau)|\hat{\phi}|^2 d\vec{r}$. The quantity W corresponds to the total oscillation energy, while $W_K \equiv \int n_0 |\vec{\nabla}\hat{\phi}|^2 d\vec{r}$, and $W_P \equiv \int (n_0/\tau)|\hat{\phi}|^2 d\vec{r}$ can be associated with the particle kinetic energy of electric-drift motion (the polarization-drift contribution is negligible), and the potential energy respectively. For monotonic density profiles, ω has the sign of the density-gradient drift frequency ω_*. One can find by iterations:

(46) $$\omega_i = -\nu_i \frac{W_K}{W} - \frac{\omega_r}{W}\int \mathrm{Re}\,\delta \frac{n_0}{\tau}|\hat{\phi}|^2 d\vec{r} \ .$$

Then the necessary instability condition is

(47) $$\omega_r \int \mathrm{Re}\,\delta \frac{n_0}{\tau}|\hat{\phi}|^2 d\vec{r} < 0 \ .$$

As $|\omega_i / \omega_r| << 1$, one can compute W_K, W_P and ω_i from the eigen-function and its gradient ($|\hat{\phi}|^2$ and $|\vec{\nabla}\hat{\phi}|^2$) obtained in the limit $\delta = 0$, $\nu_i = 0$. Experimentally observed temperature and density profiles can be taken into account in such computations.

Taking into account the explicit expression for δ,

(48) $$\omega_i = -\nu_i \frac{W_K}{W} - \frac{\omega_r}{W\Gamma_{//}}\int d\vec{r}\, \frac{n_0}{\tau}|\hat{\phi}|^2 \left[\omega_r - \omega_U - R(\tau\omega_* + l\Omega_{zi}^2)\right] .$$

In the case of constant electron temperature,

(49) $$\omega_i = \frac{1}{\Gamma_{//}}\left[\omega_r^2 + \frac{\omega_r}{k^2}\left(\omega_U + (R-1)\langle\omega_*\rangle + Rl\Omega_{zi}^2\right) - \nu_i \Gamma_{//}\right]\frac{k^2}{1+k^2} \ ,$$

where $k^2 \equiv W_K / W_P$, and $\langle Q \rangle \equiv \int n_0 Q |\phi|^2 d\vec{r} \Big/ \int n_0 |\phi|^2 d\vec{r}$.

If $W_K << W_P$ ($k^2 << 1$), then $\omega_i \approx \omega_r\left(\omega_U + (R-1)\langle\omega_*\rangle + Rl\Omega_{zi}^2\right)$. If $W_K >> W_P$ ($k^2 >> 1$), then $\omega_i \approx \omega_r^2 / \Gamma_{//} - \nu_i$, and the instability threshold is $\omega_r^2 \approx \Gamma_{//}\nu_i$.

Discussion: Theory and Experiment Compared

A rigid (shear-free) rotation, determined by the externally applied grid bias voltage $U_g = 0 \div 8V$, was reported in [Klinger et al. 1997a,b]. Such a rotation produces a Doppler shift $\omega_E = 2lA/B\sim(20 \div 60)l \times 10^3 \quad 1/s$ for each azimuthal mode. Here A determines the parabolic potential profile $\phi = Ar^2 + \phi_0$, used to approximate the potential profiles from Klinger et al. 1997b]. For the latter experimental conditions $4\Omega_E / \Omega_i = 0.5 \div 1.4$, beyond the condition $4|\Omega_E / \Omega_i| << 1$, of very slow ion rotation, which is usually adopted [E. Marden-Marshall, R.F. Ellis, and J.E. Walsh 1986], [E. Marden-Marshall, and K.L. Hall 1986], [Klinger et al. 1997b].

The following Table presents the comparison between the predictions of the exact and approximate expressions for the rotation frequency. The approximate expression $\Omega_{zi} = \Omega_E(1 - \Omega_E / \Omega_i)$ predicts a qualitatively correct growth with Ω_E and gives satisfactory qualitative estimates, when only $4\Omega_E / \Omega_i \leq 2$. The simplest evaluation $\Omega_{zi} = \Omega_E$, which is frequently applied [E. Marden-Marshall, R.F. Ellis, and J.E. Walsh 1986], [Klinger et al. 1997b], becomes less satisfactory for $4\Omega_E / \Omega_i \geq 0.4$. This evaluation overestimates the rotation frequency, while the approximate expression $\Omega_{zi} = \Omega_E(1 - \Omega_E / \Omega_i)$ underestimates it.

Ω_E / Ω_i	0.1	0.2	0.4	0.5	0.6	0.8	0.9
$\Omega_{zi}^{approx} / \Omega_i$	0.090	0.16	0.24	0.25	0.24	0.16	0.090
Ω_{zi} / Ω_i	0.092	0.17	0.31	0.37	0.42	0.52	0.57

TABLE. Predictions of the exact expression $\Omega_{zi} / \Omega = -\frac{1}{2}\left(1 \pm \sqrt{1 + 4\Omega_E / \Omega_i}\right)$ and the approximate one, $\Omega_{zi} = \Omega_E(1 - \Omega_E / \Omega_i)$.

Rotating plasma parameters are evaluated in the next Table, for various values of Ω_E / Ω_i, with $\omega^{eval} = l(\Omega_i / 4 + \Omega_E)$. One can notice that the validity condition for the low-frequency approximation is not satisfied if the ratio ω / Ω_i is taken into consideration. However the true parameter to be considered in the validity condition is not ω / Ω_i, but $(\omega - l\Omega_{zi}^{-})/\Omega_R$.

Ω_E / Ω_i	0	0.12	0.24	0.36	0.48	0.60
Ω_{zi}^- / Ω_i	0	0.11	0.20	0.28	0.35	0.42
$R = \Omega_R / \Omega_i$	1	1.2	1.4	1.6	1.7	1.8
$1/R = \rho_R / \rho_S$	1	0.82	0.71	0.64	0.59	0.54
L_N^2 / ρ_R^2	8	12	16	20	23	27
ω_* / Ω_R	0.5	0.34	0.26	0.20	0.17	0.15
$(\omega - l\Omega_{zi}^-)/\Omega_R$	0.33	0.25	0.20	0.17	0.15	0.13
$(\omega - l\Omega_{zi}^-)/\Omega_i$	0.33	0.31	0.28	0.27	0.25	0.24
ω^{eval} / Ω_i	0.5	0.74	0.98	1.2	1.5	1.7
ω / Ω_i	0.33	0.52	0.68	0.83	0.96	1.1

TABLE. Rotating plasma parameters for different values of Ω_E / Ω_i , $l = 2$. The range $\Omega_E / \Omega_i = 0.12 \div 0.48$ corresponds to the experimental conditions [Klinger et al. 1997b], [O. Grulke, T. Klinger, and A. Piel 1999], with $\omega^{exper} / \Omega_i = 0.6$. $\omega^{eval} = \omega_d + l\Omega_E$ is the approximation frequently used to evaluate the frequency of low-frequency oscillations in rotating plasmas [Klinger et al. 1997b], [O. Grülke, T. Klinger, and A. Piel 1999].

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