IC/IR/2004/9
INTERNAL REPORT
(Limited Distribution)

40th anniversary 1964 2004

United Nations Educational Scientific and Cultural Organization
and
International Atomic Energy Agency

XA0400446

THE ABDUS SALAM INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

DIFFERENTIAL INVARIANTS FOR HIGHER–RANK TENSORS. A PROGRESS REPORT

Victor Tapia[1]
Departamento de Matemáticas, Universidad Nacional de Colombia, Bogotá, Colombia
and
The Abdus Salam International Centre for Theoretical Physics, Trieste, Italy.

Abstract

We outline the construction of differential invariants for higher–rank tensors.

MIRAMARE – TRIESTE
July 2004

[1] tapiens@hotmail.com

1 Introduction

Higher-spin fields might help in a unified description of physical interactions. Higher–spin fields were introduced in [8, 10], and have been considered since then in several contexts [2, 3, 4, 5, 6, 7, 9, 13, 14, 15, 23]. One possibility to describe higher–spin fields is by means of higher–rank tensors, which has been also considered in alternative gravitational theories [18, 19, 20]. The first step in any field theoretical description of this kind is the construction of a geometrical invariant to be used as a Lagrangian. The main line of attack has been to consider higher–rank tensors in a Minkowski or Riemannian background. On the other hand, it is interesting to consider field theories constructed from the higher–rank tensors alone, that is, to develop the "differential geometry" associated to higher–rank tensors in a way similar to that in which Riemannian geometry is constructed from a second–rank tensor (a metric) [6]. Therefore, we need to construct differential invariants for higher–rank tensors. There are several general results concerning the construction of differential invariants for tensors [1, 11, 16, 22, 24]. The first step in the construction of differential invariants is to determine the number of functionally independent invariants which can be constructed from a given tensor and its derivatives. We restrict our considerations to complete symmetric higher–rank tensors. In that case, the simplest differential invariant which can be constructed contains derivatives of an order equal to the rank of the tensor. The second step is the explicit construction of these differential invariants. For first–rank tensors (vectors) the solution is the Maxwell tensor while for second–rank tensors (metrics) the solution is the Riemann–Christoffel tensor. However, for higher–rank tensors the method faces several practical obstructions due, mainly, to the fact that the inverse higher–rank tensor is an involved algebraic function of the original tensor [21]. Therefore, the few existing considerations have been restricted to linearised quantities [5]; see also [13, 14, 15].

Therefore, the construction of differential invariants for higher–rank tensors is still an open problem.

The work is organised as follows. In section 2 we outline the general method for the construction of differential invariants. A first result is that the simplest tensor differential invariant contains derivatives of the same order as the rank of the tensor. In section 3 we review the construction for first–rank tensors (vectors) and second–rank tensors (metrics). In section 4 we outline the same construction for higher–rank tensors.

2 The Number of Differential Invariants for Completely Symmetric Tensors

In this work we adopt the taxonomic definition of a tensor based on the transformation rule of its components: *a tensor is something which transforms like a tensor.* An rth–rank covariant tensor $\mathbf{G}$ is an object such that its components $G_{i_1 \cdots i_r}$ transform like

$$G_{a_1\cdots a_r}(\mathbf{y}) = X^{i_1}{}_{a_1} \cdots X^{i_r}{}_{a_r} \, G_{i_1\cdots i_r}(\mathbf{x}) \,, \tag{1}$$

where

$$X^i{}_a = \frac{\partial x^i}{\partial y^a} \,. \tag{2}$$

For later convenience let us also introduce

$$X^i{}_{ab} = \frac{\partial^2 x^i}{\partial y^a \partial y^b} \,, \tag{3}$$

with obvious extensions to higher order derivatives, and

$$Y^a{}_i = \frac{\partial y^a}{\partial x^i} \,. \tag{4}$$

Therefore, a tensor is an object such that in the transformation rule of its components only the transformation matrix $X^i{}_a$ appears. When considering derivatives of a tensor, derivatives of the transformation matrix will appear in the corresponding transformation rules; these illegal terms show that, in general, the derivative of a tensor is not a tensor. However, by means of only symmetrization operations we can construct a combination of derivatives not containing illegal terms. In order to determine the number of relations of this kind we must count the relations and the illegal terms.

The derivative of (1) is given by

$$\begin{aligned} \partial_c G_{a_1\cdots a_r}(\mathbf{y}) &= X^{i_1}{}_{a_1} \cdots X^{i_r}{}_{a_r} \, X^j{}_c \, \partial_j G_{i_1\cdots i_r}(\mathbf{x}) \\ &+ \left(X^{i_1}{}_{a_1 c} \cdots X^{i_r}{}_{a_r} + \cdots + X^{i_1}{}_{a_1} \cdots X^{i_r}{}_{a_r c} \right) G_{i_1\cdots i_r}(\mathbf{x}) \,. \end{aligned} \tag{5}$$

The number of relations $E(1, n, r)$ in (5) is the number of derivatives, n, times the number of components $T(n, r)$ of $\mathbf{G}$, given by

$$T(n,\, r) = \frac{(n + r - 1)!}{(n - 1)! r!} \,. \tag{6}$$

Therefore

$$E(1,\, n,\, r) = n \, \cdot \, T(n,\, r) \,. \tag{7}$$

The illegal terms in (5) are $(\partial^2 X)$ given by

$$(\partial^2 X)_{c a_1\cdots a_r} = X^{i_1}{}_{a_1 c} \cdots X^{i_r}{}_{a_r} \, G_{i_1\cdots i_r} = X^i{}_{a_1 c} \, Y^b{}_i \, G_{b a_2\cdots a_r} \,. \tag{8}$$

The number $U(1, n, r)$ of illegal terms is given by the number of symmetrized derivatives on X, that is $n(n + 1)/2$, times the symmetries over $r - 1$ indices in $\mathbf{G}$, that is $T(n, r - 1)$. Then,

$$U(1,\, n,\, r) = \frac{n(n+1)}{2} \cdot T(n,\, r-1)\,. \tag{9}$$

Even when the illegal terms are $X^i{}_{ab}$ they always appear in the combination shown in (8) and therefore the counting of illegal terms is as shown above. For a further derivative we obtain

$$\partial_{c_1c_2} G_{a_1\cdots a_r}(\mathrm{y}) = \left[X^{i_1}{}_{a_1c_1c_2} \cdots X^{i_r}{}_{a_r} + \cdots + X^{i_1}{}_{a_1} \cdots X^{i_r}{}_{a_rc_1c_2}\right] G_{i_1\cdots i_r}(\mathrm{x}) + \cdots\,. \tag{10}$$

The counting of relations and illegal terms, $(\partial^3 X)$, in (10) is as above and they are given by

$$\begin{aligned} E(2,\, n,\, r) &= \frac{n(n+1)}{2} \cdot T(n,\, r)\,, && (11)\\ U(2,\, n,\, r) &= \frac{n(n+1)(n+2)}{3!} \cdot T(n,\, r-1)\,. && (12) \end{aligned}$$

When considering dth–order derivatives the number of relations and the number of illegal terms, $(\partial^{d+1} X)$, are given by

$$\begin{aligned} E(d,\, n,\, r) &= \frac{(n+d-1)!}{(n-1)!d!} \cdot T(n,\, r) = \frac{(n+d-1)!}{(n-1)!d!} \cdot \frac{(n+r-1)!}{(n-1)!r!}\,, && (13)\\ U(d,\, n,\, r) &= \frac{(n+d)!}{(n-1)!(d+1)!} \cdot T(n,\, r-1) = \frac{(n+d)!}{(n-1)!(d+1)!} \cdot \frac{(n+r-2)!}{(n-1)!(r-1)!}\,. && (14) \end{aligned}$$

The possibility of finding differential invariants depends on the relative values of $E(d,n,r)$ and $U(d,n,r)$. The difference of these two quantities is

$$\Delta(d,\, n,\, r) = E(d,\, n,\, r) - U(d,\, n,\, r) = \frac{(n+d-1)!(n+r-2)!}{[(n-1)!]^2(d+1)!r!}\,(n-1)\,(d-r+1)\,. \tag{15}$$

If $d < r-1$, then $\Delta < 0$, and there are more illegal terms than relations. Therefore, it is not possible to find relations not involving the illegal terms. If $d = r$, $\Delta = 0$, and the number of relations and the number of illegal terms are equal; then it is possible to solve for the illegal terms but there is still no relation not involving them. When considering a further derivative, $d = r$, $\Delta > 0$, we have more relations than illegal terms. Therefore we obtain a differential invariant $\mathbf{R}$ with a number of components given by

$$\begin{aligned} R(n,\, r) &= E(r,\, n,\, r) - U(r,\, n,\, r)\\ &= \left(\frac{(n+r-1)!}{r!(n-1)!}\right)^2 - \frac{(n+r)!}{(n-1)!(r+1)!} \cdot \frac{(n+r-2)!}{(n-1)!(r-1)!}\\ &= \frac{(n+r-1)!(n+r-2)!}{(n-1)!(n-2)!(r+1)!r!}\,. && (16) \end{aligned}$$

Let us consider the above expression for special values of r and n. For the first values of r we obtain

$$\begin{aligned}
R(n,\,1) &= \frac{1}{2}\,n\,(n-1)\,, && (17)\\
R(n,\,2) &= \frac{1}{12}\,n^2\,(n^2-1)\,, && (18)\\
R(n,\,3) &= \frac{1}{144}\,n^2\,(n+1)^2\,(n^2+n-2)\,, && (19)\\
R(n,\,4) &= \frac{1}{2880}\,n^2\,(n+1)^2\,(n+2)^2\,(n^2+2\,n-3)\,, && (20)
\end{aligned}$$

and the resulting differential invariants are the Maxwell tensor for $r = 1$ and the Riemann–Christoffel tensor for $r = 2$. However, for the first values of n we obtain simpler expressions.

$$\begin{aligned}
R(1,\,r) &= 0\,, && (21)\\
R(2,\,r) &= 1\,, && (22)\\
R(3,\,r) &= \frac{1}{2}\,(r+2)\,[(r+2)-1]\,, && (23)\\
R(4,\,r) &= \frac{1}{12}\,(r+2)^2\,[(r+2)^2-1]\,. && (24)
\end{aligned}$$

Expression (16) can be rewritten as

$$R(n,\,r) = \frac{(N+r-1)!}{(N-1)!r!} - \frac{(N+r-3)!}{(N-1)!(r-2)!}\cdot\frac{n!}{(n-4)!4!}\,, \tag{25}$$

where $N = n(n-1)/2$, which is the number of components of a $(2r)$th–rank tensor $\mathbf{R}$ with components $R_{i_1j_1\cdots i_rj_r}$ with the following symmetries: Indices are ordered in antisymmetric couples, that is

$$R_{i_1j_1\cdots i_rj_r} = R_{[i_1j_1]\cdots[i_rj_r]}\,. \tag{26}$$

Furthermore, the tensor is completely symmetric with respect to the couples of indices $[ij]$, which is the first term in (25). The second term means that cyclic permutation over 3 indices in two couples, $n!/(n-4)!4!$, is zero; the first part of the second term, $(N+r-3)!/(N-1)!(r-2)!$, is the number of ways in which this choice can be made.

We must still consider the case $d > r$, $\Delta > 0$. In this case the corresponding differential invariants can be expressed in terms of derivatives of $\mathbf{R}$. It is however still interesting to consider the corresponding number of invariants, given by (15). In the second–rank case, $r = 2$, the corresponding expression reduces to

$$\Delta(d,\,n,\,2) = \frac{1}{2}\,n\,(d-1)\,\frac{(n+d-1)!}{(n-2)!(d+1)!}\,. \tag{27}$$

Scalar invariants are obtained from (27) just by substracting the $n(n-1)/2$ conditions which fix a local Lorentz transformation. We obtain

$$S(n,\, d) = R(n,\, d) - \frac{1}{2}\, n\,(n-1) = \frac{1}{2}\, n\,(d-1)\, \frac{(n+d-1)!}{(n-2)!(d+1)!} - \frac{1}{2}\, n\,(n-1)\,. \tag{28}$$

This result coincides with that in [16] for scalar invariants. Our formulae (27) and (28) have been obtained by means of a simple counting of relations and illegal terms and therefore our procedure is clearer than the one used in [16]. Furthermore, our formula (28), even when numerically equivalent, is simpler than that in [16].

3 Explicit Construction of Invariants

Let us start by considering a vector field $\mathbf{A}$ with components A_i. The corresponding transformation rules are

$$A_a(\mathbf{y}) = X^i{}_a\, A_i(\mathbf{x})\,. \tag{29}$$

The derivative of (29) is given by

$$\partial_b A_a(\mathbf{y}) = X^i{}_a\, X^j{}_b\, \partial_j A_i(\mathbf{x}) + X^i{}_{ab}\, A_i(\mathbf{x})\,. \tag{30}$$

Therefore, the derivative (30) is not a tensor. The number of relations in (30) is n^2, while the number of illegal terms, $(\partial^2 X)$, is $n(n+1)/2$. Since (30) is a linear algebraic system of equations there must be $n^2 - n(n+1)/2 = n(n-1)/2$ relations not involving the illegal terms $(\partial^2 X)$. In fact, we have

$$F_{ab}(\mathbf{y}) = X^i{}_a\, X^j{}_b\, F_{ij}(\mathbf{x})\,, \tag{31}$$

where

$$F_{ij} = \partial_i A_j - \partial_j A_i\,. \tag{32}$$

which we recognise as the Maxwell tensor.

Let us now consider the same construction for a second–rank symmetric tensor $\mathbf{g}$, a mtric, with components g_{ij}. The first result for a metric was obtained by Gauss [12] for $n = 2$. In 1861 Riemann constructed [17] what is today known as the Riemann–Christoffel tensor. Let us start by reminding some simple results. The inverse metric $\mathbf{g}^{-1}$ is defined as the tensor with components g^{ij} satisfying

$$g^{ik}\, g_{jk} = \delta^i_j\,. \tag{33}$$

This is not only the definition of the inverse metric $\mathbf{g}^{-1}$ but also a linear algebraic system of equations; we have n^2 equations and n^2 unknowns, therefore the system has a unique solution. The determinant of the metric is given by

$$g = \det(g_{ij}) = \frac{1}{n!}\, \epsilon^{i_1 \cdots i_n}\, \epsilon^{j_1 \cdots j_n}\, g_{i_1 j_1} \cdots g_{i_n j_n}\,. \tag{34}$$

The condition for (33) to have a solution is $g \neq 0$. In this case we can define

$$g^{ij} = \frac{1}{g}\, \frac{1}{(n-1)!}\, \epsilon^{i i_1 \cdots i_{n-1}}\, \epsilon^{j j_1 \cdots j_{n-1}}\, g_{i_1 j_1} \cdots g_{i_{n-1} j_{n-1}}\,, \tag{35}$$

which satisfies (33). Therefore, g^{ij}, defined as in (35), is the inverse metric.

The transformation rules for $\mathbf{g}$ and its derivatives are given by

$$\begin{aligned} g_{ab}(\mathbf{y}) &= X^i{}_a\, X^j{}_b\, g_{ij}(\mathbf{x})\,, & (36)\\ \partial_c g_{ab}(\mathbf{y}) &= \left(X^i{}_{ac}\, X^j{}_b + X^i{}_a\, X^j{}_{bc}\right)\, g_{ij}(\mathbf{x}) + X^i{}_a\, X^j{}_b\, X^k{}_c\, \partial_k g_{ij}(\mathbf{x})\,, & (37)\\ \partial_{dc} g_{ab}(\mathbf{y}) &= \left[X^i{}_{dca}\, X^j{}_b + X^i{}_a\, X^j{}_{dcb} + X^i{}_{ca}\, X^j{}_{db} + X^i{}_{da}\, X^j{}_{cb}\right]\, g_{ij}(\mathbf{x}) \\ &\quad + \left[\left(X^i{}_{ca}\, X^j{}_b + X^i{}_a\, X^j{}_{cb}\right)\, X^k{}_d + \left(X^i{}_{da}\, X^j{}_b + X^i{}_a\, X^j{}_{db}\right)\, X^k{}_c \right. \\ &\quad \left. + X^i{}_a\, X^j{}_b\, X^k{}_{dc}\right]\, \partial_k g_{ij}(\mathbf{x}) \\ &\quad + X^i{}_a\, X^j{}_b\, X^k{}_c\, X^l{}_d\, \partial_{lk} g_{ij}(\mathbf{x})\,. & (38) \end{aligned}$$

There are several ways of constructing the invariant $\mathbf{R}$ through different tensor manipulations. The simplest way is to solve for the terms $(\partial^2 X)$; from (37) we obtain

$$X^k{}_{ab} = X^k{}_c\, \Gamma^c{}_{ab}(\mathbf{y}) - X^i{}_a\, X^j{}_b\, \Gamma^k{}_{ij}(\mathbf{x})\,, \tag{39}$$

where

$$\Gamma^c{}_{ab} = \frac{1}{2}\, g^{cd}\, \left(\partial_a g_{bd} + \partial_b g_{ad} - \partial_d g_{ab}\right)\,. \tag{40}$$

This manipulation can be done because of the existence of an inverse metric $\mathbf{g}^{-1}$. Let us now consider a further derivative of (39). We obtain

$$\begin{aligned} X^k{}_{abc} &= X^k{}_{cd}\, \Gamma^c{}_{ab}(\mathbf{y}) + X^k{}_c\, \partial_d \Gamma^c{}_{ab}(\mathbf{y}) \\ &\quad - X^i{}_{ac}\, X^j{}_b\, \Gamma^k{}_{ij}(\mathbf{x}) - X^i{}_a\, X^j{}_{bc}\, \Gamma^k{}_{ij}(\mathbf{x}) - X^i{}_a\, X^j{}_b\, X^l{}_c\, \partial_l \Gamma^k{}_{ij}(\mathbf{x})\,. \qquad (41) \end{aligned}$$

Using (39) we can eliminate the terms $(\partial^2 X)$. On the other hand, the left–hand side of (41) is completely symmetric and this fact allows to eliminate the derivatives $(\partial^3 X)$. We arrive then to the Riemann–Christoffel tensor.

It is obvious that already for this example there is an unavoidable (and, of course, undesired) proliferation of indices. In order to reduce the overabundance of indices and terms in several equations we make some simplifying assumptions. Let us observe that the invariant we want to construct is of the form

$$\mathbf{R} = \partial^2 \mathbf{g} + \mathbf{g}^{-1} (\partial \mathbf{g})^2 . \tag{42}$$

A sufficient condition for the vanishing of this invariant is that $\mathbf{g}$ be a constant tensor. Since $\mathbf{R}$ is a tensor it will also vanish in a second system of coordinates in which the tensor $\mathbf{g}$ is no more a constant tensor. In this second system of coordinates $\mathbf{R}$ is the simplest vanishing differential invariant which can be constructed in this way. Let us therefore choose the metric in the first system of coordinates as a constant

$$g_{b_1 b_2} = X^i{}_{b_1} X^j{}_{b_2} \eta_{ij} . \tag{43}$$

From a practical point of view this choice means that we must not mind about several terms involving derivatives on the other side of the relations. The first derivatives of this expression are given by

$$\begin{aligned} \partial_c g_{ab}(\mathbf{y}) &= \left[X^i{}_{ac} X^j{}_b + X^i{}_a X^j{}_{bc} \right] \eta_{ij} , & (44) \\ \partial_{dc} g_{ab}(\mathbf{y}) &= \left[X^i{}_{dca} X^j{}_b + X^i{}_a X^j{}_{dcb} + X^i{}_{ca} X^j{}_{db} + X^i{}_{da} X^j{}_{cb} \right] \eta_{ij} . & (45) \end{aligned}$$

Of course, all the manipulations above work properly for a second–rank tensor (they are specific). However, what we need is a construction method which can be used also for higher–rank tensors. In order to construct the invariant $\mathbf{R}$ in a systematic way which will be useful for generalizations to higher–ranks let us remind that the number of components means certain symmetries. Let us therefore consider

$$\begin{aligned} \partial_{[a_2[a_1} g_{b_1]b_2]} &= \partial_{a_2 a_1} g_{b_1 b_2} - \partial_{a_2 b_1} g_{a_1 b_2} - \partial_{b_2 a_1} g_{b_1 a_2} + \partial_{b_2 b_1} g_{a_1 a_2} \\ &= 2 \left(X^i{}_{a_2 b_1} X^j{}_{a_1 b_2} - X^i{}_{a_1 a_2} X^j{}_{b_1 b_2} \right) \eta_{ij} . \end{aligned} \tag{46}$$

Let us now remind that the terms $(\partial^2 X)$ can be solved from (44). The solution is

$$X^i{}_{b_1 b_2} = X^i{}_c \Gamma^c{}_{b_1 b_2} . \tag{47}$$

Therefore

$$\begin{aligned} R_{a_1 b_1 a_2 b_2} &= \frac{1}{2} \left(\partial_{a_2 a_1} g_{b_1 b_2} - \partial_{a_2 b_1} g_{a_1 b_2} - \partial_{b_2 a_1} g_{b_1 a_2} + \partial_{b_2 b_1} g_{a_1 a_2} \right) \\ &\quad - g_{cd} \left(\Gamma^c{}_{a_2 b_1} \Gamma^d{}_{a_1 b_2} - \Gamma^c{}_{a_1 a_2} \Gamma^d{}_{b_1 b_2} \right) = 0 . \end{aligned} \tag{48}$$

Therefore, the vanishing of the differential invariant $\mathbf{R}$ is the integrability condition for $\mathbf{g}$ to be of the form (43).

Now we introduce a simplification in the notation. The indices in the fixed reference frame play no role. Therefore, the expressions above can be simplified as follows. Let us rewrite (43) as

$$g_{ab} = X_a \cdot X_b \,. \tag{49}$$

The first derivatives of this expression are given by

$$\begin{aligned} \partial_c g_{ab}(\mathbf{y}) &= X_{ac} \cdot X_b + X_a \cdot X_{bc} \,, & (50) \\ \partial_{dc} g_{ab}(\mathbf{y}) &= X_{dca} \cdot X_b + X_a \cdot X_{dcb} + X_{ca} \cdot X_{db} + X_{da} \cdot X_{cb} \,. & (51) \end{aligned}$$

The antisymmetric part of (51) is given by

$$\partial_{[a_2[a_1} g_{b_1]b_2]} = 2\,(X_{a_2b_1} \cdot X_{a_1b_2} - X_{a_1a_2} \cdot X_{b_1b_2}) \,. \tag{52}$$

which is (46) leading to (48).

4 Higher–Rank Tensors

The possibility of implementing the method exposed previously relies on the possibility of inverting several relations. However, as we will see now, the definition of an inverse tensor for higher–rank tensors is not direct. As shown in [21] only even r can be constructed consistently. In order to fix the ideas, we illustrate them in the fourth–rank case.

Let us consider completely symmetric fourth–rank tensor G_{ijkl}. The inverse tensor $\mathbf{G}^{-1}$ is a tensor with components G^{ijkl} satisfying

$$G^{ik_1k_2k_3}\, G_{jk_1k_2k_3} = \delta^i_j \,. \tag{53}$$

This is the definition of the inverse tensor but now, in contrast with (33), there are more unknowns than relations and therefore the solution is not unique. In order to avoid this undeterminacy let us define the inverse tensor in a way similar to (34). The determinant of $\mathbf{G}$ is defined by

$$G = \det(G_{ijkl}) = \frac{1}{n!}\,\epsilon^{i_1\cdots i_n} \cdots \epsilon^{l_1\cdots l_n}\, G_{i_1j_1k_1l_1} \cdots G_{i_nj_nk_nl_n} \,. \tag{54}$$

If $G \neq 0$ we can define

$$G^{ijkl} = \frac{1}{G}\,\frac{1}{(n-1)!}\,\epsilon^{im_1\cdots m_{n-1}} \cdots \epsilon^{lq_1\cdots q_{n-1}}\, G_{m_1n_1p_1q_1} \cdots G_{m_{n-1}m_{n-1}p_{n-1}q_{n-1}} \,. \tag{55}$$

This tensor satisfies (53). The expression above can be generalized to

$$
\begin{aligned}
& G^{i_1 j_1 k_1 l_1}\, G^{i_2 j_2 k_2 l_2} \\
& - \left(G^{i_2 j_1 k_1 l_1}\, G^{i_1 j_2 k_2 l_2} + G^{i_1 j_2 k_1 l_1}\, G^{i_2 j_1 k_2 l_2} + G^{i_1 j_1 k_2 l_1}\, G^{i_2 j_2 k_1 l_2} + G^{i_1 j_1 k_1 l_2}\, G^{i_2 j_2 k_2 l_1} \right) \\
& + \left(G^{i_2 j_2 k_1 l_1}\, G^{i_1 j_1 k_2 l_2} + G^{i_2 j_1 k_2 l_1}\, G^{i_1 j_2 k_1 l_2} + G^{i_2 j_1 k_1 l_2}\, G^{i_1 j_2 k_2 l_1} \right) \\
= & \frac{1}{G} \frac{1}{(n-2)!} \epsilon^{i_1 i_2 m_1 \cdots m_{n-2}} \cdots \epsilon^{l_1 l_2 q_1 \cdots q_{n-2}}\, G_{m_1 n_1 p_1 q_1} \cdots G_{m_{n-2} n_{n-2} p_{n-2} q_{n-2}} \,. \qquad (56)
\end{aligned}
$$

Contracting with $G_{i_2 j_2 k_2 l_2}$ we obtain that the inverse tensor (55) also satisfies the relation

$$
G_{(ij|mn}\, G^{mnpq}\, G_{pq|kl)} = G_{ijkl}\,. \qquad (57)
$$

By the way, this relation can be used as a better definition for the inverse tensor since now the number of equations and the number of unknowns are equal.

Let us now outline the construction of invariants. Let us start considering the transformation rule for $\mathbf{G}$, that is,

$$
G_{a_1 a_2 a_3 a_4} = X^i{}_{a_1}\, X^j{}_{a_2}\, X^k{}_{a_3}\, X^l{}_{a_4}\, G_{ijkl}\,. \qquad (58)
$$

The invariant we want to construct is of the form

$$
\mathbf{R} = \partial^4 \mathbf{G} + \mathbf{G}^{-1} (\partial^3 \mathbf{G})(\partial \mathbf{G}) + \mathbf{G}^{-1} (\partial^2 \mathbf{G})^2 + \mathbf{G}^{-2} (\partial^2 \mathbf{G})(\partial \mathbf{G})^2 + \mathbf{G}^{-3} (\partial \mathbf{G})^4\,. \qquad (59)
$$

A sufficient condition for the vanishing of this invariant is that $\mathbf{G}$ be a constant tensor. Let us therefore write

$$
G_{a_1 a_2 a_3 a_4} = X^i{}_{a_1}\, X^j{}_{a_2}\, X^k{}_{a_3}\, X^l{}_{a_4}\, \eta_{ijkl} = X_{a_1} \cdot X_{a_2} \cdot X_{a_3} \cdot X_{a_4}\,, \qquad (60)
$$

where we assume that η_{ijkl} is a constant tensor. Then, as in the second–rank case, the differential invariant we are looking for appears, as the integrability condition for $\mathbf{G}$, to be of the form (60). The first derivatives of (60) are given by

$$
\begin{aligned}
\partial_b G_{a_1 a_2 a_3 a_4} = & X_{a_1 b} \cdot X_{a_2} \cdot X_{a_3} \cdot X_{a_4} + X_{a_1} \cdot X_{a_2 b} \cdot X_{a_3} \cdot X_{a_4} \\
& + X_{a_1} \cdot X_{a_2} \cdot X_{a_3 b} \cdot X_{a_4} + X_{a_1} \cdot X_{a_2} \cdot X_{a_3} \cdot X_{a_4 b}\,, \qquad (61) \\
\partial_{b_1 b_2} G_{a_1 a_2 a_3 a_4} = & [X_{a_1 b_1 b_2} \cdot X_{a_2} \cdot X_{a_3} \cdot X_{a_4} + X_{a_1} \cdot X_{a_2 b_1 b_2} \cdot X_{a_3} \cdot X_{a_4} \\
& + X_{a_1} \cdot X_{a_2} \cdot X_{a_3 b_1 b_2} \cdot X_{a_4} + X_{a_1} \cdot X_{a_2} \cdot X_{a_3} \cdot X_{a_4 b_1 b_2}] \\
& + \Big[X_{a_1 (b_1|} \cdot X_{a_2 |b_2)} \cdot X_{a_3} \cdot X_{a_4} + X_{a_1 (b_1|} \cdot X_{a_2} \cdot X_{a_3 |b_2)} \cdot X_{a_4} \\
& + X_{a_1 (b_1|} \cdot X_{a_2} \cdot X_{a_3} \cdot X_{a_4 |b_2)} + X_{a_1} \cdot X_{a_2 (b_1|} \cdot X_{a_3 |b_2)} \cdot X_{a_4}
\end{aligned}
$$

$$
\begin{aligned}
& + X_{a_1} \cdot X_{a_2(b_1|} \cdot X_{a_3} \cdot X_{a_4|b_2)} + X_{a_1} \cdot X_{a_2} \cdot X_{a_3(b_1|} \cdot X_{a_4|b_2)} \Big] \,, \qquad (62)\\
\partial_{b_1b_2b_3} G_{a_1a_2a_3a_4} = & \; [X_{a_1b_1b_2b_3} \cdot X_{a_2} \cdot X_{a_3} \cdot X_{a_4} + X_{a_1} \cdot X_{a_2b_1b_2b_3} \cdot X_{a_3} \cdot X_{a_4} \\
& + X_{a_1} \cdot X_{a_2} \cdot X_{a_3b_1b_2b_3} \cdot X_{a_4} + X_{a_1} \cdot X_{a_2} \cdot X_{a_3} \cdot X_{a_4b_1b_2b_3}] \\
& + \Big[X_{a_1(b_1b_2|} \cdot X_{a_2|b_3)} \cdot X_{a_3} \cdot X_{a_4} + X_{a_1(b_1b_2|} \cdot X_{a_2} \cdot X_{a_3|b_3)} \cdot X_{a_4} \\
& + X_{a_1(b_1b_2|} \cdot X_{a_2} \cdot X_{a_3} \cdot X_{a_4|b_3)} + X_{a_1} \cdot X_{a_2(b_1b_2|} \cdot X_{a_3|b_3)} \cdot X_{a_4} \\
& + X_{a_1} \cdot X_{a_2(b_1b_2|} \cdot X_{a_3} \cdot X_{a_4|b_3)} + X_{a_1} \cdot X_{a_2} \cdot X_{a_3(b_1b_2|} \cdot X_{a_4|b_3)} \\
& + X_{a_1(b_1|} \cdot X_{a_2|b_2b_3)} \cdot X_{a_3} \cdot X_{a_4} + X_{a_1(b_1|} \cdot X_{a_2} \cdot X_{a_3|b_2b_3)} \cdot X_{a_4} \\
& + X_{a_1(b_1|} \cdot X_{a_2} \cdot X_{a_3} \cdot X_{a_4|b_2b_3)} + X_{a_1} \cdot X_{a_2(b_1|} \cdot X_{a_3|b_2b_3)} \cdot X_{a_4} \\
& + X_{a_1} \cdot X_{a_2(b_1|} \cdot X_{a_3} \cdot X_{a_4|b_2b_3)} + X_{a_1} \cdot X_{a_2} \cdot X_{a_3(b_1|} \cdot X_{a_4|b_2b_3)} \Big] \\
& + \Big[X_{a_1(b_1|} \cdot X_{a_2|b_2|} \cdot X_{a_3|b_3)} \cdot X_{a_4} + X_{a_1(b_1|} \cdot X_{a_2|b_2|} \cdot X_{a_3} \cdot X_{a_4|b_3)} \\
& + X_{a_1(b_1|} \cdot X_{a_2} \cdot X_{a_3|b_2|} \cdot X_{a_4|b_3)} + X_{a_1} \cdot X_{a_2(b_1|} \cdot X_{a_3|b_2|} \cdot X_{a_4|b_3)} \Big] \,, \qquad (63)\\
\partial_{b_1b_2b_3b_4} G_{a_1a_2a_3a_4} = & \; [X_{a_1b_1b_2b_3b_4} \cdot X_{a_2} \cdot X_{a_3} \cdot X_{a_4} + X_{a_1} \cdot X_{a_2b_1b_2b_3b_4} \cdot X_{a_3} \cdot X_{a_4} \\
& + X_{a_1} \cdot X_{a_2} \cdot X_{a_3b_1b_2b_3} \cdot X_{a_4} + X_{a_1} \cdot X_{a_2} \cdot X_{a_3} \cdot X_{a_4b_1b_2b_3b_4}] \\
& + \Big[X_{a_1(b_1b_2b_3|} \cdot X_{a_2|b_4)} \cdot X_{a_3} \cdot X_{a_4} + X_{a_1(b_1b_2b_3|} \cdot X_{a_2} \cdot X_{a_3|b_4)} \cdot X_{a_4} \\
& + X_{a_1(b_1b_2b_3|} \cdot X_{a_2} \cdot X_{a_3} \cdot X_{a_4|b_4)} + X_{a_1} \cdot X_{a_2(b_1b_2b_3|} \cdot X_{a_3|b_4)} \cdot X_{a_4} \\
& + X_{a_1} \cdot X_{a_2(b_1b_2b_3|} \cdot X_{a_3} \cdot X_{a_4|b_4)} + X_{a_1} \cdot X_{a_2} \cdot X_{a_3(b_1b_2b_3|} \cdot X_{a_4|b_4)} \\
& + X_{a_1(b_1|} \cdot X_{a_2|b_2b_3b_4)} \cdot X_{a_3} \cdot X_{a_4} + X_{a_1(b_1|} \cdot X_{a_2} \cdot X_{a_3|b_2b_3b_4)} \cdot X_{a_4} \\
& + X_{a_1(b_1|} \cdot X_{a_2} \cdot X_{a_3} \cdot X_{a_4|b_2b_3b_4)} + X_{a_1} \cdot X_{a_2(b_1|} \cdot X_{a_3|b_2b_3b_4)} \cdot X_{a_4} \\
& + X_{a_1} \cdot X_{a_2(b_1|} \cdot X_{a_3} \cdot X_{a_4|b_2b_3b_4)} + X_{a_1} \cdot X_{a_2} \cdot X_{a_3(b_1|} \cdot X_{a_4|b_2b_3b_4)} \Big] \\
& + \Big[X_{a_1(b_1b_2|} \cdot X_{a_2|b_3b_4)} \cdot X_{a_3} \cdot X_{a_4} + X_{a_1(b_1b_2|} \cdot X_{a_2} \cdot X_{a_3|b_3b_4)} \cdot X_{a_4} \\
& + X_{a_1(b_1b_2|} \cdot X_{a_2} \cdot X_{a_3} \cdot X_{a_4|b_3b_4)} + X_{a_1} \cdot X_{a_2(b_1b_2|} \cdot X_{a_3|b_3b_4)} \cdot X_{a_4} \\
& + X_{a_1} \cdot X_{a_2(b_1b_2|} \cdot X_{a_3} \cdot X_{a_4|b_3b_4)} + X_{a_1} \cdot X_{a_2} \cdot X_{a_3(b_1b_2|} \cdot X_{a_4|b_3b_4)} \Big] \\
& + \Big[X_{a_1(b_1b_2|} \cdot X_{a_2|b_3|} \cdot X_{a_3|b_4)} \cdot X_{a_4} + X_{a_1(b_1b_2|} \cdot X_{a_2|b_3|} \cdot X_{a_3} \cdot X_{a_4|b_4)} \\
& + X_{a_1(b_1b_2|} \cdot X_{a_2} \cdot X_{a_3|b_3|} \cdot X_{a_4|b_4)} + X_{a_1(b_1|} \cdot X_{a_2|b_2b_3|} \cdot X_{a_3|b_4)} \cdot X_{a_4} \\
& + X_{a_1(b_1|} \cdot X_{a_2|b_2b_3|} \cdot X_{a_3} \cdot X_{a_4|b_4)} + X_{a_1} \cdot X_{a_2(b_1b_2|} \cdot X_{a_3|b_3|} \cdot X_{a_4|b_4)} \\
& + X_{a_1(b_1|} \cdot X_{a_2|b_2|} \cdot X_{a_3|b_3b_4)} \cdot X_{a_4} + X_{a_1(b_1|} \cdot X_{a_2} \cdot X_{a_3|b_2b_3|} \cdot X_{a_4|b_4)} \\
& + X_{a_1} \cdot X_{a_2(b_1b_2|} \cdot X_{a_3|b_3|} \cdot X_{a_4|b_4)} + X_{a_1(b_1|} \cdot X_{a_2|b_2|} \cdot X_{a_3} \cdot X_{a_4|b_3b_4)} \\
& + X_{a_1(b_1|} \cdot X_{a_2} \cdot X_{a_3|b_2|} \cdot X_{a_4|b_3b_4)} + X_{a_1} \cdot X_{a_2(b_1|} \cdot X_{a_3|b_2|} \cdot X_{a_4|b_3b_4)} \Big] \\
& + X_{a_1(b_1|} \cdot X_{a_2|b_2|} \cdot X_{a_3|b_3|} \cdot X_{a_4|b_4)} \,. \qquad (64)
\end{aligned}
$$

Then, the derivatives above contain terms of the form

$$
\partial^4 \mathbf{G} = (\partial^4\Lambda)\,\Lambda^3 + (\partial^3\Lambda)\,(\partial\Lambda)\,\Lambda^2 + (\partial^2\Lambda)^2\,\Lambda^2 + (\partial^2\Lambda)\,(\partial\Lambda)^2\,\Lambda + (\partial\Lambda)^4 \,, \qquad (65)
$$

$$\partial^3\mathbf{G} = (\partial^3\Lambda)\,\Lambda^3 + (\partial^2\Lambda)\,(\partial\Lambda)\,\Lambda^2 + (\partial\Lambda)^3\,\Lambda\,, \tag{66}$$

$$\partial^2\mathbf{G} = (\partial^2\Lambda)\,\Lambda^3 + (\partial\Lambda)^2\,\Lambda^2\,, \tag{67}$$

$$\partial\mathbf{G} = (\partial\Lambda)\,\Lambda^3\,. \tag{68}$$

where $\Lambda = \partial X$.

The way in which the different terms are combined is given by the comments following (26). Then, we have

$$R_{i_1i_2i_3i_4j_1j_2j_3j_4} = S_{i_1i_2i_3i_4j_1j_2j_3j_4} + (\text{something})\,, \tag{69}$$

where

$$\begin{aligned} S_{i_1i_2i_3i_4j_1j_2j_3j_4} = \; & \partial_{i_1i_2i_3i_4}G_{j_1j_2j_3j_4} \\ & -(\partial_{j_1i_2i_3i_4}G_{i_1j_2j_3j_4} + \partial_{i_1j_2i_3i_4}G_{j_1i_2j_3j_4} \\ & \quad + \partial_{i_1i_2j_3i_4}G_{j_1j_2i_3j_4} + \partial_{i_1i_2i_3j_4}G_{j_1j_2j_3i_4}) \\ & +(\partial_{j_1j_2i_3i_4}G_{i_1i_2j_3j_4} + \partial_{j_1i_2j_3i_4}G_{i_1j_2i_3j_4} + \partial_{j_1i_2i_3j_4}G_{i_1j_2j_3i_4} \\ & \quad + \partial_{i_1j_2j_3i_4}G_{j_1i_2i_3j_4} + \partial_{i_1j_2i_3j_4}G_{j_1i_2j_3i_4} + \partial_{i_1i_2j_3j_4}G_{j_1j_2i_3i_4}) \\ & -(\partial_{i_1j_2j_3j_4}G_{j_1i_2i_3i_4} + \partial_{j_1i_2j_3j_4}G_{i_1j_2i_3i_4} \\ & \quad + \partial_{j_1j_2i_3j_4}G_{i_1i_2j_3i_4} + \partial_{j_1j_2j_3i_4}G_{i_1i_2i_3j_4}) \\ & +\partial_{j_1j_2j_3j_4}G_{i_1i_2i_3i_4}\,. \end{aligned} \tag{70}$$

and "something" contains derivatives of $\mathbf{G}$ of lesser order in a combination such as to cancel all transformation matrices appearing there.

It must be by now clear the kind of algebraic manipulations necessary to construct the desired invariant and that they are quite involved. Work is in progress to develop a calculational algorithm to determine the full expression of the differential invariant involving the non–linear terms and the inverse tensor (55).

5 Conclusions

We have outlined the construction of differential invariants for higher–rank tensors.

Acknowledgments

This work was partially done at the Abdus Salam International Centre for Theoretical Physics, Trieste, Italy. The work was partially supported by Dirección de Investigación — Bogotá, Universidad Nacional de Colombia.

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