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MICROCANONICAL ENSEMBLE  
EXTENSIVE THERMODYNAMICS  
OF TSALLIS STATISTICS

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Экстенсивная термодинамика микроканонического ансамбля  
в обобщенной статистической механике Цаллиса

На основе идеи Гиббса о статистических ансамблях классической и квантовой механики дано микроскопическое обоснование обобщенной равновесной статистической механики с энтропией Цаллиса. Для вывода явной зависимости равновесных распределений от переменных состояния предложен термодинамический метод, основанный на фундаментальном уравнении термодинамики и статистическом определении функций состояния макроскопической системы. В микроканоническом ансамбле показано, что если энтропийный индекс  $\xi = 1/q - 1$  является экстенсивной переменной состояния системы, то в термодинамическом пределе,  $\tilde{z} = 1/(q - 1)N = \text{const}$ , выполняется термодинамический принцип аддитивности и нулевое начало термодинамики. В частности, энтропия экстенсивна, а температура интенсивна. В этом случае статистическая механика с энтропией Цаллиса полностью удовлетворяет всем постулатам равновесной термодинамики. При этом все термодинамические соотношения микроканонического ансамбля выводятся из теоремы Эйлера. На примере микроканонического идеального газа в термодинамическом пределе было показано в явном виде выполнение теоремы Эйлера и термодинамического принципа аддитивности.

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Microcanonical Ensemble Extensive Thermodynamics  
of Tsallis Statistics

The microscopic foundation of the generalized equilibrium statistical mechanics based on the Tsallis entropy is given by using the Gibbs idea of statistical ensembles of the classical and quantum mechanics. The equilibrium distribution functions are derived by the thermodynamic method based upon the use of the fundamental equation of thermodynamics and the statistical definition of the functions of the state of the system. It is shown that if the entropic index  $\xi = 1/q - 1$  in the microcanonical ensemble is an extensive variable of the state of the system, then in the thermodynamic limit  $\tilde{z} = 1/(q - 1)N = \text{const}$  the principle of additivity and the zero law of thermodynamics are satisfied. In particular, the Tsallis entropy of the system is extensive and the temperature is intensive. Thus, the Tsallis statistics completely satisfies all the postulates of the equilibrium thermodynamics. Moreover, evaluation of the thermodynamic identities in the microcanonical ensemble is provided by the Euler theorem. The principle of additivity and the Euler theorem are explicitly proved by using the illustration of the classical microcanonical ideal gas in the thermodynamic limit.

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## INTRODUCTION

The equilibrium statistical mechanics and thermodynamics are well defined theories in modern physics [1,2]. Applications of these theories are restricted by investigation of the so-called thermodynamic or statistical systems which are constrained by several rigid requirements [3]. One of the first attempts to construct the generalized equilibrium statistical mechanics based on the mathematical redefinition of the Boltzmann–Gibbs statistical entropy and the principles of the information theory belongs to C. Tsallis [4]. Until recently, there has been a great deal of interest in studying nonextensive thermodynamics due to its relevance in many fields of physics [5,6]. However, many fundamental features regarding the violation of the zero law of thermodynamics and the principle of additivity remain unclear [7,8]. Note that these difficulties have resulted in the occurrence of a large number of variants of the Tsallis generalized statistical mechanics [9].

The statistical mechanics investigates thermodynamic systems which are defined solely by the specification of macroscopic variables on the basis of the theory of probability and the microscopic laws of the classical and quantum mechanics. The evolution of the macroscopic system with a large number of degrees of freedom is impossible to describe by only dynamic methods. Therefore, the Gibbs idea of statistical ensembles is usually used [10]. All information about the macrostate of the system is contained in the phase distribution function, which evolves according to the Liouville equation, or in the statistical operator, whose evolution with time is described by the von Neumann equation. To derive the phase distribution function and the statistical operator is the primary goal of the nonequilibrium statistical mechanics.

In particular, the equilibrium statistical mechanics implies that one uses the Gibbs equilibrium statistical ensembles. In the state of thermodynamic equilibrium of the system, the phase distribution function and the statistical operator do not depend on time. Therefore, they are functions only of the first integrals of motion of the dynamic system. In this case, the mechanical laws and the Liouville and von Neumann equations do not allow one to determine unequivocally the equilibrium distribution function and the statistical operator [2,10]. Therefore, an obvious dependence of the equilibrium distributions on the macroscopic variables of the state of the system is defined by introducing additional postulates. The traditional way is based on the Gibbs postulate of the equiprobability of the dynamic states of the isolated system [1]. The alternative way rests on the Jaynes principle of a maximum of the information entropy [11]. The statistical mechanics constructed on the Gibbs equilibrium distributions, which corresponds

to the Boltzmann–Gibbs statistical entropy, completely satisfies all postulates of the equilibrium thermodynamics.

Standard treatments of the Tsallis statistics point out that the entropic index  $q$  is an additional intensive parameter, which has a fixed value for different thermodynamic systems [9]. This concept leads to shortcomings of thermodynamics and needs to be reconsidered. As shown further, these problems can be resolved by the assumption that the parameter  $\xi = 1/q - 1$  is the extensive argument of the statistical entropy.

The paper is organized as follows. In the second section, the Gibbs equilibrium statistical ensembles in quantum and classical mechanics are considered. In the third section, the principal prepositions of the equilibrium thermodynamics are resumed. In the fourth section, the definitions of the information and statistical entropies are given. The microcanonical equilibrium distribution function and statistical operator are deduced in the fifth section. In the sixth section, the performance of the thermodynamic principles in the microcanonical ensemble of the Tsallis statistics are proved. The developed formalism is exemplified in the seventh section by treating the classical microcanonical ideal gas.

## 1. EQUILIBRIUM STATISTICAL ENSEMBLES

In the present section, we review the basic postulates of the microscopic foundation of the statistical mechanics and the dynamic methods describing the quantum and classical many-body systems [2, 10].

Evolution with time of the macroscopic system with a large number of degrees of freedom can never be explained by purely dynamical laws by virtue of several reasons [10]. First of all, for classical systems there exists an uncertainty of the initial states. A small discrepancy in the initial conditions results eventually in a large uncertainty of a final dynamic state. It is the subject of the so-called chaotic dynamics. Secondly, the real systems are not completely isolated. Therefore, some degrees of freedom and external influences are not included in the equations of motion. Thus, in classical mechanics a macrostate of a system is imperfectly known at the microscopic level. The system can be found in any dynamic state compatible with the external macroscopic conditions. Therefore, for the macroscopic system it is possible to maintain only the probabilistic description of dynamic processes. For this reason, in the statistical mechanics the Gibbs idea of statistical ensembles is straightforward. The macroscopic state of the system is represented as a set of a large number of copies of the dynamic system under identical macroscopic conditions. Each system of the ensemble is represented by a point in phase space  $(x, p)$ , which moves along its own trajectory, according to the Hamilton equations of motion. Note that in the classical mechanics a dynamic state of the system with  $f$  degrees of freedom is defined by a point

$(x, p) = (x_1, \dots, x_f, p_1, \dots, p_f)$  in the  $2f$ -measured phase space  $\Gamma$ , i.e. by a set of the generalized coordinates  $(x) = (x_1, \dots, x_f)$  and the momenta  $(p) = (p_1, \dots, p_f)$ .

Any physical observable  $A$  of the macroscopic system is represented as the expectation value  $\langle A \rangle^t$  of the dynamic variable  $A(x, p, t)$ :

$$\langle A \rangle^t = \int A(x, p, t) \varrho(x, p, t) d\Gamma, \quad (1)$$

where  $\varrho(x, p, t)$  is the phase distribution function which is proportional to the density of probability distribution of systems of the ensemble in phase space. The distribution function must satisfy the normalization condition

$$\int \varrho(x, p, t) d\Gamma = 1. \quad (2)$$

Here  $d\Gamma$  is an infinitesimal element of phase space and  $dw(x, p, t) = \varrho(x, p, t) d\Gamma$  is a probability to find the system of the ensemble in an element  $d\Gamma$  near a phase point  $(x, p)$  at the moment of time  $t$ . For a dynamic variable  $A(x, p, t)$  the equation of motion is given by

$$\frac{dA}{dt} = \frac{\partial A}{\partial t} + \{A, H\}, \quad (3)$$

where  $\{\varphi_1, \varphi_2\}$  is the Poisson bracket for two arbitrary phase functions  $\varphi_1, \varphi_2$ . The evolution with time of a phase distribution function is governed by

$$\frac{\partial \varrho}{\partial t} = \{H, \varrho\}, \quad (4)$$

called the Liouville equation, with the initial conditions  $\varrho(x, p, t)|_{t=t_0} = \varrho(x, p, t_0)$ . The number of points of phase space representing systems of the statistical ensemble remains unchanged in time during the Hamiltonian evolution. All points located at the moment of time  $t$  in an element of phase space  $d\Gamma$  appear at the moment of time  $t'$  in some other element  $d\Gamma'$ . According to Liouville theorem, the volume of a region in phase space remains constant in the process of movement of phase points. Therefore, the phase distribution function is constant along the phase trajectories,  $\varrho(x, p, t) = \varrho(x', p', t')$ ; besides, the total time derivative of the distribution function is equal to zero,  $d\varrho/dt = 0$ . So that to find the expectation value of an observable (1) it is necessary to solve the Hamilton equations for each phase point  $(x, p)$  of the ensemble and find solutions of the equation of motion (3) for a dynamic variable  $A(x, p, t)$  and of the Liouville equation (4).

The description of the quantum many-body system based on the Schrödinger equation is a hopeless task for several reasons [2, 10]. Indeed, the exact solutions of the Schrödinger equation for most physical systems cannot be obtained

explicitly. The macroscopic system is experimentally impossible to be prepared in a pure quantum state. Moreover, a real system is not completely isolated. Thus, in the Hamiltonian it is impossible to take into account the contribution of all degrees of freedom connected with the external influences. Therefore, to describe the quantum systems the mixed states are considered. A macrostate thus appears as a set of possible microstates, which are set up by state vectors  $|\Psi_r(t)\rangle$ ,  $r = 1, 2, \dots$ , each with its own probability  $w_r$  for its occurrence. The probabilities are positive and normalized,

$$\sum_r w_r = 1, \quad w_r \geq 0. \quad (5)$$

Note that in the quantum mechanics a dynamic state of the quantum system is defined by a vector of state  $|\Psi(t)\rangle$ , which is an element of the abstract Hilbert space  $\mathcal{E}_H$ . A vector of state is normalized to unity. The evolution with time of the dynamic state is determined by the Schrödinger equation with the Hamiltonian  $H$ , which is the linear hermitian operator acting on the vectors of state  $|\Psi(t)\rangle$ . In the quantum mechanics, the dynamic variables are represented by linear hermitian operators  $A$  acting on the elements of the Hilbert space. Note that the dynamic states  $|\Psi(t)\rangle$  represent pure quantum states.

The expectation value of any dynamic variable  $A$  is defined by the following expression:

$$\langle A \rangle^t = \text{Tr} [A \varrho(t)], \quad (6)$$

where  $\varrho(t)$  is the statistical operator which allows one to determine the expectation value of a physical observable regardless of the choice of the set of quantum states  $\{|\Psi_r(t)\rangle\}$ :

$$\varrho(t) = \sum_r w_r |\Psi_r(t)\rangle \langle \Psi_r(t)|. \quad (7)$$

The statistical operator is normalized to unity:

$$\text{Tr} \varrho(t) = 1, \quad (8)$$

it is hermitian  $\varrho^\dagger(t) = \varrho(t)$  and is positive, i.e. has no negative eigenvalues as probabilities  $w_r \geq 0$  are eigenvalues of the statistical operator,  $\varrho(t)|\Psi_r(t)\rangle = w_r |\Psi_r(t)\rangle$ . The evolution with time of a statistical operator is governed by

$$\frac{\partial \varrho(t)}{\partial t} = \frac{1}{i\hbar} [H, \varrho(t)], \quad (9)$$

called the von Neumann equation. Here, the brackets designate the commutator of two operators. The von Neumann equation allows one to find the statistical operator at any time  $t$  if it is known during some initial moment  $t_0$ . The equation

of motion for a dynamic variable  $A_t$ , which explicitly depends on time, can be written as

$$\frac{dA_t}{dt} = \frac{\partial A_t}{\partial t} + \frac{1}{i\hbar}[A_t, H_t]. \quad (10)$$

Further, we shall touch upon the issue concerning the microscopic foundation of the equilibrium statistical mechanics based on the Gibbs idea of the equilibrium statistical ensembles. In the state of thermal equilibrium of the macroscopic system, the phase distribution function and the statistical operator should not depend on the time  $t$ . Therefore, for classical and quantum equilibrium distributions  $\varrho_{\text{eq}}$  the Liouville and von Neumann equations, respectively, become

$$\{H, \varrho_{\text{eq}}\} = 0, \quad [H, \varrho_{\text{eq}}] = 0. \quad (11)$$

From these equations it follows that the equilibrium distribution function and the statistical operator are the first integrals of motion and, consequently, they must depend only on the first integrals of motion of the system. Moreover, if these quantities are unequivocal and additive, then there exist only four such integrals of motion: energy  $H$ , the total momentum vector  $\mathbf{P}$ , the total angular momentum vector  $\mathbf{M}$ , and the number of particles  $N$ . In an equilibrium state, the total momentum and the total angular momentum are equal to zero, and they cannot be taken into account if the system is in a motionless vessel. Thus, for the classical and quantum systems with the fixed number of particles the phase distribution function and the statistical operator results in the expressions

$$\varrho_{\text{eq}}(x, p) \equiv \varrho_{\text{eq}}(H(x, p)), \quad \varrho_{\text{eq}} \equiv \varrho_{\text{eq}}(H), \quad (12)$$

which are normalized to unity. The expectation value of an observable for the classical and quantum system, respectively, are given by the following formulae:

$$\langle A \rangle = \int A(x, p) \varrho_{\text{eq}}(x, p) d\Gamma, \quad \langle A \rangle = \text{Tr} [A \varrho_{\text{eq}}]. \quad (13)$$

Note that the mechanical laws and the Liouville or von Neumann equations do not determine unequivocally the equilibrium distribution function and the equilibrium statistical operator. To express the equilibrium distributions from the independent variables of state, the introduction of additional postulates is required. A traditional way to construct the equilibrium distributions is based on the Gibbs postulate of the equiprobability of all accessible dynamic states of the isolated system [1]. An alternative way for this is based on the statistical definition of the entropy and the use of the Jaynes principle explored in the information theory [11]. In the present study, we suggest a new method based on the laws of the equilibrium thermodynamics.

## 2. EQUILIBRIUM THERMODYNAMICS

As already mentioned, the obvious dependence of the equilibrium distribution function and the statistical operator from the macroscopic variables of state will have to be justified eventually by considerations based on equilibrium thermodynamics. Therefore, in this section, we briefly recall the general laws of the macroscopic thermodynamics and give the definition to the thermodynamic entropy.

The thermodynamic systems, which are the object of the equilibrium thermodynamics, are formed by the contents of a geometrical volume of macroscopic dimensions, the boundary of which separates the outside world from the system [12]. The type of interaction between the system and the external environment is determined by the exchanges of energy (heat and work) and matter through the boundaries of this system or its macroscopic parts. Mathematically, the macroscopic state of the thermodynamic system is fixed by the set of independent variables of state. The functions of the state of the system depend on the variables of state and characterize the global properties of this system. In thermodynamics, there are variables which have mechanical analogues and which have only specifically thermodynamic character. The work  $\delta W$  is connected with a change of mechanical parameters of the system. On the other hand, the heat transfer through heat-conducting walls is concerned with a change of specifically thermodynamic parameters of the system,  $\delta Q = CdT$ , where  $C$  is the heat capacity. The thermodynamic system is given completely by the equations of state and the caloric equation. Note that in the equilibrium thermodynamics the system at all time passes through the equilibrium states as a quasi-static reversible process.

Thermodynamic systems must satisfy some obligatory conditions [3]. First, these are the systems of a large number of particles interacting with each other and with external fields. Usually, the scale of measurement of the number of particles in the system is the Avogadro number  $N_0$ . Thus, the number of particles of the system is limited not only from below  $N \gg 1$  but also from above. The macroscopic system is commensurable with the researcher and his experimental devices of the laboratory sizes for which the ratio  $N/N_0$  is a finite number. Systems of the scale of the universe or its parts cannot be thermodynamic. Second, for every thermodynamic system the zero law of thermodynamics is fulfilled, i.e. for such a system there exists a state of thermal equilibrium, which eventually is reached by the system at the fixed external conditions. In the macroscopic theory, the zero principle is a generalization of daily experience and supervision over the thermodynamic systems. It is necessary to note two important properties of the state of thermal equilibrium. First, this is a dynamic state. In the system there exist a chaotic thermal motion of particles and a different sort of fluctuation of both the parameters of state of the system and the fluxes of particles, energy, etc. However, these fluctuations are so small that the system practically does not

leave the state of equilibrium. Second, for the equilibrium state the property of transitivity is valid. This property guaranties the existence of the special thermal measure, the temperature  $T$ , which is a general characteristic of all thermodynamic systems in equilibrium contact and which does not depend on the place and the method of measurements.

Third, for thermodynamic systems the principle of additivity is valid: all variables belong to two classes of additivity, according to the reaction of a given physical one to the division of the equilibrium system into the equilibrium macroscopic parts, for example, into two parts

$$\text{System}(1 + 2) \longrightarrow \text{System}(1) + \text{System}(2). \quad (14)$$

The extensive variables can be split into two parts, and they should be proportional to the actual amount of matter present:

$$\mathcal{F}_{1+2} = \mathcal{F}_1 + \mathcal{F}_2. \quad (15)$$

On the other hand, the intensive variables have to keep its values and cannot depend on the size of the system

$$\phi_{1+2} = \phi_1 = \phi_2. \quad (16)$$

As an example, we may consider the thermodynamic systems which may be fixed in terms of the macroscopic variables of state  $T, V, N$ . In this case, the thermodynamic principle of additivity is implemented if intensive quantities are functions of intensive arguments, and extensive variables are proportional to the number of particles of the system multiplied by the intensive quantity. Such dependence of extensive and intensive variables is provided by the thermodynamic limit [3]. In this respect, all expressions have to be exposed to a formal limiting procedure  $N \rightarrow \infty, V \rightarrow \infty, v = V/N = \text{const}$ , and only main asymptotics on  $N$  should be kept. Then the extensive variables  $\mathcal{F}$  can be written ( $\alpha > 0$ )

$$\mathcal{F}(T, V, N) \Big|_{v=\text{const}}^{N \rightarrow \infty} = N(f(T, v) + O(N^{-\alpha})) \stackrel{\text{as}}{\cong} Nf(T, v), \quad (17)$$

whereas the intensive variables  $\phi$  take the following form:

$$\phi(T, V, N) \Big|_{v=\text{const}}^{N \rightarrow \infty} = \phi(T, v) + O(N^{-\alpha}) \stackrel{\text{as}}{\cong} \phi(T, v), \quad (18)$$

where  $v = V/N$  is the specific volume and  $f = \mathcal{F}/N$  is the specific  $\mathcal{F}$ . Note that the thermodynamic limit is a one-limiting procedure. The transitions not coordinated among themselves  $N \rightarrow \infty$  and  $V \rightarrow \infty$  have no physical sense, as in this case we would get results for either the superdense system or the empty one.

Fourth, in relation to the thermodynamic systems the first, the second and the third principles of thermodynamics are fulfilled, being the mathematical basis of the macroscopic theory. The first principle postulates the energy conservation law. In an open system, the energy  $dE$  supplied by the exterior is equal to the sum of the heat  $\delta Q$ , the mechanical work  $\delta W$ , performed at the boundaries of the system, and the exchange of matter  $dE_{\text{mat}}$  with the environment

$$dE = \delta Q - \delta W + dE_{\text{mat}}, \quad (19)$$

where  $\delta W = pdV + Xdz$ ;  $z = (z_1, \dots, z_k)$  and  $V$  are the «thermodynamic coordinates»;  $X = (X_1, \dots, X_k)$  and  $p$  play the role of the associated «forces»,  $dE_{\text{mat}} = \mu dN$  is the change of energy due to the exchange of matter, and  $\mu = \{\mu_i\}$ ,  $N = \{N_i\}$  are the chemical potentials and the number of particles for each kind  $i$ , respectively. The second principle of thermodynamics in the axiomatic formulation of R.J.Clausius postulates the existence of a function of state, called entropy, which is given by the explicit formula

$$dS_{\text{th}} = \frac{\delta Q}{T} \quad (20)$$

and which possesses the following properties: the entropy of the system is an extensive property, conjugate to the temperature  $T$ . The second principle is not the general law of nature. This one is fair only in relation to the thermodynamic systems. The absolute value of entropy is determined from the third law of thermodynamics or the Nernst theorem, according to which, in the Planck rigid formulation, the entropy of the system tends to zero as  $T \rightarrow 0$ :

$$\lim_{T \rightarrow 0} S_{\text{th}} = 0. \quad (21)$$

The first and the second principles of thermodynamics for the quasi-static reversible processes can now be combined to give the fundamental equation of thermodynamics:

$$TdS_{\text{th}} = dE + pdV + Xdz - \mu dN, \quad (22)$$

where the differential sign  $d\mathcal{F}$  of any quantity  $\mathcal{F}$  is understood as a macroscopic infinitesimal change of this measure at transition of the system from equilibrium state 1 to equilibrium state 2,  $d\mathcal{F} = d\mathcal{F}_2 - d\mathcal{F}_1$ .

The second law of thermodynamics for nonequilibrium states, also formulated by R.J.Clausius, refers to the irreversible processes in which the change in entropy satisfies the inequality

$$dS_{\text{th}} > \frac{\delta Q'}{T}, \quad (23)$$

where  $\delta Q'$  is the heat absorbed by the system which passes through nonequilibrium states such that  $dS_{\text{th}} = S_{\text{th}2} - S_{\text{th}1}$ . This principle gives the direction of

a real process allowing one to investigate the properties of equilibrium states as extreme ones [3]. For the irreversible transformation  $1 \rightarrow 2$ , we have

$$\delta Q = T dS_{\text{th}} > \delta Q' = dE' + p' dV + X' dz - \mu' dN, \quad (24)$$

where the prime designates the quantities corresponding to irreversible processes. In particular, in this respect, we shall consider the isolated system set up by the variables of the state  $(E, V, z, N)$ . The infinitesimal changes of the variables of the state are  $dE = dE' = 0, dV = 0, dz = 0, dN = 0$ . Thus the right-hand side of inequality (24) is equal to zero, and the irreversible processes lead to increase of entropy (absolute temperature  $T$  is always positive):

$$dS_{\text{th}} > 0, \quad (25)$$

which will grow until the system at the fixed values of  $(E, V, z, N)$  does not achieve the equilibrium state, in which the entropy is maximal  $S_{\text{max}} = S_{\text{equilibrium}} = S_{\text{th}}(E, V, z, N)$ . Note that a necessary condition of an extremum of the thermodynamic entropy is a condition of thermal equilibrium,  $(\delta S_{\text{th}})_{EVzN} = 0$ , and a maximum of this extremum expresses a condition of stability of the state of thermal equilibrium,  $(\delta^2 S_{\text{th}})_{EVzN} < 0$ . Here variations are made on those parameters of the system which can accept nonequilibrium values under the specified fixed conditions.

### 3. TSALLIS INFORMATION AND STATISTICAL ENTROPY

Historically, the theory of information borrowed many concepts from the statistical mechanics. Now it is a well defined theory, and, following Jaynes, its postulates are widely applied as initial in the statistical mechanics [10].

Let us define the Tsallis information entropy, which recently has received wide popularity due to the property of nonextensivity and which is used for construction of the so-called generalized statistical mechanics [4, 9]. Let  $\{p_i\}$  be the discrete distribution of probabilities for  $W$  independent elementary events satisfying the condition

$$\sum_{i=1}^W p_i = 1. \quad (26)$$

Then the Tsallis information entropy for statistical distributions  $\{p_i\}$  is defined in the following manner [4]:

$$S_{\text{inf}} = -k \sum_{i=1}^W \frac{p_i - p_i^q}{1 - q}, \quad (27)$$

where  $k$  is the Boltzmann constant and  $q \in \mathbf{R}$  is the real parameter accepting values  $0 < q < \infty$ . Note that the extension to arbitrary continuous distribution of probabilities is straightforward. In the limit  $q \rightarrow 1$  we come to the well-known expression for the Boltzmann–Gibbs–Shannon entropy,  $S_{\text{inf}}^{(\text{BGS})} = -k \sum_{i=1}^W p_i \ln p_i$ . Note that the information entropy (27) is known as Havrda–Charvat–Daróczy–Tsallis entropy (see [13]). However, in this paper, we shall use the short name for it.

The information entropy is considered to be a measure of uncertainty of information concerning the statistical distribution  $\{p_i\}$ . The entropy  $S_{\text{inf}} \geq 0$ , and it is zero for  $q > 0$  when one of the probabilities  $p_i$  is equal to unit and all the others are equal to zero, i.e., when we have certainty of information. On the other hand, for the fixed number of events  $W$  entropy  $S_{\text{inf}}$  has an extremum equal to  $S_{\text{inf}} = k[W^{1-q} - 1]/[1 - q]$  (a maximum for  $q > 0$  and a minimum for  $q < 0$ ) when these events are equiprobable,  $p_i = 1/W$ . This limiting case has the maximal uncertainty and, hence, contains a minimum of information about results of the experiment. The function  $S_{\text{inf}}$  is convex (concave) for  $q > 0$  ( $q < 0$ ) that is a basis for the thermodynamic stability. If two systems 1 and 2 are independent in the viewpoint of the theory of probability ( $p_{ij}^{(1+2)} = p_i^{(1)} \times p_j^{(2)}$ ), then the Tsallis information entropy is nonextensive:

$$S_{\text{inf}}^{(1+2)} = S_{\text{inf}}^{(1)} + S_{\text{inf}}^{(2)} + (1 - q) \frac{1}{k} S_{\text{inf}}^{(1)} S_{\text{inf}}^{(2)}. \quad (28)$$

In the case when  $q < 1$ ,  $q = 1$  or  $q > 1$ , the function  $S_{\text{inf}}$  is superextensive, extensive, and subextensive, respectively [5].

Let us introduce for further convenience a new representation for the Tsallis information entropy

$$S_{\text{inf}} = k\xi \sum_{i=1}^W p_i (1 - p_i^{1/\xi}), \quad (29)$$

where a new parameter  $\xi$  is expressed through the variable  $q$  in the following form:

$$\xi = \frac{1}{q - 1}. \quad (30)$$

Thus, the parameter  $\xi$  takes the values  $-\infty \leq \xi \leq -1$  for  $0 < q \leq 1$  and  $0 < \xi \leq \infty$  for  $1 \leq q < \infty$ . In particular, in the limiting case for the value of the parameter  $q = 1$ , we have  $\xi = \pm\infty$ .

In the classical mechanics the Tsallis statistical entropy for a continuous random variable can be written as

$$S(t) = k\xi \int \varrho(x, p, t) [1 - \varrho^{1/\xi}(x, p, t)] d\Gamma, \quad (31)$$

where  $\varrho(x, p, t)$  is the phase distribution function normalized to unity (2). In the quantum mechanics, the Tsallis statistical entropy is expressed through the statistical operator  $\varrho(t)$  describing the mixed quantum states:

$$S(t) = k\xi \text{Tr} \{ \varrho(t) [1 - \varrho^{1/\xi}(t)] \}. \quad (32)$$

In particular, calculating a trace in the diagonal representation, where  $\langle n | \varrho(t) | n' \rangle = w_n(t) \delta_{nn'}$ , the Tsallis entropy can be expressed through the probabilities  $w_n(t)$ :

$$S(t) = k\xi \sum_n w_n(t) [1 - w_n^{1/\xi}(t)]. \quad (33)$$

Let us show that the Tsallis statistical entropy is not additive. For this purpose, we assume that the distribution function  $\varrho^{(1+2)}(x_1, p_1, x_2, p_2, t)$  describes two independent classical ensembles with the distribution functions  $\varrho^{(1)}(x_1, p_1, t)$  and  $\varrho^{(2)}(x_2, p_2, t)$ . Using the normalization conditions for the distribution functions  $\varrho^{(1)}(t)$  and  $\varrho^{(2)}(t)$  and taking into account the factorization relation  $\varrho^{(1+2)}(t) = \varrho^{(1)}(t)\varrho^{(2)}(t)$  valid for two independent classical ensembles, we obtain

$$S^{(1+2)}(t) = S^{(1)}(t) + S^{(2)}(t) - \frac{1}{k\xi} S^{(1)}(t) S^{(2)}(t), \quad (34)$$

where the statistical entropy for each system is expressed in the form (31) with the corresponding distribution function and an infinitesimal element of phase space.

In a quantum case, nonadditivity of the statistical entropy (34) is proved similarly. Note that in this case the statistical operator  $\varrho^{(1+2)}(t)$  describing two independent mixed quantum ensembles is the direct product of the respective statistical operators,  $\varrho^{(1+2)}(t) = \varrho^{(1)}(t) \otimes \varrho^{(2)}(t)$ .

It is easy to show that the Tsallis statistical entropy suffers from grave shortcoming. The statistical entropy  $S(t)$  against the thermodynamic one  $S_T$  in the isolated system does not depend on time and, hence, cannot grow at relaxation of the system to the equilibrium. Let us prove this statement. As the total time derivative  $t$  from the phase distribution function is equal to zero,  $d\varrho/dt = 0$ , valid from the Liouville equation (4) and Liouville theorem, the total time derivative from the classical entropy (31) yields immediately the equality

$$\frac{dS(t)}{dt} = k\xi \int \frac{d\varrho(x, p, t)}{dt} [1 - (1 + \frac{1}{\xi})\varrho^{1/\xi}(x, p, t)] d\Gamma = 0. \quad (35)$$

For the quantum ensembles, the Tsallis statistical entropy does not depend on time. Note that for the Gibbs statistical entropy this problem is inherent as well [10].

#### 4. MICROCANONICAL ENSEMBLE

In this section, the microcanonical distribution function and the statistical operator will be expressed through the variables of state of the isolated system  $(E, V, z, N)$ . Let us consider the equilibrium statistical ensemble of the closed energetically isolated systems of  $N$  particles at the constant volume  $V$  and the thermodynamic coordinate  $z$ . It is supposed that all systems have identical energy  $E$  within  $\Delta E \ll E$ .

To begin with, we turn to instances of the classical case. The Tsallis equilibrium statistical entropy (31) represents a function of the parameter  $\xi$  and a functional of the equilibrium phase distribution function  $\varrho_{\text{eq}}(x, p)$ :

$$S(\xi, \{\varrho_{\text{eq}}\}) = k\xi \int_D \varrho_{\text{eq}}(x, p) (1 - \varrho_{\text{eq}}^{1/\xi}(x, p)) d\Gamma_N, \quad (36)$$

where  $d\Gamma_N = dx dp$  is an infinitesimal element of phase space. Let the phase distribution function  $\varrho_{\text{eq}}(x, p)$  be distinct from zero only in the region of phase space  $D$ , which is defined by inequalities  $E \leq H(x, p) \leq E + \Delta E$  and be normalized to unity:

$$\int_D \varrho_{\text{eq}}(x, p) d\Gamma_N = 1. \quad (37)$$

The phase distribution function depends on the first additive integrals of motion of the system. In particular, it is a function of the Hamiltonian,  $\varrho_{\text{eq}}(x, p) = \varrho_{\text{eq}}(H(x, p))$ . Moreover, the Hamilton function  $H(x, p)$  has the parametrical dependence upon the number of particles  $N$  and volume  $V$  of the system. However, it is not a function from the variables  $\xi$  and  $z$ , i.e. the microstates of the system are independent of the quantities  $\xi, z$ .

For an isolated system, in the state of thermal equilibrium the thermodynamic entropy  $S_{\text{th}}(E, V, z, N)$  has its maximal value. Hence, the fundamental equation of thermodynamics (22) for the quasiequilibrium processes is implemented. Changes of the variables of state at transition from one equilibrium state to another nearby state are equal to zero,  $dE = 0, dV = 0, dz = 0$  and  $dN = 0$ . Therefore, from the basic equation of thermodynamics (22) it follows immediately that the thermodynamic entropy at the fixed values of  $E, V, z, N$  is constant (see (25)):

$$(dS_{\text{th}})_{EVzN} = 0. \quad (38)$$

To express the phase distribution function  $\varrho_{\text{eq}}(x, p)$  through the variables of state  $(E, V, z, N)$ , let us replace the equilibrium thermodynamic entropy  $S_{\text{th}}$  of the macroscopic system with the Tsallis statistical one (36),  $S_{\text{th}}(E, V, z, N) \iff S(\xi, \{\varrho_{\text{eq}}\})$ , and substitute it in Eq. (38). Taking into account Eqs. (37) and (38),

one finds

$$dS = \frac{\partial S}{\partial \xi} d\xi + \int_D \frac{\delta S}{\delta \varrho_{\text{eq}}} d\varrho_{\text{eq}} d\Gamma_N = 0, \quad (39)$$

$$\int_D d\varrho_{\text{eq}} d\Gamma_N = 0, \quad (40)$$

where the symbol  $d$  before the functions  $S$ ,  $\xi$  and  $\varrho_{\text{eq}}$  is the total differential in variables  $(E, V, z, N)$ . One should note that the unequivocal conformity between statistical and thermodynamic entropies is satisfied for the case where the parameters  $\xi$  and  $\{\varrho_{\text{eq}}\}$  are the functions of the variables of state  $(E, V, z, N)$  of the isolated system. The Hamiltonian  $H(x, p)$  and the microstates of the system do not depend on the variables  $\xi$  and  $z$ . Then, the parameter  $\xi$  is a function of  $z$ :

$$\xi = \frac{1}{q-1} = z. \quad (41)$$

Since  $d\xi = 0$  and  $d\varrho_{\text{eq}} = 0$ , we get the solutions to the system of Eqs. (39) and (40):

$$\frac{\delta S(z, \{\varrho_{\text{eq}}\})}{\delta \varrho_{\text{eq}}} = k\alpha, \quad (42)$$

where  $\alpha$  is a certain constant, and  $k$  is the Boltzmann constant, which was introduced for convenience. Substituting Eq. (36) into (42), we obtain

$$\varrho_{\text{eq}}^{1/z}(x, p; E, V, z, N) = \frac{z - \alpha}{z + 1}. \quad (43)$$

The parameter  $\alpha$  has been eliminated by using Eqs. (36) and (37):

$$\varrho_{\text{eq}}(x, p; E, V, z, N) = \left[1 - \frac{S}{kz}\right]^z. \quad (44)$$

Equations (44) and (37) together give

$$\left[1 - \frac{S}{kz}\right]^{-z} = \int_D d\Gamma_N = \int \Delta(H(x, p) - E) d\Gamma_N \equiv W(E, V, N), \quad (45)$$

where  $\Delta(\varepsilon)$  is the function distinct from zero only in the interval  $0 \leq \varepsilon \leq \Delta E$ , where it is equal to unit. The statistical weight  $W(E, V, N)$  is meant as a dimensionless phase volume, i.e. the number of dynamic states inside a layer  $\Delta E$ . Based on this, we get the equipartition probability from Eq. (44) as a function of the thermodynamic ensemble variables, energy  $E$ , volume  $V$ , number of particle  $N$ , and parameter  $z$  [10]:

$$\varrho_{\text{eq}}(x, p; E, V, z, N) = W^{-1}(E, V, N) \Delta(H(x, p) - E). \quad (46)$$

Thus, using Eq. (45), we can write the entropy as (cf. [4, 14])

$$S(E, V, z, N) = kz[1 - W^{-1/z}(E, V, N)] = kz[1 - e^{-S_G(E, V, N)/kz}], \quad (47)$$

where  $S_G$  is the Gibbs entropy [1, 10] for the microcanonical ensemble  $(E, V, N)$ :

$$S_G(E, V, N) = k \ln W(E, V, N). \quad (48)$$

The quantum microcanonical ensemble and the corresponding equilibrium distribution function are in some respects analogous to the familiar classical ones. Let the probability distribution for quantum states of the system be different from zero only in the layer  $E \leq E_i \leq E + \Delta E$  and be normalized to unity:

$$\sum_i w_i = 1, \quad E \leq E_i \leq E + \Delta E. \quad (49)$$

The Tsallis equilibrium statistical entropy is a function of the parameter  $\xi$  and probabilities  $\{w_i\}$ :

$$S(\xi, \{w_i\}) = k\xi \sum_i w_i (1 - w_i^{1/\xi}). \quad (50)$$

By using Eqs. (38) and (49) the quantum microcanonical distribution is found from a maximum principle of thermodynamic entropy of the isolated system  $(E, V, z, N)$ :

$$dS = \frac{\partial S}{\partial \xi} d\xi + \sum_i \frac{\partial S}{\partial w_i} dw_i = 0, \quad (51)$$

$$\sum_i dw_i = 0, \quad (52)$$

where the symbol  $d$  is the total differential in variables  $(E, V, z, N)$  and the statistical entropy depends on the parameters  $\xi$  and  $\{w_i\}$ . The parameter  $\xi$  is equal to a variable  $z$ ,  $\xi = z$ . For the fixed variables of state  $(E, V, z, N)$ , we have  $dE = 0$ ,  $dV = 0$ ,  $dz = 0$ , and  $dN = 0$ . Then, the system of Eqs. (51) and (52) will have the solution if the following equality is fulfilled:

$$\frac{\partial S(z, \{w_i\})}{\partial w_i} = k\alpha. \quad (53)$$

Substituting Eq. (50) into (53), we get

$$w_i^{1/z}(E, V, z, N) = \frac{z - \alpha}{z + 1} = \text{const}, \quad E \leq E_i \leq E + \Delta E. \quad (54)$$

The parameter  $\alpha$  has been excluded by means of Eqs. (49), (50), and (54):

$$w_i(E, V, z, N) = \left[1 - \frac{S}{kz}\right]^z. \quad (55)$$

Substituting Eq. (55) into (49), we obtain

$$\left[1 - \frac{S}{kz}\right]^{-z} = \sum_i \Delta(E_i - E) \equiv W(E, V, N). \quad (56)$$

The statistical weight  $W(E, V, N)$  is equal to the number of quantum states in the layer  $\Delta E$ . The quantum microcanonical distribution (55) becomes

$$w_i(E, V, z, N) = W^{-1}(E, V, N)\Delta(E_i - E). \quad (57)$$

The statistical operator corresponding to the microcanonical distribution of probabilities of quantum states (57) can be written as [10]

$$\rho_{\text{eq}}(E, V, z, N) = W^{-1}(E, V, N)\Delta(H - E), \quad (58)$$

where the operator function  $\Delta(H - E)$  is determined in the diagonal representation by the matrix elements  $\langle k|\Delta(H - E)|k'\rangle = \Delta(E_k - E)\delta_{kk'}$ . The quantum statistical entropy is calculated similarly to the classical one (47) with statistical weight (56). Note that the classical and quantum microcanonical distributions (46) and (57) are extreme equilibrium ones which correspond to a maximum of the Tsallis statistical entropy [4]. The distribution functions (46) and (57) obtained by the thermodynamic method described here are identical with ones obtained by the Jaynes principle. The index  $q$  for the Jaynes principle is a fixed parameter and does not depend on the variables of state of the system. In this case, the Tsallis statistics does not satisfy the zero law of thermodynamics [8].

## 5. THERMODYNAMICS OF MICROCANONICAL ENSEMBLE

It is well-known from the conventional statistical mechanics that in the thermal equilibrium the Gibbs entropy of the microcanonical ensemble is an extensive variable, and it has all peculiarities of the thermodynamic entropy in the thermodynamic limit [3, 10]. Mathematically, this implies that the Gibbs entropy  $S_G$  is a homogeneous function of variables  $E, V$  and  $N$  of the first order, i.e. one has the following property [12]:

$$S_G(\lambda E, \lambda V, \lambda N) = \lambda S_G(E, V, N), \quad (59)$$

where  $\lambda$  is a certain constant. After substitution of Eq. (48) into (59), it is easy to check up that the statistical weight  $W$  must satisfy the following requirement:

$$W(\lambda E, \lambda V, \lambda N) = W^\lambda(E, V, N). \quad (60)$$

Taking into account Eqs. (47) and (59), one finds the following peculiarity of the Tsallis entropy

$$S(\lambda E, \lambda V, \lambda z, \lambda N) = \lambda S(E, V, z, N), \quad (61)$$

which shows that the Tsallis entropy in the microcanonical ensemble is a homogeneous function of variables  $E, V, z, N$  of the first order. In other words, it is extensive. It is essential to make clear that the homogeneity property of quantities (59)–(61) is realized only in the thermodynamic limit.

Differentiating Eq. (61) with respect to  $\lambda$ , and putting  $\lambda = 1$ , we obtain the well-known Euler theorem for the homogeneous functions:

$$E \left( \frac{\partial S}{\partial E} \right)_{V,z,N} + V \left( \frac{\partial S}{\partial V} \right)_{E,z,N} + z \left( \frac{\partial S}{\partial z} \right)_{E,V,N} + N \left( \frac{\partial S}{\partial N} \right)_{E,V,z} = S. \quad (62)$$

Using the thermodynamic relations following from the fundamental equation of thermodynamics (22) in case of the isolated thermodynamic system ( $E, V, z, N$ )

$$\begin{aligned} \left( \frac{\partial S}{\partial E} \right)_{V,z,N} &= \frac{1}{T}, & \left( \frac{\partial S}{\partial V} \right)_{E,z,N} &= \frac{p}{T}, \\ \left( \frac{\partial S}{\partial z} \right)_{E,V,N} &= \frac{X}{T}, & \left( \frac{\partial S}{\partial N} \right)_{E,V,z} &= -\frac{\mu}{T}, \end{aligned} \quad (63)$$

we get the Euler theorem [12]:

$$TS = E + pV + Xz - \mu N. \quad (64)$$

Applying the differential operator with respect to the ensemble variables ( $E, V, z, N$ ) on Eq. (64), we obtain the fundamental equation of thermodynamics

$$TdS = dE + pdV + Xdz - \mu dN \quad (65)$$

and the Gibbs–Duhem relation [12]

$$SdT = Vdp + zdX - Nd\mu. \quad (66)$$

Equation (66) means that the variables  $T, \mu, X$  and  $p$  are not independent. The fundamental equation of thermodynamics (65) provides the first principle

$$\delta Q = dE + pdV + Xdz - \mu dN \quad (67)$$

and the second law of thermodynamics

$$dS = \frac{\delta Q}{T}. \quad (68)$$

Here  $\delta Q$  is a heat transfer by the system to the environment for quasi-static transition of the system from one equilibrium state to another nearby state.

Let us investigate the homogeneity properties of the variables  $T, p, \mu$  and  $X$ . Substituting Eq.(47) into (63), we obtain the following expressions for the temperature  $T$  [8]:

$$T(E, V, z, N) = T_G(E, V, N) W^{1/z}(E, V, N) = T_G(E, V, N) e^{S_G(E, V, N)/kz} \quad (69)$$

and the variable  $X$

$$X(E, V, z, N) = kT_G(E, V, N)[e^{S_G(E, V, N)/kz} - 1 - S_G(E, V, N)/kz]. \quad (70)$$

The pressure and the chemical potential of the system are equivalent with the pressure  $p_G$  and the chemical potential  $\mu_G$  of the Gibbs statistics, respectively,  $p(E, V, z, N) = p_G(E, V, N)$  and  $\mu(E, V, z, N) = \mu_G(E, V, N)$ . These equations were derived by using the thermodynamical relations for the temperature  $T_G$ , the pressure  $p_G$  and the chemical potential  $\mu_G$  of the Gibbs statistics, and taking into account Eq. (48):

$$\frac{1}{T_G} = \left( \frac{\partial S_G}{\partial E} \right)_{V, N}, \quad \frac{p_G}{T_G} = \left( \frac{\partial S_G}{\partial V} \right)_{E, N}, \quad -\frac{\mu_G}{T_G} = \left( \frac{\partial S_G}{\partial N} \right)_{E, V}. \quad (71)$$

The Gibbs quantities  $T_G, p_G, \mu_G$  are the homogeneous functions of the variables of state  $(E, V, N)$  of the zero order. This can be proved by using Eqs.(71) and (59). Then, a combination of Eqs.(69) and (59) allows us to write the relation for the temperature  $T$ :

$$T(\lambda E, \lambda V, \lambda z, \lambda N) = T(E, V, z, N). \quad (72)$$

Similarly to Eq.(72), the relations for the pressure  $p(\lambda E, \lambda V, \lambda z, \lambda N)$ , the chemical potential  $\mu(\lambda E, \lambda V, \lambda z, \lambda N)$ , and the variable  $X(\lambda E, \lambda V, \lambda z, \lambda N)$  are fulfilled. Thus, the temperature  $T$ , the pressure  $p$ , the chemical potential  $\mu$ , and quantity  $X$  are the homogeneous functions of the variables  $E, V, z, N$  of the zero order. So they are intensive variables [12].

Let us prove in more detail the thermodynamic principle of additivity [3]. For instance, we assume that  $\lambda = 1/N$  and introduce the following specific variables:

$$\varepsilon = \frac{E}{N}, \quad v = \frac{V}{N}, \quad \tilde{z} = \frac{z}{N} = \frac{1}{(q-1)N}. \quad (73)$$

Thus, Eqs. (61) and (72) for the entropy and the temperature of the system, by using (73) with respect to  $\lambda = 1/N$ , can be rewritten as

$$s(\varepsilon, v, \tilde{z}) = \frac{1}{N} S(E, V, z, N) \quad (74)$$

and

$$T(\varepsilon, v, \tilde{z}) = T(E, V, z, N), \quad (75)$$

where  $s(\varepsilon, v, \tilde{z})$  is the specific entropy,  $s = S/N$ , which depends only on the intensive variables  $\varepsilon, v$  and  $\tilde{z}$ . For the pressure  $p$ , the chemical potential  $\mu$ , and  $X$ , we have equations similar to that for the temperature (75). So, comparing Eqs. (74) and (75) with the thermodynamic equations (17) and (18), we conclude that the entropy  $S$  is an extensive variable, as it is proportional to the number of particles  $N$  multiplied by an intensive variable  $s$ , but the temperature  $T$ , the pressure  $p$ , the chemical potential  $\mu$ , and  $X$  are intensive variables.

Let us divide the system into two parts (1 and 2) and require that the total number of particles of the system should be equal to the sum of the number of particles of each subsystem separately and the specific quantities (73) should be equal among themselves

$$N_{1+2} = N_1 + N_2, \quad \varepsilon_{1+2} = \varepsilon_1 = \varepsilon_2, \quad v_{1+2} = v_1 = v_2, \quad \tilde{z}_{1+2} = \tilde{z}_1 = \tilde{z}_2. \quad (76)$$

Then, the variables  $E, V$  and  $z$  are extensive. Taking into account Eq. (76), one finds

$$s_{1+2}(\varepsilon_{1+2}, v_{1+2}, \tilde{z}_{1+2}) = s_1(\varepsilon_1, v_1, \tilde{z}_1) = s_2(\varepsilon_2, v_2, \tilde{z}_2). \quad (77)$$

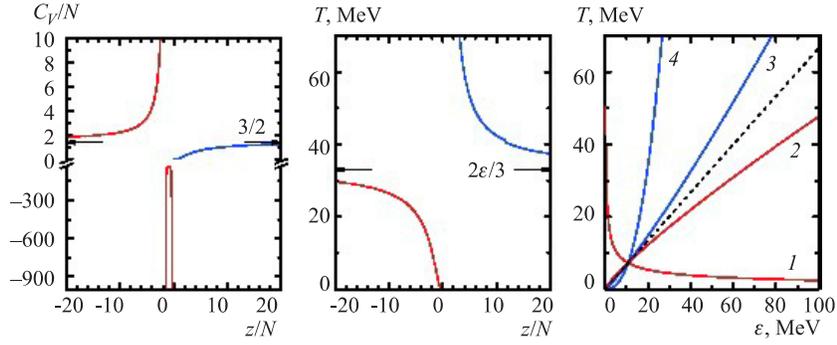
Multiplying it by the first equation from (76) and using (74), we get

$$S_{1+2}(E_{1+2}, V_{1+2}, z_{1+2}, N_{1+2}) = S_1(E_1, V_1, z_1, N_1) + S_2(E_2, V_2, z_2, N_2). \quad (78)$$

Thus, in the microcanonical ensemble the Tsallis entropy is an extensive variable. Furthermore, Eqs. (75) and (76) allow us to write

$$T_{1+2}(E_{1+2}, V_{1+2}, z_{1+2}, N_{1+2}) = T_1(E_1, V_1, z_1, N_1) = T_2(E_2, V_2, z_2, N_2). \quad (79)$$

So, in the thermodynamic limit, the zero law of thermodynamics and the thermodynamic principle of additivity (see. (17) and (18)) for the Tsallis statistics in the microcanonical ensemble are valid. Here, the thermodynamic limit denotes the limiting statistical procedure  $N \rightarrow \infty$  at  $\varepsilon = \text{const}$ ,  $v = \text{const}$ , and  $\tilde{z} = \text{const}$  with keeping the main asymptotics on  $N$ . Note that the correct thermodynamic limit,  $(q-1)N = \text{const}$ , for the Tsallis statistics has already been discussed in the paper of Botet et al. [15, 16].



The dependence of the heat capacity (left) and the temperature (center) on the specific  $\tilde{z} = z/N$  for the classical ideal gas of  $N$  nucleons at the values of the specific energy  $\varepsilon = 50$  MeV and the specific volume  $v = 3/\rho_0$ . The temperature as a function of the specific energy (right) for the different values of  $\tilde{z} = -1, -10, 10,$  and  $1$  (the curves 1, 2, 3, and 4, respectively) at  $v = 3/\rho_0$ . The dashed line corresponds to the Gibbs statistics

## 6. THE PERFECT GAS

The thermodynamic principle of additivity can thoroughly be investigated in the framework of a classical nonrelativistic ideal gas. In the microcanonical ensemble  $(E, V, z, N)$ , the statistical weight (45) of the perfect gas of  $N$  identical nucleons is given by [17]

$$\begin{aligned}
 W(E, V, N) &= \frac{V^N}{N!} \int \frac{d^3 p_1 \dots d^3 p_N}{(2\pi\hbar)^{3N}} \delta\left(\sum_{i=1}^N \frac{\vec{p}_i^2}{2m} - E\right) = \\
 &= \frac{V^N}{N!} \left(\frac{m}{2\pi\hbar^2}\right)^{\frac{3}{2}N} \frac{E^{\frac{3}{2}N-1}}{\Gamma(\frac{3}{2}N)}, \quad (80)
 \end{aligned}$$

where  $m$  is the nucleon mass. In the thermodynamic limit ( $N \gg 1$ ,  $\varepsilon = E/N = \text{const}$ ,  $v = V/N = \text{const}$ ) from Eq. (80), it follows immediately that [3]

$$W^{1/N}(E, V, N) = v \left(\frac{m\varepsilon e^{5/3}}{3\pi\hbar^2}\right)^{3/2} \equiv w(\varepsilon, v). \quad (81)$$

So Eq. (81) proves relation (60) for the statistical weight  $W$  with  $\lambda = 1/N$ . Then, the Tsallis entropy (47) is reduced to

$$S(E, V, z, N) = Ns(\varepsilon, v, \tilde{z}), \quad s(\varepsilon, v, \tilde{z}) = k\tilde{z} \left[1 - w^{-1/\tilde{z}}\right]. \quad (82)$$

Comparing Eq. (82) with (17), we conclude that the Tsallis entropy is an extensive variable. Note that in the limit  $\tilde{z} \rightarrow \pm\infty$ , we obtain the formula for the Gibbs specific entropy [16]:

$$s(\varepsilon, v, \tilde{z})|_{\tilde{z} \rightarrow \pm\infty} = k \ln w \equiv s_G(\varepsilon, v). \quad (83)$$

Substituting (82) into (63), we get

$$T(E, V, z, N) = \frac{2}{3} \frac{\varepsilon}{k} w^{1/\tilde{z}} \equiv T(\varepsilon, v, \tilde{z}). \quad (84)$$

The temperature (84) is a function of the specific variables  $\varepsilon, v$  and  $\tilde{z}$ . Therefore, it is an intensive variable by virtue of Eq. (18). In the limit  $\tilde{z} \rightarrow \pm\infty$ , we obtain the well-known formula for the Gibbs statistics

$$T(\varepsilon, v, \tilde{z})|_{\tilde{z} \rightarrow \pm\infty} = \frac{2}{3} \frac{\varepsilon}{k} = T_G(\varepsilon, v). \quad (85)$$

In a similar way, the pressure  $p$ , the chemical potential  $\mu$ , and  $X$  become

$$p(E, V, z, N) = \frac{2}{3} \frac{\varepsilon}{v} \equiv p(\varepsilon, v, \tilde{z}), \quad (86)$$

$$\mu(E, V, z, N) = \frac{2}{3} \varepsilon \left[ \frac{5}{2} - \ln w \right] \equiv \mu(\varepsilon, v, \tilde{z}), \quad (87)$$

$$X(E, V, z, N) = -\frac{2}{3} \varepsilon \left[ 1 + \frac{1}{\tilde{z}} \ln w - w^{1/\tilde{z}} \right] \equiv X(\varepsilon, v, \tilde{z}). \quad (88)$$

Note that the pressure  $p$  and the chemical potential  $\mu$  for the classical ideal gas in the microcanonical ensemble do not depend on the parameter  $\tilde{z}$ , and they are equal to respective quantities of the Gibbs statistics. Then, Eqs. (82), (84), and (86)–(88) yield the Euler theorem (64) in terms of the specific variables

$$Ts = \varepsilon + pv - \mu + X\tilde{z}. \quad (89)$$

In the limit  $\tilde{z} \rightarrow \pm\infty$ , the pressure  $p$  and the chemical potential  $\mu$  remain unchanged but the variable  $X(\varepsilon, v, \tilde{z}) = 0$ . So, by the example of the classical ideal gas the principle of additivity for the Tsallis statistics is proved. The Euler theorem (64) or (89) shows that in the Tsallis statistics the quantities  $z = 1/(q-1)$  and  $X$  should be the variable of the state and the associated «force», respectively.

At this point one important quantity must be noted, the heat capacity  $C_V = 1/(\partial T/\partial E)_{V,z,N}$ . In the framework of the ideal gas of  $N$  nucleons it can be written as

$$C_V = \frac{3}{2} k N w^{-1/\tilde{z}} \left( 1 + \frac{3}{2} \frac{1}{\tilde{z}} \right)^{-1}. \quad (90)$$

Figure shows the specific heat (left) and the temperature (center) vs. the parameter  $\tilde{z}$ . The calculations are done for the system of nucleons at the specific energy  $\varepsilon = 50$  MeV and the specific volume  $v = 3/\rho_0$ , where  $\rho_0 = 0.168 \text{ fm}^{-3}$ . It is of great interest that both the heat capacity and the temperature sharply change their shape in the region of small values of  $\tilde{z}$  and considerably defer from their Gibbs limit, which in the figure is indicated by arrows. In the region of  $-3/2 < \tilde{z} < 0$ , the heat capacity is negative. It is remarkable that such a behaviour has really been caused by the decrease of the temperature with  $\varepsilon$ . This dependence can be seen even better in right panel of Figure which shows the temperature vs. the specific energy of the system for different values of the parameter  $\tilde{z}$ .

## 7. CONCLUSIONS

In this paper, we have explored the microscopic foundation of the generalized equilibrium statistical mechanics based on the Tsallis statistical entropy. The viewpoint utilized here considers that the microcanonical ensemble is most convenient to analyze the fundamental questions of the statistical mechanics. We summarize our main principles.

Here, the Gibbs idea of the statistical ensembles defined within the framework of the quantum and classical mechanics was used. In this approach, the equilibrium phase distribution function and the statistical operator do not depend on time, and they are functions of the additive first integrals of motion of the system by virtue of performance of Liouville and von Neumann equations. Additionally, these main quantities are functions of the macroscopic variables of state of the system. To derive the distribution functions, in contrast with the Jaynes principle, the new thermodynamic method based on the fundamental equation of thermodynamics and statistical definition of the functions of the state of the system was given.

In this paper, we have made the following claim. The index  $\xi$  of the Tsallis entropy should be an extensive variable of the state of the system. As a result of this assumption, we obtain that in the microcanonical ensemble the Tsallis entropy represents the homogeneous function of the variables  $E, V, z, N$  of the first order. The temperature of the system is an intensive variable, and, consequently, the zero law of thermodynamics is satisfied. Other functions of state of the system are either extensive or intensive. Thus, in the thermodynamic limit,  $\tilde{z} = 1/(q - 1)N = \text{const}$ , in the Tsallis statistics the thermodynamic principle of additivity is carried out. Note that the Tsallis information entropy is nonextensive because the parameter  $\xi$  is a certain intensive constant. Also it is necessary to note that the Tsallis statistical entropy as well as the Gibbs one has an essential lack. Both the entropies do not depend on time while the thermodynamic entropy grows up to achieve its maximal value in the state of thermal equilibrium. The extensive

property of the Tsallis entropy in the microcanonical ensemble yields the Euler theorem which permits one to find the fundamental equation of thermodynamics and the Gibbs–Duhem relation. Thus, the first and the second principles of thermodynamics are fulfilled. Note that in the limit,  $\tilde{z} \rightarrow \pm\infty$ , all expressions of the Tsallis statistics take the form of the conventional Gibbs statistical mechanics. So the Tsallis statistical mechanics in the microcanonical ensemble satisfies all postulates of the equilibrium thermodynamics.

Finally, the classical nonrelativistic ideal gas of  $N$  identical nucleons in the microcanonical ensemble was considered to illustrate the principles which were elucidated in the general theory. It has been shown that in the thermodynamic limit the statistical weight, the entropy, the temperature, and other quantities are the homogeneous functions of the first and zero order of the variables of state, respectively. Note that for ideal gas the Euler theorem was accomplished and in the limit,  $\tilde{z} \rightarrow \pm\infty$ , all expressions resembled ones of the Gibbs statistics.

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