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PARA-HERMITIAN AND PARA-QUATERNIONIC MANIFOLDS

Stefan Ivanov[1]
University of Sofia "St. Kl. Ohridski", Faculty of Mathematics and Informatics,
Blvd. James Bourchier 5, 1164 Sofia, Bulgaria
and
The Abdus Salam International Centre for Theoretical Physics, Trieste, Italy

and

Simeon Zamkovoy
University of Sofia "St. Kl. Ohridski", Faculty of Mathematics and Informatics,
Blvd. James Bourchier 5, 1164 Sofia, Bulgaria.

Abstract

A set of canonical parahermitian connections on an almost para-Hermitian manifold is defined. A Para-hermitian version of the Apostolov-Gauduchon generalization of the Goldberg-Sachs theorem in General Relativity is given. It is proved that the Nijenhuis tensor of a Nearly para-Kähler manifolds is parallel with respect to the canonical connection. Salamon's twistor construction on quaternionic manifold is adapted to the para-quaternionic case. A locally conformally hyper-para-Kähler (hypersymplectic) flat structure with parallel Lee form on the Kodaira-Thurston complex surfaces modeled on $S^1 \times \widetilde{SL(2,\mathbb{R})}$ is constructed. Anti-self-dual locally conformally hyper-para-Kähler (hypersymplectic) neutral metrics with non vanishing Weyl tensor are obtained on the Inoe surfaces. An example of anti-self-dual neutral metric which is not locally conformally hyper-para-Kähler (hypersymplectic) is constructed.

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[1]ivanovsp@fmi.uni-sofia.bg

1. Introduction

We study the geometry of structures on a differentiable manifold related to the algebra of paracomplex numbers as well as to the algebra of paraquaternions together with a naturally associated metric which is necessarily of neutral signature. These structures lead to the notion of almost para-Hermitian manifold, in even dimension, as well as to the notion of almost paraquaternionic Hermitian manifolds in dimensions divisible by four. Some of these spaces play a role in string theory [37, 38, 11] and integrable systems [24].

Almost para-Hermitian geometry is a topic with many analogies with the almost Hermitian geometry and also with differences.

In the present note we show that a lot of local and some of the global results in almost Hermitian manifolds carry over, in the appropriately defined form, to the case of almost para-Hermitian spaces.

We define a set of canonical parahermitian connections on an almost para-Hermitian manifold and use them to describe properties of 4-dimensional para-Hermitian and 6-dimensional Nearly para-Kähler spaces.

We find a para-hermitian analogue of the Apostolov-Gauduchon generalization [6] of the Goldberg-Sachs theorem in General Relativity (see e.g. [49]) which relates the Einstein condition to the structure of the positve Weyl tensor in dimension 4. We prove the result using essentially the Chern connection.

It turns out that any conformal class of neutral metrics on an oriented 4-manifold is equivalent to the existence of a local almost hyper-paracomplex structure, i.e. a collection of anti-commuting almost complex structure and almost product structure. Using the properties of the Bismut connection we easily derive that the integrability of the almost hyper-paracomplex structure leads to the anti-self-duality of the corresponding conformal class of neutral metrics. Applying this result to invariant hyper-paracomplex structure on 4-dimensional Lie groups [3, 21] we find explicit anti-self-dual non Weyl flat neutral metrics on compact solve 4-manifolds which are locally conformally hyper-para-Kähler (hypersymplectic). Some of them seem to be new. In particular, we obtain an explicit invariant locally conformally hyper-para-Kähler (hypersymplectic) structure with non-zero Weyl curvature on the Inoe surfaces modeled on Sol in the sense of [57] (Theorem 6.11). Thus, we equip the Inoe surfaces with an invariant anti-self-dual neutral metric with non-zero Weyl tensor. It is well known that these surfaces do not support any symplectic 2-form and therefore any (hyper) (para) Kähler structure (see also [50]).

We endow the 4-dimensional non-compact hyperbolic Hopf manifold with a hyper- paracomplex structure which is conformally para-Kähler flat. It turns out that the non-compact hyperbolic Hopf space admits compact quotients by a discrete subgroup, the compact hyperbolic Hopf manifolds. These manifolds are known as a Kodaira-Thurston complex surfaces modeled on $S^1 \times \widetilde{SL(2,\mathbb{R})}$ in the sense of [57]. They have odd first Betti number, Kodaira dimension equal to 1 [57], do not admit any symplectic 2-form [17] and consequently no (hyper)-(para)-Kähler structure (see also [50]). We obtain an invariant locally conformally hyper-para-Kähler (hypersymplectic) flat structure with parallel Lee form on the Kodaira-Thurston complex surfaces modeled on $S^1 \times \widetilde{SL(2,\mathbb{R})}$ (Theorem 6.7).

We construct an anti-self-dual neutral metric which is not locally conformally hyper-para-Kähler (hypersymplectic) adapting the Ashtekar's [7] formulation of the self-duality Einstein equations to the case of neutral metric and the Joyce's construction [39] of hyper-complex structure from holomorphic functions.

We prove that the Nijenhuis tensor of a Nearly para-Kähler manifoldis is parallel with respect to the canonical connection. In the particular dimension six, we show that these spaces are Einsteinian but the zero scalar curvature case cannot be excluded. To obtain examples of Nearly para-Kähler manifolds we involve twistor machinery. We adapt Salamon's twistor construction on quaternionic manifold [52, 53, 54] to the para-quaternionic situation considering the reflector space of a paraquaternionic manifold as a higher dimensional analogue of the reflector space of a 4-dimensional manifold with a metric of neutral signature described in [38]. We show that the reflector space of an Einstein self-dual non-Ricci flat 4 manifold as well as the reflector space of quaternionic para-Kähler manifold admit a Nearly para-Kähler as well as an almost para-Kähler structure adapting the discussion in [18]. We present homogeneous as well as non locally homogeneous examples of 6 dimensional Nearly para-Kähler and almost para-Kähler manifold.

2. Preliminaries

Let V be a real vector space of even dimension $2n$. An endomorphism $P : V \to V$ is called a *paracomplex structure* on V if $P^2 = 1$ and the eigenspaces V^{+1}, V^{-1} corresponding to the eigenvalues 1 and -1, respectively are of the same dimension n, $V = V^+ \oplus V^-$. Consider the algebra

$$\mathbb{A} = \{x + \epsilon y, x, y \in \mathbb{R}, \epsilon^2 = 1\}$$

of paracomplex numbers over $\mathbb{R}$. As in the ordinary complex case, $\mathbb{A}^n$ is identified with $(\mathbb{R}^{2n}, P)$, where $Pv = \epsilon v$. P is called *the canonical paracomplex structure* on $\mathbb{R}^{2n}$.

The notions of (almost) paracomplex, para-Hermitian, paraholomorphic etc. objects are defined in the ususal way over the paracomplex numbers $\mathbb{A}$ instead of the complex numbers $\mathbb{C}$. A survey on paracomplex geometry is presented in [22].

A (1,1)-tensor field P on a 2n-dimensional smooth manifold M is said to be an *almost product structure* if $P^2 = 1$. In this case the pair (M, P) is called *almost product manifold*. An *almost paracomplex manifold* is an almost product manifold (M, P) such that the two eigenbundles T^+M and T^-M associated with the two eigenvalues ± 1 of P have the same rank. Equivalently, a splitting of the tangent bundle $TM = TM^+ \oplus TM^-$ of the subbundles $TM^\pm$ of the same fibre dimension is called an almost paracomplex structure. A smooth section of TM^+ is called *(1,0)-vector field* while a smooth section of TM^- is said to be *(0,1)-vector field* with respect to the almost paracomplex structure. Such a structure may alternatively be defined as a G-structure on M with structure group $GL(n,\mathbb{R})\times GL(n,\mathbb{R})$.

The Nijenhuis tensor N of P is defined by [58]

$$4N(X,Y)=[PX,PY]+[X,Y]-P[PX,Y]-P[X,PY]. \tag{2.1}$$

The structure P is said to be *paracomplex* if $N=0$ [44] which is equivalent to the distributions on M defined by $TM^{\pm}$ to be both completely integrable [41]. The paracomplex manifold can also be characterized by the existence of an atlas with paraholomorphic coordinate maps i.e. the coordinate maps satisfying the para Cauchy-Rieman equations [44] (see also [41]).

An almost para-Hermitian manifold (M,P,g) is a smooth manifold endowed with an almost paracomplex structure P and a pseudo-Riemannian metric g compatible in the sense that

$$g(PX,Y)+g(X,PY)=0,$$

equivalently $g(PX,PY)=-g(X,Y)$. It follows that the metric is *neutral* i.e. it has signature (n,n) and the eigenbundles $TM^{\pm}$ are totally isotropic with respect to g. Equivalently, an almost para-Hermitian manifold is a smooth manifold whose structure group can be reduced to the real representation of the paraunitary group $U(n,\mathbb{A})$ isomorphic to $(GL(n,\mathbb{R})\times GL(n,\mathbb{R}))\cap O(n,n)\cong GL(n,\mathbb{R})$.

Further we consider an orthonormal basis $e_1,...,e_n,e_{n+1}=Pe_1,...,e_{2n}=Pe_n$ and denote $\epsilon_i=sign(g(e_i,e_i))=\pm 1, \quad \epsilon_i=1, i=1,...,n, \quad \epsilon_j=-1, j=n+1,...,2n.$

The fundamental 2-form F of an almost para-Hermitian manifold is defined by

$$F(X,Y)=g(X,PY).$$

The covariant derivative of F with respect to the Levi-Civita connection is expressed in terms of dF and N in the following way (see e.g. [41])

$$\begin{gathered}2(\nabla^g F)(X;Y,Z)=-2g((\nabla^g_X P)Y,Z)=\\ dF(X,Y,Z)+dF(X,PY,PZ)+4N(PX;Y,Z).\end{gathered} \tag{2.2}$$

The Lee form θ is defined by $\theta=\delta F\circ P$, where $\delta=-*d*$ is the co-differential with respect to g. For 1-form α we adapt the notation $P\alpha(X)=-\alpha(PX)$. Thus, $\theta=-P\delta F$. We also have

$$\theta(X)=\sum_{i=1}^{2n}\epsilon_i(\nabla^g F)(e_i;e_i,PX)=\frac{1}{2}\sum_{i=1}^{2n}dF(e_i,Pe_i,X)=\sum_{i=1}^{n}dF(e_i,Pe_i,X).$$

Almost para-Hermitian manifolds are classified by using the decomposition of the space $\nabla^g F$ under the action of the structure group of invariant and irreducible subspaces [12, 28]. We recall that the class of para-Kähler manifolds is defined by the condition $\nabla^g F=0$ (equivalently, $dF=0$). If the paracomplex structure is integrable ($N=0$) then we have the notion of para-Hermitian manifold. Para-Hermitian manifolds can be characterized also by the condition $(\nabla^g_{PX}P)PY+(\nabla^g_X P)Y=0$ [48]. The class of *almost para-Kähler manifolds* is defined by $dF=0$. The condition $\nabla^g F$ to be a 3-form characterizes *Nearly para-Kähler manifolds*. A para-Hermitian manifolds locally conformally equivalent to a para-Kähler space are determined by $dF = \theta \wedge F, \quad d\theta = 0$ [28, 14].

A number of examples of almost para-Hermitian manifolds including the non-compact hyperbolic Hopf and hyperbolic Calabi-Eckmann manifolds [13] are collected in [22]. Another source of examples comes from the k-symmetric spaces, i.e. homogeneous spaces defined by a Lie group automorphism of order k [10]. Almost para-Hermitian manifolds are also called *almost bilagrangian* [38, 42]. They arise in relation with the existence of Killing spinors of an indefinite neutral metric [42].

3. Parahermitian connections

A linear connection ∇ on an almost para-Hermitian manifold (M, g, P) is said to be *parahermitian connection* if it preserves the para- Hermitian structure, i.e. $\nabla g = \nabla P = 0$.

In this section we define canonical parahermitian connections in a (formally) similar way as it was done in [31] for an almost Hermitian manifold.

We start with type decomposition of an element $B \in \Lambda^2(TM)$. We denote $g(X, B(Y, Z)) := B(X; Y, Z)$. Let $Bi(B) : \Lambda^2(TM) \to \Lambda^3$ be the Bianchi projector

$$3Bi(B)(X;Y,Z) = B(X;Y,Z) + B(Y;Z,X) + B(Z;X,Y).$$

Further, we say that B is

- of type (1,1) if B(PX,PY)=-B(X,Y);
- of type (0,2) if B(PX,Y)=-PB(X,Y);
- of type (2,0) if B(PX,Y)=PB(X,Y).

We will denote the corresponding type-subspaces by $\Lambda^{1,1}$, $\Lambda^{0,2}$, $\Lambda^{2,0}$, respectively, such that $B = B^{1,1} \oplus B^{0,2} \oplus B^{2,0}$. The projections are given by

$$\begin{aligned} B^{1,1}(X,Y) &= \frac{1}{2}\left(B(X,Y) - B(PX,PY)\right), \\ B^{0,2}(X,Y) &= \frac{1}{4}\left(B(X,Y) + B(PX,PY) - PB(PX,Y) - PB(X,PY)\right), \\ B^{2,0}(X,Y) &= \frac{1}{4}\left(B(X,Y) + B(PX,PY) + PB(PX,Y) + PB(X,PY)\right) \end{aligned}$$

We define an involution $In : \Lambda^2(TM) \to \Lambda^2(TM)$ by $In(B)(X;Y,Z,) = B(X;PY,PZ)$.

We may consider a 3-form ψ as a totally skew-symmetric section of $\Lambda^2(TM)$. It thus admits two different type decompositions:

1. decomposition as a 3-form: $\psi = \psi^+ \oplus \psi^-$, where ψ^+ denotes the (1,2)+(2,1)-part and ψ^--the (3,0)+(0,3)-part of ψ given by

$$\begin{aligned} \psi^+(X,Y,Z) &= \frac{1}{4}\left(3\psi(X,Y,Z) - \psi(X,PY,PZ) - \psi(PX,Y,PZ) - \psi(PX,PY,Z)\right), \\ \psi^-(X,Y,Z) &= \frac{1}{4}\left(\psi(X,Y,Z) + \psi(X,PY,PZ) + \psi(PX,Y,PZ) + \psi(PX,PY,Z)\right). \end{aligned}$$

2. A type decomposition as an element of $\Lambda^2(TM)$.

The two decompositions are related by $\psi^- = \psi^{0,2}, \quad \psi^+ = \psi^{2,0} + \psi^{1,1}$.

Let ∇ be any parahermitian connection. Then we have

$$g(\nabla_X Y, Z) - g(\nabla^g_X Y, Z) = A(X; Y, Z), \tag{3.3}$$

where $A \in \Lambda^2(TM)$ since $\nabla g = 0$.

The torsion of $\nabla, T(X,Y) = \nabla_X Y - \nabla_Y X - \nabla_{[X,Y]} \in \Lambda^2(TM)$ and

$$T = -A + 3Bi(A), \quad A = -T + \frac{3}{2}Bi(T), \quad Bi(A) = \frac{1}{2}Bi(T). \tag{3.4}$$

We determine ∇ in terms of its torsion.

Denote $d^a F(X,Y,Z) := -dF(PX, PY, PZ)$ we easily obtain the following

Proposition 3.1. *On an almost para-Hermitian manifold we have:*

a) The Nijenhuis tensor is of type (0,2). In particular it is trace-free, $tr(N) = 0$. The skew-symmetric part of N is given by

$$Bi(N) = \frac{1}{3}(d^a F)^-; \tag{3.5}$$

b) The component $(\nabla^g F)^{1,1} = 0$.

c) The component $(\nabla^g F)^{0,2}$ is determined by N:

$$\begin{aligned}(\nabla^g F)^{0,2}(X;Y,Z) = dF^-(X,Y,Z) + 2N(PX;Y,Z) = \\ N(PX;Y,Z) - N(PY;Z,X) - N(PZ;X,Y).\end{aligned} \tag{3.6}$$

d) The component $(\nabla^g F)^{2,0}$ is determined by dF^+:

$$(\nabla^g F)^{2,0}(X;Y,Z) = \frac{1}{2}\left(dF^+(X,Y,Z) + dF^+(X,PY,PZ)\right) \tag{3.7}$$

We describe the parahermitian connections in the next

Theorem 3.2. *Let ∇ ba a parahermitian connection. Then*

$$T^{0,2} = -N, \qquad Bi(T^{2,0}) - Bi(T^{1,1}) = -\frac{1}{3}(d^a F)^+ \tag{3.8}$$

For any 3-form ψ^+ of type (1,2)+(2,1) and any section B_b of $\Lambda^{1,1}(TM)$ satisfying $Bi(B_b) = 0$ there exists a unique parahermitian connection whose torsion T is given by the formula

$$T = -N - \frac{1}{8}(d^a F)^+ - \frac{3}{8}In(d^a F)^+ + \frac{9}{8}\psi^+ + \frac{3}{8}In(\psi^+) + B_b. \tag{3.9}$$

The corresponding parahermitian connection is then equal to $\nabla^g + A$, where A is obtained from T by (3.4).

Proof. Since $\nabla P = 0$ we get the first equality in (3.8) by straightforward calculations. We calculate $T^{2,0} - T^{1,1} = N + In(T), \quad 3Bi(In(T)) = -d^a F$. Apply (3.5) to derive $3\left(Bi(T^{2,0}) - Bi(T^{1,1})\right) = -d^a F + (d^a F)^- = -(d^a F)^+$ which completes the proof of (3.8).

Denote by ψ^+ the (1,2)+(2,1)-form $Bi(T^{2,0}) + Bi(T^{1,1})$ and use (3.8) to get

$$Bi(T^{2,0}) = \frac{1}{2}\left(\psi^+ - \frac{1}{3}(d^a F)^+\right), \quad Bi(T^{1,1}) = \frac{1}{2}\left(\psi^+ + \frac{1}{3}(d^a F)^+\right). \tag{3.10}$$

The linear connection ∇ preserves the almost paracomplex structure if and only if A satisfies $A(X;PY,Z)+A(X;Y,PZ)=(\nabla^g F)(X;Y,Z)$ By means of (3.4) the last equality is equivalent to

$$-T(X;PY,Z)-T(X;Y,PZ)+\frac{3}{2}\left(Bi(T)(X;PY,Z)+Bi(T)(X;Y,PZ)\right)= \\ =(\nabla^g F)(X;Y,Z). \tag{3.11}$$

The first consequence of (3.10) and (3.11) is that the (1,1)-part of T which satisfies the Bianchi identity is free, denote it by B_b. Take the (0,2) and (2,0) parts of (3.11), apply (3.6), (3.7) and use (3.8), (3.10) to get formula (3.9). □

Corollary 3.3. *Let (M,g,P) be a 2n-dimensional almost para-Hermitian manifold. There exists parahermitian connection on M with totally skew-symmetric torsion if and only if the Nijenhuis tensor is totally skew-symmetric. In this case the connection is unique and the torsion T is given by*

$$T=(d^aF)^+-N \tag{3.12}$$

Proof. Assume T is a 3-form. Then N is a 3-form due to (3.8) and $B_b=0$. We claim $\psi^+=(d^aF)^+$. Indeed, $\psi^+=\frac{3}{4}T+\frac{1}{4}d^aF$. On the other hand, $\psi^+=Bi(T^{2,0})+Bi(T^{1,1})=T+N=T+\frac{1}{3}(d^aF)^-$. Hence, the claim follows. Substituting $\psi^+=(d^aF)^+$ into (3.9) we get (3.12). The corollary follows from Theorem 3.2 □

We shall call this connection *Bismut connection*.

Definition 3.4. A parahermitian connection is called canonical if its torsion T satisfies the following conditions

$$T_b^{1,1}=0, \quad (Bi(T))^+=-\frac{2t-1}{3}(d^aF)^+ \tag{3.13}$$

for some real parameter t. We denote the corresponding connection by ∇^t.

Combining (3.9) with (3.13) we get that the torsion T^t of ∇^t is given by

$$T^t=-N-\frac{3t-1}{4}(d^aF)^+-\frac{t+1}{4}In(d^aF)^+.$$

Any canonical connection is connected with the Levi-Civita connection by

$$g(\nabla^t_XY,Z)=g(\nabla^g_XY,Z)-\frac{1}{2}g(\nabla^g_XP)(PY,Z)- \\ \frac{t}{4}\left((d^aF)^+(X,Y,Z)-(d^aF)^+(X,PY,PZ)\right). \tag{3.14}$$

The parahermitian connection with torsion 3-form is the canonical connection given by $t=-1$. Another remarkable connection is the canonical connection obtained for $t=0$ [59],

$$g(\nabla^0_XY,Z)=g(\nabla^g_XY,Z)-\frac{1}{2}g(\nabla^g_XP)(PY,Z), \quad T^0=-N+\frac{1}{4}(d^aF)^+-\frac{1}{4}In(d^aF)^+.$$

Note that if $dF^+=0$ then the real line of the canonical connections degenerates to a point ∇^0 with torsion $T^0=-N$. Almost para-Hermitian manifolds satisfying the condition $dF^+=0$

are called *quasi-para-Kähler* or *(1,2)-symplectic.* In view of Proposition 3.1, quasi-para-Kähler manifolds are characterized by [59], $(\nabla^g_{PX}F)(PY,Z) - ((\nabla^g_X F)(Y,Z) = 0$.

3.1. **Canonical connection on para-Hermitian manifold.** We apply our previous discussion to a para-Hermitian manifold, $N = 0$.

Theorem 3.5. *Let (M, g, P) be a 2n-dimensional para-Hermitian manifold.*

a) There exists unique parahermitian connection ∇^1 on M with torsion $T^1 \in \Lambda^{2,0}(TM)$ i.e. T^1 satisfies

$$T^1(PX,Y) = PT^1(X,Y).$$

This connection is the canonical connection obtained by $t = 1$ and given by

$$g(\nabla^1_X Y, Z) = g(\nabla^g_X Y, Z) - \frac{1}{2}dF(PX,Y,Z). \tag{3.15}$$

b) The curvature $R^1 := [\nabla^1, \nabla^1] - \nabla^1_{[,]}$ is of type (1,1) in the sense that

$$R^1(PX,PY) = -R^1(X,Y).$$

Proof. From $N = 0$ we get $(d^aF)^- = 0, d^aF = (d^aF)^+$. Apply Theorem 3.2. We have $B_b = 0, \psi^+ = Bi(T) = -\frac{1}{3}d^aF$ since $T^1 \in \Lambda^{2,0}(TM)$. Hence, this is the canonical connection obtained for $t = 1$ which proves a).

To prove b) we consider the paracomplex coordinate system $(x^1, ..., x^n, \bar{x}^1, ...\bar{x}^n)$ around a point $p \in M$ such that $\frac{\partial}{\partial x^1}, ..., \frac{\partial}{\partial x^n}$ is an $+$-eigen basis of T_pM^+ and $\frac{\partial}{\partial \bar{x}^1}, ..., \frac{\partial}{\partial \bar{x}^n}$ is an $-$-eigen basis of T_pM^-, i.e. $P\frac{\partial}{\partial x^i} = \frac{\partial}{\partial x^i}, \quad P\frac{\partial}{\partial \bar{x}^i} = -\frac{\partial}{\partial \bar{x}^i}$. Then the metric and the fundamental 2-form are given by $g = 2g_{i\bar{j}}dx^i dx^{\bar{j}}, \quad F = F_{i\bar{j}}dx^i \wedge dx^{\bar{j}}, \quad F_{i\bar{j}} = -F_{\bar{j}i} = -g_{i\bar{j}}$.

Summation in repeated indexes is always assumed. We adapt the following convention: for a tensor K of type (p,q), the symbol $\overline{K^{j_1,...,\bar{j}_p}_{i_1,...\bar{i}_q}}$ means $K^{\bar{j}_1,...,j_p}_{\bar{i}_1,...i_q}$.

We easily derive the expressions

$$dF_{ijk} = dF_{\bar{i}\bar{j}\bar{k}} = 0, \quad dF_{ij\bar{k}} = \frac{\partial g_{i\bar{k}}}{\partial x^j} - \frac{\partial g_{j\bar{k}}}{\partial x^i}, \quad dF_{\bar{i}\bar{j}k} = \frac{\partial g_{k\bar{j}}}{\partial x^{\bar{i}}} - \frac{\partial g_{k\bar{i}}}{\partial x^{\bar{j}}} = -\overline{dF_{ij\bar{k}}}.$$

The Koszul formula gives for the local components Γ^k_{ij} of the Levi-Civita connection the expressions

$$\Gamma^k_{ij} = \frac{1}{2}g^{k\bar{s}}\left(\frac{\partial g_{i\bar{s}}}{\partial x^j} + \frac{\partial g_{j\bar{s}}}{\partial x^i}\right), \quad \Gamma^{\bar{k}}_{\bar{i}\bar{j}} = \frac{1}{2}g^{\bar{k}s}\left(\frac{\partial g_{\bar{i}s}}{\partial x^{\bar{j}}} + \frac{\partial g_{\bar{j}s}}{\partial x^{\bar{i}}}\right),$$
$$\Gamma^k_{i\bar{j}} = \frac{1}{2}g^{k\bar{s}}dF_{\bar{j}\bar{s}i} = \Gamma^k_{\bar{j}i}, \quad \Gamma^{\bar{k}}_{i\bar{j}} = \frac{1}{2}g^{s\bar{k}}dF_{si\bar{j}} = \Gamma^{\bar{k}}_{\bar{j}i}, \quad \Gamma^{\bar{k}}_{ij} = \Gamma^k_{\bar{i}\bar{j}} = 0. \tag{3.16}$$

The local components C^k_{ij} of ∇^1 are calculated from (3.15) and (3.16)

$$C^k_{ij} = g^{k\bar{s}}\frac{\partial g_{j\bar{s}}}{\partial x^i}, \quad C^{\bar{k}}_{\bar{i}\bar{j}} = g^{s\bar{k}}\frac{\partial g_{s\bar{j}}}{\partial x^{\bar{i}}}, \quad C^k_{i\bar{j}} = C^k_{\bar{i}j} = C^{\bar{k}}_{ij} = C^k_{\bar{i}\bar{j}} = C^{\bar{k}}_{i\bar{j}} = C^{\bar{k}}_{\bar{i}j} = 0 \tag{3.17}$$

The curvature tensor R^1 has the property $R^1 \circ P = P \circ R^1$ since $\nabla^1 P = 0$. To prove b) it is sufficient to show $R^1_{ijk\bar{l}} = R^1_{\bar{i}\bar{j}\bar{k}l} = 0$ which is a direct consequence of (3.17). $\square$

Further we shall call ∇^1 *the Chern connection*. This connection coincides with the canonical compatible connection of the tangent bundle viewed as a parahermitian, paraholomorphic bundle of rank n defined in [26].

Corollary 3.6. *The curvature R^g of the Levi-Civita connection of a para-Hermitian manifold satisfies the identities $R^g_{ijkl} = R^g_{\bar{i}\bar{j}\bar{k}\bar{l}} = 0$, equivalently*

$$R^g(X,Y,Z,V) + R^g(PX,PY,PZ,PV) + R^g(X,Y,PZ,PV) + R^g(X,PY,Z,PV)+$$
$$R^g(X,PY,PZ,V) + R^g(PX,PY,Z,V) + R^g(PX,Y,PZ,V) + R^g(PX,Y,Z,PV) = 0.$$

The curvature R^1 and the torsion T^1 of the Chern connection are given by

$$R^1_{i\bar{j}k\bar{l}} = -g_{s\bar{l}} \frac{\partial C^s_{ik}}{\partial x^{\bar{j}}} = -g_{s\bar{l}} \frac{\partial}{\partial x^{\bar{j}}} \left(g^{s\bar{m}} \frac{\partial g_{k\bar{m}}}{\partial x^i} \right), \quad T_{\bar{k}ij} = dF_{ij\bar{k}}.$$

3.2. **Ricci forms of the canonical connections.** For a linear connection ∇ with curvature tensor R on an almost para-Hermtitan manifold of dimension $2n$ we have Ricci type tensors:

- the Ricci tensor $\rho(X,Y) := \sum_{i=1}^{2n} \epsilon_i R(e_i, X, Y, e_i)$;
- the *-Ricci tensor $\rho^*(X,Y) := \sum_{i=1}^{2n} \epsilon_i R(e_i, X, PY, Pe_i)$;
- the Ricci form $r(X,Y) := -\frac{1}{2} \sum_{i=1}^{2n} \epsilon_i R(X,Y,e_i,Pe_i) = -\sum_{i=1}^{n} R(X,Y,e_i,Pe_i)$.

The scalar curvatures are defined to be the corresponding trace:

- the scalar curvature $s = tr_g \rho = \sum_{i=1}^{2n} \epsilon_i \rho(e_i, e_i)$,,
- the $*$-scalar curvature $s^* = tr_g \rho^* = 2\sum_{i=1}^{2n} \epsilon_i \rho(e_i, e_i)$,
- the trace of the Ricci form $\tau = tr_g r = \sum_{i,j=1}^{n} r(e_i, Pe_i)$.

For the Levi-Civita connection, we have the properties (see [48])
$\rho^{g*}(X,Y) = r^g(X,PY)$, $\quad \rho^{g*}(X,Y) + \rho^{g*}(PY,PX) = 0$ and consequently, $s^{g*} = \tau^g$.

To find relations between the Ricci forms of the canonical hermitian connection we consider the paraholomorphic canonical bundle $\Lambda^n_{\mathbb{A}}(TM)$. Any linear connection preserving the structure P, i.e. preserving the eigensubbundles TM^+ and TM^-, induces a connection on the line bundle $\Lambda^n_{\mathbb{A}}(TM)$ with curvature equal to (-) its Ricci form. Let s be a section of $\Lambda^n_{\mathbb{A}}(TM)$. From (3.14) we infer that $\nabla^t s = \nabla^0 s + \frac{t}{2} P\theta \otimes s$. Consequently $r^t = r^0 - \frac{t}{2} d(P\theta)$. In particular the Ricci forms of the Bismut and Chern connection are related by

$$r^{-1} = r^1 + d(P\theta). \tag{3.18}$$

4. Para-Hermitian 4-manifold

In this section we find a para-hermitian analogue of the Apostolov-Gauduchon generalization [6] of the Goldberg-Sachs theorem in General Relativity (see e.g. [49]). We prove the result using the properties of the Chern connection.

Let (M,g) be an oriented pseudo-Riemannian 4-manifold with neutral metric g of signature $(+,+,-,-)$. This is equivalent, on one hand to the existing of an almost paracomplex structure, and on the other, to the existence of two kinds of almost complex structures. In a compact case the second property leads to topological obstructions to the existence of neutral metric expressed in terms of the signature and the Euler characteristic [46].

The bundle $\Lambda^2 M$ of real 2-forms of a neutral Riemannian 4-manifold splits

$$\Lambda^2 M = \Lambda^+ M \oplus \Lambda^- M, \tag{4.19}$$

where $\Lambda^+ M$, resp. $\Lambda^- M$ is the bundle of self-dual, resp. anti-self-dual 2-forms, i.e. the eigen-subbundle with respect to the eigenvalue $+1$, resp. -1, of the Hodge $*$-operator acting as an involution on $\Lambda^2 M$. We may also consider the connected component $SO^+(2,2)$ of the structure group $SO(2,2)$. This group has the splitting $S^+(2,2) = SL(2) \times SL(2)$ which defines two real vector bundles (of rank 2) S^+ and S^- and $TM = S^M \oplus S^-$ which induces the splitting (4.19).

We will freely identify vectors and co-vectors via the metric g.

The self-dual part $W^+ = \frac{1}{2}(W + *W)$ of the Weyl tensor W is viewed as a section of the bundle $\mathbb{W}^+ = Sym_0 \Lambda^+ M$ of symmetric traceless endomorphisms of $\Lambda^+ M$.

Let P be an almost paracomplex structure compatible with the metric g such that (g, P) defines an almost para-Hermitian structure. Then the fundamental 2-form F is a section of $\Lambda^+ M$ and has constant norm 2. Conversely, any smooth section of $\Lambda^+ M$ with constant norm 2 is the fundamental 2-form of an almost paracomplex structure. Our considerations in this section are complementary to that in [5] in the sense that a section of $\Lambda^+ M$ with norm -2 can be considered as a Kähler form of an almost complex structure, the case investigated in [5].

We have the following orthogonal splitting for $\Lambda^+ M$

$$\Lambda^+ M = \mathbb{R}.F \oplus \Lambda_0^+ M, \tag{4.20}$$

where $\Lambda_0^+ = \Lambda^{0,2} M \oplus \Lambda^{2,0} M$ denotes the bundle of P-invariant real 2-forms ϕ, $\phi(PX, PY) = \phi(X, Y)$.

In accordance with (4.20) the bundle $\mathbb{W}^+$ splits into three pieces as follows:

$$\mathbb{W}^+ = \mathbb{W}_1^+ \oplus \mathbb{W}_2^+ \oplus \mathbb{W}_3^+,$$

where

- $\mathbb{W}_1^+ = \mathbb{M} \times \mathbb{R}$ is the subbundle of elements preserving (4.20) and acting by the homothety on the two factors, hence the trivial line bundle generating by the elements $\frac{3}{4} F \otimes F - \frac{1}{2} id$;
- $\mathbb{W}_2^+$ is the subbundle of elements which exchange the two factors in (4.20): each element $\phi \in \Lambda_0^+ M$ is identified with the element $\frac{1}{2}(F \otimes \phi + \phi \otimes F)$;
- $\mathbb{W}_3^+$ is the subbundle of elements preserving the splitting (4.20) and acts trivially on the first factor $\mathbb{R}.F$, i.e. it is the space of those endomorhisms of Λ_0^+ which are P-invariant.

Thus, W^+ can be written in the form

$$W^+ = f\left(\frac{3}{4} F \otimes F - \frac{1}{2} id\right) + \frac{1}{2}(F \otimes \phi + \phi \otimes F) + W_3^+, \tag{4.21}$$

where f is some real function.

In dimension 4 the Lee form θ determines dF completely by

$$dF = \theta \wedge F. \tag{4.22}$$

In particular, $d\theta$ is trace-free, $\sum_{i=1}^{2} d\theta(e_i, Pe_i) = 0$. Hence, the self-dual part $d\theta^+$ of $d\theta$ is a section of $\Lambda_0^+ M$.

To any 4-dimensional almost para-Hermitian manifold with a Lee form θ one can associate the canonical Weyl structure, i.e. a torsion-free connection ∇^w determined by the equation $\nabla^w g = \theta \otimes g$. The conformal scalar curvature k of an almost para-Hermitian structure is defined to be the scalar curvature of the canonical Weyl structure with respect to the metric g. Then (see e.g. [30])

$$k = s - \frac{3}{2}\left(g(\theta,\theta) + 2\delta\theta\right). \tag{4.23}$$

The conformal scalar curvature is conformally invariant of weight -2, i.e. if $g' = f^{-2}g$ then $k' = f^2 k$.

4.1. **Curvature of para-Hermitian 4-manifold.** Let (M, g, P) be a 4-dimensional para-Hermitian manifold. The Chern connection ∇^1 and the Levi-Civita connection are related by $g(\nabla^1_X Y, Z) = g(\nabla^g_X Y, Z) - \frac{1}{2}(\theta \wedge F)(PX, Y, Z)$ due to (3.15) and (4.22). Consequently,

$$\begin{aligned} R^1(X,Y,Z,V) &= R^g(X,Y,Z,V) - \frac{1}{2} d(P\theta)(X,Y)F(V,Z) + \\ &\frac{1}{2}\left(L(Y,Z)g(V,X) - L(X,Z)g(V,Y) + L(X,V)g(Y,Z) - L(Y,V)g(Z,X)\right), \end{aligned} \tag{4.24}$$

where the tensor L can be expressed in the following form

$$L(X,Y) = (\nabla^g_X \theta)Y + \frac{1}{2}\theta(X)\theta(Y) - \frac{1}{4} g(\theta,\theta) g(X,Y). \tag{4.25}$$

The curvature R^1 is of type (1,1) according to Theorem 3.5. Then (4.24),(4.25) imply, in local para-holomorphic coordinates, that

$$\begin{aligned} R^g_{ijk\bar{l}} &= -\frac{1}{2}\left(L_{jk} g_{i\bar{l}} - L_{ik} g_{j\bar{l}} + d\theta_{ij} g_{k\bar{l}}\right), \quad R^g_{\bar{i}\bar{j}\bar{k}l} = \overline{R^g_{ijk\bar{l}}}; \\ R^g_{ij\bar{k}\bar{l}} &= -\frac{1}{2}\left(L_{j\bar{k}} g_{i\bar{l}} - L_{i\bar{k}} g_{j\bar{l}} - L_{j\bar{l}} g_{i\bar{k}} + L_{i\bar{l}} g_{j\bar{k}}\right), \quad R^g_{\bar{i}\bar{j}kl} = \overline{R^g_{ij\bar{k}\bar{l}}}, \\ L_{ij} &= \nabla^g_i \theta_j + \frac{1}{2}\theta_i\theta_j, \qquad L_{i\bar{j}} = \nabla^g_i \theta_{\bar{j}} + \frac{1}{2}\theta_i\theta_{\bar{j}} - \frac{1}{4}|\theta|^2 g_{i\bar{j}}. \end{aligned} \tag{4.26}$$

We take the traces in (4.24), (4.26), (3.18) and use (4.23) to prove our technical

Proposition 4.1. *The Ricci tensors and the scalar curvatures of a 4-dimensional para-Hermitian manifold satisfy the conditions*

$$\rho^g_{jk} = R^g_{ijk\bar{i}} + R^g_{ikj\bar{i}} = -\frac{1}{2}\left(\nabla^g_j\theta_k + \nabla^g_k\theta_j + \theta_j\theta_k\right), \quad \rho^g_{\bar{j}\bar{k}} = \overline{\rho^g_{jk}},$$
$$\rho^{g*}_{jk} = -R^g_{ijk\bar{i}} + R^g_{ikj\bar{i}} = -\frac{1}{2}d\theta_{jk}, \quad \rho^{g*}_{\bar{j}\bar{k}} = \overline{\rho^{g*}_{jk}}$$
$$\rho^g_{j\bar{k}} + \rho^{g*}_{j\bar{k}} = 2R^g_{ij\bar{k}\bar{i}} = \left(\frac{1}{2}\delta\theta + \frac{1}{4}g(\theta,\theta)\right)g_{j\bar{k}},$$
$$s + s^* = 2\delta\theta + g(\theta,\theta),$$
$$k = -\frac{1}{2}(s + 3s^*) = -\tau^{-1}.$$

In particular, the conformal scalar curvature is equal to $(-)$ *the trace of the Ricci form of the Bismut connection. Therefore, the trace of the Ricci form of the Bismut connection is a conformal invariant of weight* -2.

We note that the expression of the (1,1)-part of the sum of the two Ricci tensors and the formula for the sum of the two scalar curvatures in Proposition 4.1 was obtained in [48].

The structure of W^+ on a 4-dimensional para-Hermitian manifold is similar to that of the hermitian manifold presented in [6]. We describe it in the following

Lemma 4.2. *On a 4-dimensional para-Hermitian manifold the third component* W^+_3 *of* W^+ *vanishes identically and the positive Weyl tensor is given by*

$$W^+ = \frac{k}{8}F\otimes F - \frac{k}{12}id - \frac{1}{4}\psi\otimes F - \frac{1}{4}F\otimes\psi, \tag{4.27}$$

where the two form ψ *is determined by the self-dual part of* $d\theta_+, \psi_{ij} = d\theta_{ij}$.

Proof. On a 4-dimensional pseudo-Riemannian manifold the Weyl tensor is expressed in terms of the normalized Ricci tensor $h = -\frac{1}{2}\left(\rho - \frac{s}{6}g\right)$ as follows

$$\begin{aligned} W(X,Y,Z,V) = R^g(X,Y,Z,V) - h(X,Z)g(Y,V) + \\ h(Y,Z)g(X,V) - h(Y,V)g(X,Z) + h(X,V)g(Y,Z). \end{aligned} \tag{4.28}$$

The condition $W^+_3 = 0$ is a consequence of Corollary 3.6. According to (4.21) we have $W(F) = W^+(F) = fF + \psi$. We calculate from (4.28) applying Lemma 4.1 that

$$W_{ijk\bar{k}} = -\frac{1}{2}d\theta_{ij}, \quad W_{i\bar{j}k\bar{k}} = \frac{k}{6}g_{i\bar{j}}.$$

Hence, the lemma follows. □

Another glance at (3.18) leads to the expression $r^{-1}_{ij} = -d\theta_{ij}, \quad r^{-1}_{\bar{i}\bar{j}} = d\theta_{\bar{i}\bar{j}}$ since the Ricci form of the Chern connection is of type (1,1). A combination of the last equalities with Lemma 4.2 implies

Proposition 4.3. *A 4-dimensional para-Hermitian manifold is anti-self-dual* ($W^+ = 0$) *if and only if the Ricci form of the Bismut connection is an anti-self-dual 2-form.*

Consider the codifferential of the positive Weyl tensor δW^+ as an element of $\Lambda^2_0(T^*M)$. Then we have the splitting

$$\delta W^+ = (\delta W^+)^+ \oplus (\delta W^+)^-,$$

where $(\delta W^+)^+$ is a section of $\Lambda_0^{2,0+1,1}(T^*M)$ while $(\delta W^+)^-$ is a section of $\Lambda_0^{0,2}(T^*M)$. In particular, $(\delta W^+)^- = 0$ if and only if the co-differential of the whole Weyl tensor vanishes on any three (1,0)-vectors.

Theorem 4.4. *Let (M, g, P) be a 4-dimensional para-Hermitian manifold. Then the following conditions are equivalent:*

- *a) The 2-form $d\theta$ is anti-self-dual, $d\theta^+ = 0$;*
- *b) $W_2^+ = 0$, equivalently, the fundamental 2-form is an eigenform of W^+;*
- *c) $(\delta W^+)^- = 0$, equivalently, $(\delta W)(X^{1,0}; Y^{1,0}, Z^{1,0}) = 0$.*

Proof. The equivalence a) $\Leftrightarrow$ b) is proved in Lemma 4.2.

The second Bianchi identity reads as

$$\delta W(X;Y,Z) = (\nabla^g_Y h)(Z,X) - (\nabla^g_Z h)(Y,X)$$

On (1,0) vectors it gives due to (3.16) that

$$(\delta W^+)(X^{1,0};Y^{1,0},Z^{1,0}) = (\nabla^g_{Y^{1,0}}\rho)(Z^{1,0},X^{1,0}) - (\nabla^g_{Z^{1,0}}\rho)(Y^{1,0},X^{1,0}). \tag{4.29}$$

Assume $d\theta_{ij} = 0$. Then Lemma 4.1, the Ricci identities and (4.26) imply $\nabla^g_i\rho_{jk} - \nabla^g_j\rho_{ik} = 0$. Hence, $(\delta W^+)^- = 0$ due to (4.29). The implication a) $\Rightarrow$ c) is proved.

Let $(\delta W^+)^- = 0$. We use the Chern connection. For its local components and torsion tensor we have

$$C^k_{ij} = \Gamma^k_{ij} + \frac{1}{2}(\theta_i\delta^k_j - \theta_j\delta^k_i), \quad T^k_{ij} = \theta_i\delta^k_j - \theta_j\delta^k_i.$$

Equation (4.29), in terms of the Chern connection, takes the form

$$\nabla^1_i\rho_{jk} - \nabla^1_j\rho_{ik} = \frac{3}{2}\left(\theta_j\rho_{ik} - \theta_i\rho_{jk}\right). \tag{4.30}$$

The Ricci identities for the Chern connection, $\nabla^1_i\nabla^1_j\theta_k - \nabla^1_j\nabla^1_i\theta_k = \theta_j\nabla^1_i\theta_k - \theta_i\nabla^1_j\theta_k$, the first equality in Lemma 4.1 and (4.30) yield

$$\nabla^1_i d\theta_{jk} - \nabla^1_j d\theta_{ik} = \theta_k d\theta_{ij} - \frac{3}{2}(\theta_i d\theta_{jk} - \theta_j d\theta_{ik}).$$

Make a cyclic permutation in the latter then add the two and subtract the third of the obtained equalities to get

$$\nabla^1_i d\theta_{jk} = -2\theta_i d\theta_{jk} + \frac{1}{2}(\theta_j d\theta_{ki} - \theta_k d\theta_{ji}). \tag{4.31}$$

Take the covariant derivative in (4.31) and apply (4.31) to the obtained result to derive

$$\begin{aligned}\nabla^1_l\nabla^1_i d\theta_{jk} &= -2\nabla^1_l\theta_i d\theta_{jk} + \frac{1}{2}(\nabla^1_l\theta_j d\theta_{ki} - \nabla^1_l\theta_k d\theta_{ji}) \\ &+4\theta_i\theta_l d\theta_{jk} + \frac{5}{4}(\theta_i\theta_j d\theta_{lk} - \theta_i\theta_k d\theta_{lj}) - (\theta_j\theta_l d\theta_{ki} - \theta_k\theta_l d\theta_{ji})\end{aligned} \tag{4.32}$$

The Ricci identity $\nabla^1_i\nabla^1_j d\theta_{kl} - \nabla^1_j\nabla^1_i d\theta_{kl} = -\theta_i\nabla^1_j d\theta_{kl} + \theta_j\nabla^1_i d\theta_{kl}$ and (4.32) imply

$$2d\theta_{li}d\theta_{jk} + \frac{3}{4}(\theta_i\theta_j d\theta_{kl} + \theta_i\theta_k d\theta_{lj} - \theta_j\theta_l d\theta_{ki} - \theta_k\theta_l d\theta_{ij}) =$$
$$\frac{1}{2}(d\theta_{ij}\nabla^1_l\theta_k + d\theta_{ki}\nabla^1_l\theta_j - d\theta_{kl}\nabla^1_i\theta_j - d\theta_{lj}\nabla^1_i\theta_k).$$

Change $l \leftrightarrow j, \quad i \leftrightarrow k$ into the latter equality and sum up the results to obtain

$$4d\theta_{li}d\theta_{jk} = d\theta_{lk}d\theta_{ij} + d\theta_{lj}d\theta_{ki}.$$

From the last equality we easily infer $5d\theta_{li}d\theta_{jk} = 0$. Hence, $d\theta_{jk} = 0$. □

Corollary 4.5. *Assume that the Ricci tensor ρ of a para-Hermitian 4-manifold is P-anti-invariant, i.e. $\rho_{ij} = 0$. Then $d\theta$ is anti-self-dual 2-form, $d\theta_{ij} = 0$, equivalently $W^+_2 = 0$.*

In particular, on a para-Hermitian Einstein 4-manifold the fundamental 2-form is an eigenform of the positive Weyl tensor.

We consider the question of integrability of totally isotropic real 2-plane supplementary distributions on an oriented 4-dimensional neutral Riemannian manifold. Any such splitting of the tangent bundle defines an almost paracomplex structure compatible with the neutral metric such that we get an almost para-Hermitian 4-manifold. The integrability of 2-plane supplementary distributions is equivalent to the integrability of the almost paracomplex structure. A necessary condition is the vanishing of the third component W^+_3 of the positive Weyl tensor which is equivalent to the vanishing of the whole Weyl tensor on the 2-plane distribution. Note that this is equivalent to the vanishing of the whole curvature on the 2-plane i.e. the identity in Corollary 3.6 holds. This leads to the fourth order polynomial equation [4] (see also [5]) which cannot have always real-root solutions and in this case there are no integrable 2-plane supplementary distributions. In the case of existence, we give sufficient conditions in the following

Theorem 4.6. *Let (M, g) be an oriented neutral Riemannian 4-manifold with nowhere vanishing positive Weyl tensor W^+. Suppose that P is an almost paracomplex structure such that W^+ vanishes on each eigen-subbundle determined by P, i.e. the component W^+_3 of W^+ with respect to P vanishes. Then any of the two following three conditions imply the third:*

i) $W^+_2 = 0$;

ii) $(\delta W^+)^- = 0$;

iii) the paracomplex structure P is integrable.

Proof. Observe that any smooth section F of Λ^+M with constant norm 2 is the fundamental 2-form of an almost paracomplex structure. Replacing M by a two-fold covering, if necessary, the positive Weyl tensor W^+ can be written in the form (4.21), where f is a smooth function and $W^+_3 = 0$.

According to Theorem 4.4 we have to show that i) and ii) imply iii).

Assume $W_2^+ = 0$. Then $W^+ = \frac{3}{4} fF \otimes F - \frac{1}{2} fid$. Using the definition of the Lee form, we easily calculate that

$$(4.33) \qquad (\delta W^+)_X = \left(\frac{1}{2} Pdf(X) - \frac{3}{4} fP\theta(X) \right) F - \frac{3}{4} f \nabla^g_{PX} F + \frac{1}{4} \left(df \wedge X + Pdf \wedge PX \right).$$

The (0,2)-part of (4.33) gives $0 = (\delta W^+)^- = (\nabla^g F)^{0,2}$. Using (3.6) we infer $N = 0$. □

Corollary 4.7. *Let (M, g, P) be an almost para-Hermitian 4-manifold.*

i) Suppose $W^+ \neq 0$ everywhere and $W_2^+ = W_3^+ = 0$. Then $(\delta W^+)^+ = 0$ is equivalent to $d\left(|W^+|^{-\frac{2}{3}} F\right) = 0$.

ii) Suppose (M, g, P) is a para-Hermitian 4-manifold. If it has nowhere vanishing positive Weyl tensor then $\delta W^+ = 0$ if and only if $g' = |W^+|^{-\frac{2}{3}} g$ is a para-Kähler metric. The Ricci tensor ρ^g of g is P-anti-invariant if and only if the vector field $Pgrad_{g'} f$, where $f = |W^+|^{-\frac{1}{3}}$ is a Killing vector field with respect to the para-Kähler metric g'.

In particular, a para-Hermitian Einstein 4-manifold is either with everywhere vanishing positive Weyl tensor or is globally conformal to a para-Kähler space. In the latter case there exists non zero Killing vector field with respect to the para-Kähler metric.

Proof. The (2,0)+(1,1)-part of (4.33) yields

$$(\delta W^+)^+_X = \frac{3}{2} \left(\frac{1}{3} Pdf(X) - \frac{1}{2} fP\theta(X) \right) F + \frac{3}{4} \left[\left(\frac{1}{3} df - \frac{1}{2} f\theta \right) \wedge X + \left(\frac{1}{3} Pdf - \frac{1}{2} fP\theta \right) \wedge PX \right].$$

Assume $W_2^+ = 0$. Then the function f is nowhere vanishing otherwise W^+ will have zeros. Moreover $|W^+|^2 = (W^+(F,F))^2 = 4f^2$. The equation $(\delta W^+)^+ = 0$ is equivalent to $\theta = \frac{2}{3} dlnf$. Thus we prove i). The condition ii) is a consequence of i) and Theorem 4.6. □

5. Nearly para-Kähler manifolds

An almost para-Hermitian manifold is called *Nearly para-Kähler (nearly bilagrangian)* if the almost para-hermitian structure satisfies the identity

$$(\nabla^g_X P)X = 0, \quad \Leftrightarrow (\nabla^g_X F)(Y,Z) + (\nabla^g_Y F)(X,Z) = 0.$$

An example of Nearly para-Kählerian 6-manifold is given in [13].

We denote the unique canonical connection ∇^0 on a Nearly para-Kähler manifold by ∇.

Applying the statements in Proposition 3.1, we get

Proposition 5.1. *A Nearly para-Kähler manifold is quasi-Kähler, $dF^+ = 0$, the Nijenhuis tensor N is a 3-form and the torsion T of the unique canonical connection is determined by the Nijenhuis tensor, $T = -N = P\nabla P$.*

Many properties of Nearly para-Kähler manifolds are, in some sense, formally very similar to those of Nearly Kähler manifolds mainly studied by A. Gray [33, 34, 35]. Below we follow roughly [35, 43] (see also [15]).

Proposition 5.2. *On a Nearly para-Kähler manifold the following identity holds*

$$R^g(X,Y,Z,V) + R^g(X,Y,PZ,PV) = g((\nabla^g_X P)Y, (\nabla^g_Z P)V). \tag{5.34}$$

Proof. The nearly para-Kähler condition implies $(\nabla^g_X P)(Y, PY) = 0$. Then, we easily get that $R^g(X,Y,X,Y) + R^g(X,Y,PX,PY) = g((\nabla^g_X P)Y, (\nabla^g_X P)Y)$. Polarize the latter equality and use the Bianchi identity to obtain (5.34). □

Our crucial result in this section is the following

Theorem 5.3. *On a Nearly para-Kähler manifold the Nijenhuis tensor is parallel with respect to the canonical connection* ∇,

$$\nabla N = -\nabla T = 0.$$

Proof. The curvature R^g of the Levi-Civita connection and the curvature R of the canonical connection are related by

$$\begin{aligned} R^g(X,Y,Z,V) &= R(X,Y,Z,V) - \frac{1}{2}(\nabla_X T)(Y,Z,V) + \frac{1}{2}(\nabla_Y T)(X,Z,V) \\ &-\frac{1}{2}g\left(T(X,Y),T(Z,V)\right) - \frac{1}{4}g\left(T(Y,Z),T(X,V)\right) - \frac{1}{4}g\left(T(Z,X),T(Y,V)\right) \end{aligned} \tag{5.35}$$

Since $R \circ P = P \circ R$, equation (5.35) leads to

$$\begin{gathered} R^g(X,Y,Z,V) + R^g(X,Y,PZ,PV) = \\ -(\nabla_X T)(Y,Z,V) + (\nabla_Y T)(X,Z,V) - g(T(X,Y),T(Z,V)). \end{gathered}$$

Comparing the latter equality with (5.34), we derive $(\nabla_X T)(Y,Z,V) - (\nabla_Y T)(X,Z,V) = 0$. Take the cyclic sum and add the result to conclude $\nabla T = 0$. □

Corollary 5.4. *On a Nearly para-Kähler manifold the following identities hold*

$$R^g(X,Y,Z,V) + R^g(X,Y,PZ,PV) + R^g(PX,Y,PZ,V) + R^g(PX,Y,Z,PV) = 0,$$

$$R^g(X,Y,Z,V) = R^g(PX,PY,PZ,PV);$$

$$R(X,Y,Z,V) = R(Z,V,X,Y) = -R(PX,PY,Z,V) = -R(X,Y,PZ,PV);$$

$$\rho^g(PX,PY) = -\rho^g(X,Y), \quad \rho^{g*}(PX,PY) = -\rho^{g*}(X,Y) = -\rho^{g*}(Y,X),$$

$$\rho(X,Y) = \rho(Y,X), \qquad r(PX,PY) = -r(X,Y),$$

$$\rho^g(X,Y) - \rho(X,Y) = \frac{1}{2}\sum_{i=1}^n g(T(X,e_i),T(Y,e_i)), \tag{5.36}$$

$$\rho^g(X,Y) + \rho^{g*}(X,Y) = 2\sum_{i=1}^n g(T(X,e_i),T(Y,e_i)), \tag{5.37}$$

$$\rho^{g*}(X,Y) = r^g(X,PY) = r(X,PY) - \frac{1}{2}\sum_{i=1}^n g(T(X,e_i),T(Y,e_i)), \tag{5.38}$$

$$3\rho^g(X,Y) - \rho^{g*}(X,Y) = 4\rho(X,Y), \tag{5.39}$$

$$\rho^g(X,Y) + 5\rho^{g*}(X,Y) = 4r(X,PY). \tag{5.40}$$

Proof. Put $\nabla T = 0$ into (5.35) to get

$$R^g(X,Y,Z,V) = R(X,Y,Z,V) \tag{5.41}$$
$$-\frac{1}{2}g\left(T(X,Y),T(Z,V)\right) - \frac{1}{4}g\left(T(Y,Z),T(X,V)\right) - \frac{1}{4}g\left(T(Z,X),T(Y,V)\right)$$

All the identities in the corollary are easy consequences of (5.41). □

5.1. **Nearly para-Kähler manifolds of dimension 6.** We recall that a Nearly para-Kähler manifold is said to be of *constant type* $\alpha \in \mathbb{R}$ if

$$g((\nabla^g_X P)Y, (\nabla^g_X P)Y) = \alpha\left(g(X,X)g(Y,Y) - g^2(X,Y) + g^2(PX,Y)\right). \tag{5.42}$$

In the Nearly Kähler case the constant type condition occurs only in dimension 6 and any 6-dimensional Nearly Kähler manifold is an Einstein manifold with positive scalar curvature of positive constant type [35]. It is observed in [42] that in the Nearly para-Kähler case the constant type phenomena occurs but the zero-value of α cannot be excluded. We describe the structure of the Ricci tensor in the next

Theorem 5.5. *Any 6-dimensional strict nearly para-Kähler manifold is an Einstein manifold of constant type $\alpha \in \mathbb{R}$ and the following relations hold*

$$\rho^g = 5\alpha g, \quad \rho^{g*} = -\alpha g, \quad \rho = 4\alpha g. \tag{5.43}$$

Consequently, the Riemannian scalar curvature $s^g = 30\alpha$.

In particular, if $\alpha = 0$ then the manifold is Ricci flat.

Proof. Let $e_1, e_2, e_3, Pe_1, Pe_2, Pe_3$ be an orthonormal local basis of smooth vector fields. The torsion T of the canonical connection (or equivalently, the Nijenhuis tensor N) is a 3-form of type (3,0)+(0,3). Therefore we may write $T(e_1,e_2) = ae_3 + bPe_3$, where a and b are smooth functions which turn to be constants because the torsion is ∇-parallel. It is easy to calculate that (5.42) holds with $\alpha = a^2 - b^2$. Moreover, we get the formula

$$\sum_{i=1}^{3} g(T(X,e_i),T(Y,e_i)) = 2(a^2-b^2)g(X,Y) = 2\alpha g(X,Y). \tag{5.44}$$

The strict Nearly para-Kähler condition implies $(a,b) \neq (0,0)$. In particular, the (3,0)+(0,3)-form T is non-degenerate. On the other hand, $\nabla T = 0$ due to Theorem 5.3. Hence, the Ricci 2-form of the canonical connection vanishes as a curvature of a flat line bundle. The condition $r = 0$ and Corollary 5.4 complete the proof. □

Remark 5.6. Using similar arguments as in the Nearly Kähler situation [35] we derive that a Nearly para-Kähler manifold of non-zero constant type must be of dimension 6.

6. Examples, twistors and reflectors on para quaternionic manifolds

To obtain examples of Nearly para-Kähler manifolds we involve twistor machinery. We are going to adapt Salamon's twistor construction on quaternionic manifold [52, 53, 54] to the para quaternionic spaces.

6.1. **Para quaternionic manifolds.** Both quaternions H and para quaternions $\tilde{H}$ are real Clifford algebras, $H = C(2,0)$, $\tilde{H} = C(1,1) \cong C(0,2)$. In other words, the algebra $\tilde{H}$ of para quaternions is generated by the unity 1 and the generators J_1, J_2, J_3 satisfying the *para quaternionic identities*,

$$J_1^2 = J_2^2 = -J_3^2 = 1, \qquad J_1J_2 = -J_2J_1 = J_3. \tag{6.45}$$

We recall the notion of almost para quaternionic manifold introduced by Libermann [44]. An *almost para-hyper-complex structure* (Libermann called it *almost quaternionic strucuture of the second kind*) on a smooth manifold consists of two almost product structures J_1, J_2 and an almost complex structure J_3 which mutually anti-commute, i.e. these structures satisfy the para quaternionic identities (6.45). Such a structure is also called *complex product structure* [3, 1].

An *almost hyper-paracomplex structure* on a 4n-dimensional manifold M is a triple $\tilde{H} = (J_\alpha), \alpha = 1,2,3$, where J_α,$\alpha = 1,2$ are almost paracomplex structures $J_\alpha : TM \to TM$, and $J_3 : TM \to TM$ is an almost complex structure, satisfying the para quaternionic identities (6.45). When each $J_\alpha, \alpha = 1,2,3$ is an integrable structure, $\tilde{H}$ is said to be a *hyper-paracomplex structure* on M. Such a structure is also called sometimes *pseudo-hyper-complex* [24]. Any hyper-paracomplex structure admits a unique torsion-free connection ∇^{ob} preserving J_1, J_2, J_3 [3, 1] called *the Obata connection.*

In fact an almost hyper-paracomplex structure is hyper-paracomplex if and only if any two of the three structures J_α are integrable due to the following

Proposition 6.1. *The Nijenhuis tensors of an almost hyper-paracomplex structure are related by:*

$$2N_\alpha(X,Y) = N_\beta(J_\gamma X, J_\gamma Y) - J_\gamma N_\beta(J_\gamma X, Y) - J_\gamma N_\beta(X, J_\gamma Y) - J_\gamma^2 N_\beta(X,Y) +$$

$$N_\gamma(J_\beta X, J_\beta Y) - J_\beta N_\gamma(J_\beta X, Y) - J_\beta N_\gamma(X, J_\beta Y) - J_\beta^2 N_\gamma(X,Y)$$

Proof. The formula follows by definitions with long but standard computations. □

We note that during the preparation of the manuscript the formula in Proposition 6.1 appeared in the context of Lie algebras in [21].

An *almost para quaternionic structure* on M is a rank-3 subbundle $P \subset End(TM)$ which is locally spanned by almost para-hypercomplex structure $\tilde{H} = (J_\alpha)$; such a locally defined triple $\tilde{H}$ will be called an admissible basis of P. A linear connection D on TM is called *para quaternionic connection* if D preserves P, i.e. there exist locally defined 1-forms $\omega_\alpha, \alpha = 1,2,3$ such that

$$DJ_1 = -\omega_3 \otimes J_2 + \omega_2 \otimes J_3, \quad DJ_2 = \omega_3 \otimes J_1 + \omega_1 \otimes J_3, \quad DJ_3 = \omega_2 \otimes J_1 + \omega_1 \otimes J_2. \tag{6.46}$$

Consequently, the curvature R^D of D satisfies the relations

$$
\begin{aligned}
[R^D, J_1] &= -A_3 \otimes J_2 + A_2 \otimes J_3, \\
[R^D, J_2] &= A_3 \otimes J_1 + A_1 \otimes J_3, \qquad (6.47)\\
[R^D, J_3] &= A_2 \otimes J_1 + A \otimes J_2,
\end{aligned}
$$

$$A_1 = d\omega_1 + \omega_2 \wedge \omega_3, \quad A_2 = d\omega_2 + \omega_3 \wedge \omega_1, \quad A_3 = d\omega_3 + \omega_1 \wedge \omega_2.$$

An almost para quaternionic structure is said to be a *para quaternionic* if there is a torsion-free quaternionic connection. A *P-hermitian metric* is a pseudo Riemannian metric which is compatible with the almost hyper-paracomplex structure $\tilde{H} = (J_\alpha), \alpha = 1,2,3$ in the sense that the metric g is skew-symmetric with respect to each $J_\alpha, \alpha = 1,2,3$, i.e.

$$g(J_1., J_1.) = g(J_2., J_2.) = -g(J_3., J_3.) = -g(.,.). \qquad (6.48)$$

The metric g is necessarily of neutral signature (2n,2n).

An almost para quaternionic (resp. para quaternionic) manifold with P-hermitian metric is called an *almost para quaternionic Hermitian* (resp. *para quaternionic Hermitian*) manifold. If the Levi-Civita connection of a para quaternionic Hermitian manifold is para quaternionic connection then the manifold is said to be *para quaternionic Kähler* manifold. This condition is equivalent to the statement that the holonomy group of g is contained in $Sp(n,\mathbb{R})Sp(1,\mathbb{R})$ for $n \geq 2$ [29, 56]. A typical example is the para quaternionic projective space endowed with the standard para quaternionic Kähler structure [19]. Any para quaternionic Kähler manifold of dimension $4n \geq 8$ is known to be Einstein [29, 56]. If on a para quaternionic Kähler manifold there exists an admissible basis $(\tilde{H})$ such that each $J_\alpha, \alpha = 1,2,3$ is parallel with respect to the Levi-Civita connection then the manifold is called *hyper-para-Kähler*. Such manifolds are also called *hypersymplectic* [36], *neutral hyper-Kähler* [40, 27]. The equivalent characterization is that the holonomy group of g is contained in $Sp(n,\mathbb{R})$ if $n \geq 2$ [56].

When $n \geq 2$, the para quaternionic condition, i.e. the existence of torsion-free para quaternionic connection is a strong condition which is equivalent to the 1-integrability of the associated $GL(n,\tilde{H})Sp(1,\mathbb{R}) \cong GL(2n,\mathbb{R})Sp(1,\mathbb{R})$- structure [1, 3]. Such a structure is a type of a para-conformal structure [9].

6.2. **Hyper-paracomplex structures on 4-manifold.** For $n = 1$ an almost para quaternionic structure is the same as oriented neutral conformal structure and turns out to be always para quaternionic [24, 29, 56, 21]. The existence of a (local) hyper-paracomplex structure is a strong condition because of the next

Theorem 6.2. *If on a 4-manifold there exists a (local) hyper paracomplex structure then the corresponding neutral conformal structure is anti-self-dual.*

Proof. Let $(g,(J_\alpha),\alpha=1,2,3)$ be an almost hyper-paracomplex Hermitian structure with fundamental 2-form F_α associated to each J_α. Define (2,0)+(0,2)-form with respect to J_1 by $\Phi_1 = F_2 + F_3$. Denote by $\theta_1,\theta_2,\theta_3$ the corresponding Lee forms (defined by $\theta_\alpha = \delta F_\alpha \circ J_\alpha$).

Lemma 6.3. *The structure $(g,(J_\alpha),\alpha=1,2,3)$ is a hyper-paracomplex structure, if and only if the three Lee forms coincide, $\theta_1=\theta_2=\theta_3$.*

Proof. The Levi-Civita connection satisfies (6.46). Consequently the Nijenhuis tensors obey

$$N_\alpha = -B_\alpha\otimes J_\beta + J_\beta\otimes B_\alpha - J_\alpha B_\alpha\otimes J_\gamma + J_\gamma\otimes J_\alpha B_\alpha, \quad B_\alpha = \omega_\beta - J_\alpha^3\omega_\gamma. \tag{6.49}$$

Simple calculations using (6.46) give

$$\theta_1 = -J_2\omega_2 + J_3\omega_3, \quad \theta_2 = J_1\omega_1 + J_3\omega_3, \quad \theta_3 = -J_2\omega_2 + J_1\omega_1.$$

The last three identities and (6.49) yield

$$J_1(\theta_2-\theta_1) = B_3, \quad J_2(\theta_2-\theta_3) = B_1, \quad J_3(\theta_3-\theta_1) = B_2.$$

Another glance at (6.49) completes the proof of the lemma. □

Suppose that each $J_\alpha,\alpha=1,2,3$ is integrable. Denote the common Lee form by θ and take the 3-form T to be the Hodge-dual to θ with respect to g. We have the identities $T = *\theta = -\theta\circ J_1\wedge F_1 = -\theta\circ J_2\wedge F_2 = +\theta\circ J_3\wedge F_3$. Then the Bismut connections of the three structures coincide, i.e. the linear connection $\nabla^b := \nabla^g + \frac{1}{2}T$ preserves the metric and each $J_\alpha,\alpha=1,2,3$. Therefore, each fundamental two form is parallel with respect to this connection, $\nabla^b F_\alpha = 0, \alpha=1,2,3$ and so is Φ_1. Then the Ricci form of the Bismut connection vanishes. Hence, $W^+=0$ due to Proposition 4.3 □

Note that the integrability condition, Lemma 6.3, in the case of hyper-complex structure, is due to F. Battaglia and S. Salamon (see [32]).

Example 6.4. We consider the so-called *hyperbolic Hopf manifold* $\mathbb{R}\times S^2_2(1)$ isomorphic to the Lie group $\mathbb{R}\times SL(2,\mathbb{R})$. The Lie algebra $\mathbb{R}\times sl(2,\mathbb{R})\cong gl(2,\mathbb{R})$ has a basis $\{W,X,Y,Z\}$ with Z central and non-zero brackets given by

$$[X,Y]=W, \quad [Y,W]=-X, \quad [W,X]=Y.$$

An almost paracomplex structure on $\mathbb{R}\times S^2_2(1)$ is constructed in [13]. The Lie algebra $\mathbb{R}\times sl(2,\mathbb{R})$ supports a hyper-paracomplex structure given by [3, 21]

$$J_3Z = X, \quad J_3Y = W, \quad J_2Z = Y, \quad J_2X = -W.$$

We pick a compatible neutral metric g, in the corresponding conformal class, defined such that the basis $\{W,X,Y,Z\}$ is an orthonormal basis, X,Z have norm 1 while Y,W have norm -1, i.e. $g(X,X)=g(Z,Z)=-g(W,W)=-g(Y,Y)=1$.

Proposition 6.5. *The invariant hyper-para-Hermitian structure on the hyperbolic Hopf manifold $\mathbb{R}\times SL(2,\mathbb{R})$, described above, is non-flat conformally equivalent to a flat hyper-para-Kähler (hypersymplectic) structure.*

More precisely, the Lee form $\theta = -Z$ is ∇^g-parallel and therefore closed and the Obata connection is the Levi-Civita connection of the flat hyper-para-Kähler (hypersymplectic) metric $g^{ob} = e^{-t}g$, where t is the local coordinate on $\mathbb{R}$.

Proof. The Koszul formula gives the following non-zero terms:

$$2\nabla^g_X Y = W, \quad 2\nabla^g_Y W = -X, \quad 2\nabla^g_W X = Y,$$
$$2\nabla^g_X W = -Y, \quad 2\nabla^g_Y X = -W, \quad 2\nabla^g_W Y = X.$$

It is easy to check that g is not flat and $\theta = -Z = -dt$ satisfies $\nabla^g\theta = 0$. The Levi-Civita connection of the conformal metric $g' = e^{-t}g$ is determined by

$$2\nabla^{g'}_A B := 2\nabla^g_A B - \theta(A)B - \theta(B)A + g(A,B)\theta.$$

It is straightforward to verify that $\nabla^{g'}$ preserves J_1, J_2, J_3. Hence, it is the Obata connection and the metric g' is hyper-para-Kähler (hypersymplectic). It is not difficult to calculate that the connection $\nabla^{g'}$ is flat. □

Remark 6.6. An invariant Weyl-flat hyper-parahermitian structure on $\mathbb{R}\times S^2_2(1)$ is just the neutral product of the standard Lorenz metric of constant sectional curvature on the unit hyperbolic sphere $S^2_2(1)$ and the flat metric on $\mathbb{R}$. In coordinates (x,y,z,t) it has the form

$$ds^2 = (\cosh y)^2(\cosh z)^2 dxdx + dtdt - (\cosh z)^2 dydy - dzdz.$$

The universal cover $\widetilde{SL(2,\mathbb{R})}$ of the Lie group $SL(2,\mathbb{R})$ admits a discrete subgroup Γ such that the quotient space $(\widetilde{SL(2,\mathbb{R})}/\Gamma)$ is a compact 3-manifold [47, 51, 55]. Such a space has to be Seifert fibre space [55] and all the quotients are classified in [51]. The compact 4-manifold $M = S^1 \times (\widetilde{SL(2,\mathbb{R})}/\Gamma)$ admits a complex structure and is known as Kodaira-Thurston surface modeled on $S^1 \times \widetilde{SL(2,\mathbb{R})}$ [57]. We have

Theorem 6.7. *The Kodaira-Thurston surfaces $K = S^1\times(\widetilde{SL(2,\mathbb{R})}/\Gamma)$ modeled on $S^1\times\widetilde{SL(2,\mathbb{R})}$ admit an invariant locally conformally hyper-para-Kähler (hypersymplectic) flat structure which is not globally conformal to a hyper-para-Kähler (hypersymplectic) one. The Lee form of the structure is ∇^g-parallel.*

The surfaces K have odd first Betti number and do not admit any symplectic 2-form. Therefore K does not support any global (para)-Kähler structure.

Proof. The first part is a consequence of Proposition 6.5 since the hyper-parahermitian structure on $\mathbb{R}\times\widetilde{SL(2,\mathbb{R})}$ described in the proposition descends to K. This structure is not globally conformal to a hyper-para-Kähler (hypersymplectic) structure since the closed Lee θ is actually a 1 form on the circle S^1 and therefore cannot be exact.

The result of Wall [57] states that for any discrete subgroup Γ the manifold K has odd Betti number and possesses a complex structure which has Kodaira dimension 1. The work of Biquard [17] implies that a complex surface of this type does not admit symplectic structure. □

The 4-dimensional Lie algebras admitting a hyper-paracomplex structure were classified recently in [21]. It is shown in [21] that exactly 10 types of Lie algebras admit a hyper-paracomplex structure. Theorem 6.2 tells us that the corresponding neutral metrics are anti-self-dual. We show below that some of them are not conformally flat.

Note that all 4-dimensional Lie groups admitting anti-self-dual non conformally flat Riemannian metric are classified in [23].

Example 6.8. We keep the notations in [21].

i) Consider the solvable Lie algebra PHC5 with a basis $\{X, Y, Z, W\}$, non-zero bracket $[X, Y] = X$ and hyper-paracomplex structure given by

$$J_3 Z = W, \quad J_3 X = Y, \quad J_2 Z = W, \quad J_2 X = Y - Z, \quad J_2 Y = X + W.$$

Consider the oriented basis $A = X, \quad B = Y, \quad C = Y - Z, \quad D = -X - W$ and pick a compatible neutral metric g with non-zero values on the basis $\{A, B, C, D\}$ given by $g(A, A) = g(B, B) = -g(C, C) = -g(D, D) = 1$. The metric g on the corresponding simply connected solvable Lie group is conformally hyper-para-Kähler (hypersymplectic) since the Lee form $\theta = B - C$ is closed and therefore exact. It is anti-self-dual metric with non-zero Weyl tensor because its curvature $R^g(A, B, C, D) = 1$. In local coordinates $\{x, y, z, t\}$ it is given by

$$ds^2 = e^{2y} dxdx + dydy - e^{-y}(dxdt + dtdx) + (dydz + dzdy).$$

ii) Consider the solvable Lie algebras PHC6, PHC9, PHC10 defined by non-zero brackets:

PHC6 $[X, Y] = Z, [X, W] = X + aY + bZ, [W, Y] = Y$

PHC9 $[Z, W] = Z, [X, W] = cX + aY + bZ, [Y, W] = Y, c \neq 0$

PHC10 $[Y, X] = Z, [W, Z] = cZ, [W, X] = \frac{1}{2}X + aY + bZ, [W, Y] = (c - \frac{1}{2})Y, c \neq 0$

These algebras admit a hyper-paracomplex structure defined by

$$J_3 Z = Y, \quad J_3 X = W, \quad J_2 Z = Y, \quad J_2 X = W - Z, \quad J_2 W = X + Y.$$

Consider the oriented frame $A = X, B = W, C = W - Z, D = -X - Y$. A compatible metric g is defined such that the frame $\{A, B, C, D\}$ is orthonormal with $g(A, A) = g(B, B) = -g(C, C) = -g(D, D) = 1$. The Lee forms of these hyper-parahermitian structures are closed and the curvature satisfies

PHC6 $R^g(A, B, C, D) = (1 - a)$;

PHC9 $R^g(A, B, C, D) = \frac{1}{2}(2c^2 - 3c - 2ac + 2a + 1)$;

PHC10 $R^g(A, B, C, D) = \frac{1}{2}(c^2 + 2ac - c)$;

Clearly there are constants (a,b,c) such that the corresponding Lie algebras admit anti-self-dual neutral metric with non-zero Weyl tensor.

It turns out that the conformal structure $[g]$ induced by the invariant hyper-paracomplex structure on the corresponding simply connected 4-dimensional Lie group is actually generated by a hyper-para-Kähler (hypersymplectic) structure since the Lee form θ is closed (and therefore exact) in all 10 possible cases described in [21]. In some cases the Lee form is zero and the structure on the Lie algebra is hyper-para-Kähler (hypersymplectic), in particular, the Lie algebra defined in [3] and obtained from $PHC9$ for $c=-1, \quad a=b=0$. The corresponding solvable Lie group possesses an invariant hyper-para-Kähler (hypersymplectic) structure with non-zero Weyl tensor since the Lee form vanishes and the curvature has non-zero value on an orthonormal basis. Summarizing, we get

Proposition 6.9. *Any one of the nine simply connected solvable Lie groups corresponding to a solvable 4-dimensional Lie algebra admitting hyper-paracomplex structure supports a hyper-para-Kähler (hypersymplectic) structure.*

Remark 6.10. The hyper-para-Kähler (hypersymplectic) structures on the nine solvable Lie groups of Proposition 6.9 are invariant in the only four of the nine cases due to the recent classification of the hyper-para-Kähler (hypersymplectic) 4-dimensional Lie algebras [2].

Due to the Malcev theorem [45], the 4-dimensional nilpotent Lie group H has a discrete subgroup Γ such that the quotient $M=H/\Gamma$ is a compact nil-manifold, the Kodaira Thurston surface. It is known that it admits hyper-para-Kähler (hypersymplectic) structure [40], see also [27].

The solvable Lie group G corresponding to some of the solvable Lie algebras admitting a hyper-paracomplex structure has a discrete subgroup Γ such that the quotient $M=H/\Gamma$ is a compact solve-manifold. Thus, we obtain compact examples of anti-self-dual neutral metrics which all are locally conformally hyper-para-Kähler (hypersymplectic) and most of them are with non vanishing Weyl tensor.

We remark two important particular cases.

Sol_1^4. Let us take $a=b=0$ in the Lie algebra $PHC6$. The Lee form $\theta=B-C$ is not ∇^g-parallel but closed and the Weyl curvature does not vanish because $R(A,B,C,D)=1$. The corresponding solvable Lie group is known to be Sol_1^4. In coordinates x,y,z,t the left invariant vector fields A,B,C,D can be expressed as follows

$$A=e^{-t}\frac{\partial}{\partial x},\quad B=\frac{\partial}{\partial t},\quad C=\frac{\partial}{\partial t}-\frac{\partial}{\partial z},\quad D=-e^{-t}\frac{\partial}{\partial x}-e^{t}\frac{\partial}{\partial y}-e^{t}x\frac{\partial}{\partial z}.$$

The invariant neutral anti-self-dual metric with non-zero Weyl tensor has the form

$$ds^2=e^{2t}dxdx+dtdt-(dxdy+dydx)+(dtdz+dzdt)-x(dtdy+dydt).$$

Sol_0^4. Let us take $c=-2, \quad a=b=0$ in the Lie algebra $PHC9$. The Lee form $\theta=B-C$ is not ∇^g-parallel but closed and the Weyl curvature does not vanish because $R(A,B,C,D)=15/2$. The corresponding solvable Lie group is known to be Sol_0^4. In coordinates x,y,z,t the left invariant vector fields A,B,C,D can be expressed as follows

$$A=e^{-2t}\frac{\partial}{\partial x},\quad B=\frac{\partial}{\partial t},\quad C=\frac{\partial}{\partial t}-e^{t}\frac{\partial}{\partial z},\quad D=-e^{-2t}\frac{\partial}{\partial x}-e^{t}\frac{\partial}{\partial y}.$$

The invariant neutral anti-self-dual metric with non-zero Weyl tensor has the form

$$ds^2=e^{4t}dxdx+dtdt-e^{t}(dxdy+dydx)+e^{-t}(dtdz+dzdt).$$

The geometric structures modeled on these groups appear as two of the possible geometric structures on 4-manifold [57]. Its compact quotients by a discrete group Γ constitute the Inoe surfaces

[57]. It is well known that the Inoe surfaces have zero second Betti number and therefore do not admit any symplectic structure. Descending the hyper-paracomplex structures on Sol_1^4, Sol_0^4 described above to the quotient, we obtain

Theorem 6.11. *The Inoe surfaces admit an invariant locally conformally hyper-para-Kähler (hypersymplectic) structure and do not admit any global one. The Weyl curvature of the corresponding conformal structure of neutral signature is not zero and therefore, the Inoe surfaces support an invariant anti-self-dual not Weyl flat metric of neutral signature.*

Remark 6.12. According to Milnor's result [47] if a solvable Lie group admits a compact quotient by a discrete subgroup it has to be unimodular. A glance at the possible Lie algebras supporting a hyper-paracomplex structure listed in [21] leads to only one more possibility, namely the Lie group corresponding to $PHC6, (a,b) \neq (0,0)$ is unimodular and may have compact quotients.

Example 6.13. We construct a local hyper-paracomplex structure which is not conformally equivalent to a hyper-para-Kähler (hypersymplectic), i.e. its Lee form $d\theta \neq 0$.

We adapt the Ashtekar's [7] formulation of the self-duality Einstein equations to the case of neutral metric and then use the Joyce's construction [39] of hyper-complex structure from holomorphic functions.

Let V_1, V_2, V_3, V_4 be vector fields on an oriented 4-manifold M forming an oriented basis for TM at each point. Then $V_1, ..., V_4$ define a neutral conformal structure $[g]$ on M. Define an almost hyper-paracomplex structure (J_2, J_3) by the equations

$$J_3V_1 = -V_2, \quad J_3V_3 = V_4, \qquad J_2V_1 = -V_4, \quad J_2V_2 = V_3.$$

Suppose that $V_1, ..., V_4$ satisfy the three vector field equations

$$[V_1, V_2] + [V_3, V_4] = 0, \quad [V_1, V_3] + [V_2, V_4] = 0, \quad [V_1, V_4] - [V_2, V_3] = 0. \tag{6.50}$$

It is easy to check that these equations imply the integrability of (J_1, J_2), i.e. (J_1, J_2) is a hyper-paracomplex structure which is compatible with the neutral conformal structure $[g]$. Hence, $[g]$ is anti-self-dual due to Theorem 6.2.

The neutral Ashtekar's equation (6.50) may be written in a complex form

$$[V_1 + iV_2, V_1 - iV_2] + [V_3 + iV_4, V_3 - iV_4] = 0, \quad [V_1 + iV_2, V_3 - iV_4] = 0. \tag{6.51}$$

Let M be a complex surface, let (z^1, z^2) be local holomorphic coordinates, and define $V_1, ..., V_4$ by

$$V_1 + iV_2 = f_1\frac{\partial}{\partial z^1} + f_2\frac{\partial}{\partial z^2}, \quad V_3 + iV_4 = f_3\frac{\partial}{\partial z^1} + f_4\frac{\partial}{\partial z^2},$$

where f_j is a complex function on M. Substituting into (6.51) we find the equations are satisfied identically if f_j is a holomorphic function with respect to the complex structure on M. So we can construct a hyper-paracomplex structure, with the opposite orientation, out of four holomorphic functions $f_1, ..., f_4$.

Taking $f_1 = f, f_2 = f_3 = 0, f_4 = 1$ we obtain a local hyper-paracomplex structure. Consider a particular neutral metric $g \in [g]$ such that

$$g(V_1,V_1)=g(V_2,V_2)=-g(V_3,V_3)=-g(V_4,V_4)=1, g(V_j,V_k)=0, j\neq k.$$

The corresponding common Lee form is given by

$$\theta=\frac{1}{f}\frac{\partial f}{\partial z^2}dz^2+\frac{1}{\bar f}\frac{\bar\partial \bar f}{\partial \bar z^2}d\bar z^2.$$

Then $d\theta\neq 0$ provided $\frac{\partial f}{\partial z^1}\neq 0$.

6.3. **Twistor and reflector spaces on para quaternionic Kähler manifold.** Consider the space $\tilde H_1$ of imaginary para quaternions. It is isomorphic to the Lorenz space $\mathbb{R}^2_1$ with a Lorenz metric of signature (-,-,+) defined by $< q,q' >= Re(q\overline{q'})$, where $\overline{q}=-q$ is the conjugate imaginary para quaternion. In $\mathbb{R}^2_1$ there are two kinds of 'unit spheres', namely the sphere $S^2_1(1)$ of radius 1 (the 1-sheeted hyperboloid) which consists of all imaginary para quaternions of norm 1 and the sphere $S^2_1(-1)$ of radius (-1) (the 2-sheeted hyperboloid) which contains all imaginary para quaternions of norm (-1). The sphere S^2_1 carries a natural para-hermitian structure while the sphere $S^2_1(-1)$ carries a natural hermitian structure of signature (1,1), both induced by the restriction of the Lorenz metric and the cross-product on $\tilde H_1\cong\mathbb{R}^2_1$ defined by

$$X\times Y=\sum_{i\neq k}x^iy^kJ_iJ_k$$

for vectors $X=x^iJ_i,\quad Y=y^kJ_k$. Namely, for a tangent vector $X=x^iJ_i$ to the sphere $S^2_1(1)$ at a point $q_+=q^k_+J_k$ (resp. tangent vector $Y=y^k_-J_k$ to the sphere $S^2_1(-1)$ at a point $q_-=q^k_-J_k$ we define $PX:=q_+\times X$ (resp. $JY=q_-\times Y$). It is easy to check that PX is again tangent vector to $S^2_1(1),\quad P^2X=X,\quad <PX,PX>=-<X,X>$ (resp. JY is tangent vector to $S^2_1(-1),\quad J^2Y=-Y,\quad <JY,JY,>=<Y,Y,>$.

We start with a para quaternionic Kähler manifold $(M,g,\tilde H=(J_\alpha))$. The vector bundle P carries a natural Lorenz structure of signature (-,-,+) such that (J_1,J_2,J_3) forms an orthonormal local basis of P. The are two kinds of "unit sphere" bundles according to the existence of the one sheeted hyperboloid $S^2_1(1)$ and the two-sheeted hyperboloid $S^2_1(-1)$. The twistor space $Z^+(M)$ (resp. $Z^-(M)$) is the unit sphere bundle with fibre $S^2_1(1)$ (resp. $S^2_1(-1)$). In other words, the fibre of $Z^+(M)$ consists of all almost paracomplex structures (resp. all almost complex structures) compatible with the given para quaternionic Kähler structure. The bundle $Z^+(M)$ over a 4-dimensional manifold with a neutral metric was constructed in [38] and called *the reflector space*. Further, we keep their notation.

Denote by $\pi^\pm$ the projection of $Z^\pm(M)$ onto M, respectively. Keeping in mind the formal similarity with the quaternionic geometry where there are two natural almost complex structures [8, 25], we observe the existence of two naturally arising almost paracomplex structures on $Z^+(M)$ (resp. two almost complex structures on $Z^-(M)$ [18]) defined as follows:

The Levi-Civita connection on P preserves the Lorenz metric and induces a linear connection on $Z^\pm$ i.e. a splitting of the tangent bundle $TZ^\pm=\mathbb{H}^\pm\otimes\mathbb{V}^\pm$, respectively, where $\mathbb{V}^\pm$ is the vertical distribution tangent to the fibre $S^2_1(1)$, (resp. $S^2_1(-1)$) and $\mathbb{H}^\pm$ a supplementary horizontal distribution induced by the Levi-Civita connection. By definition, the horizontal transport associated to $\mathbb{H}^\pm$ preserves the canonical Lorenz metric of the fibres $S^2_1(1)$ (resp. $S^2_1(-1)$); and also their orientation; as a corollary, it preserves the canonical paracomplex structure on $S^2_1(1)$ (resp. the canonical complex structure on $S^(2_1(-1))$ described above. Since the vertical distribution $\mathbb{V}^{\pm}$ is tangent to the fibres, this paracomplex structure (resp. complex structure) induces an endomorphism $\tilde{P}$ with $\tilde{P}^2 = id$ (resp. $\tilde{J}$ with $\tilde{J}^2 = -id$) on $\mathbb{V}^{\pm}_z$ for each $z \in Z^+(M)$ (resp. $Z^-(M)$). On the other hand, each point z on $Z^+(M)$ (resp. $Z^-(M)$) is by definition a paracomplex structure on $T_{\pi(z)}Z^+(M)$ (resp. a complex structure on $T_{\pi(z)}Z^-(M)$) which may be lifted into an endomorphism $\bar{P}$ on $\mathbb{H}^+_z$ with $\bar{P}^2 = id$ (resp. $\bar{J}$ on $\mathbb{H}^-_z$ with $\bar{J}^2 = -id$). We define almost paracomplex structures $\mathbb{P}_1, \mathbb{P}_2$ on $Z^+(M)$ and almost complex structure $\mathbb{J}_1, \mathbb{J}_2$ on $Z^-(M)$ by

$$\begin{aligned}
&\mathbb{P}_1(\mathbb{V}^+) = \mathbb{V}^+, \quad \mathbb{P}_1|_{\mathbb{V}^+} = \tilde{P}, \qquad \mathbb{P}_1(\mathbb{H}^+) = \mathbb{H}^+, \quad \mathbb{P}_1|_{\mathbb{H}^+} = \bar{P},\\
&\mathbb{P}_2(\mathbb{V}^+) = \mathbb{V}^+, \quad \mathbb{P}_2|_{\mathbb{V}^+} = -\tilde{P}, \qquad \mathbb{P}_2(\mathbb{H}^+) = \mathbb{H}^+, \quad \mathbb{P}_2|_{\mathbb{H}^+} = \bar{P};\\
&\mathbb{J}_1(\mathbb{V}^-) = \mathbb{V}^-, \quad \mathbb{J}_1|_{\mathbb{V}^-} = \tilde{J}, \qquad \mathbb{J}_1(\mathbb{H}^-) = \mathbb{H}^-, \quad \mathbb{J}_1|_{\mathbb{H}^-} = \bar{J},\\
&\mathbb{J}_2(\mathbb{V}^-) = \mathbb{V}^-, \quad \mathbb{J}_2|_{\mathbb{V}^-} = -\tilde{J}, \qquad \mathbb{J}_2(\mathbb{H}^-) = \mathbb{H}^-, \quad \mathbb{J}_2|_{\mathbb{H}^-} = \bar{J}).
\end{aligned}$$

Define a pseudo Riemannian metrics on $Z^+(M)$ (resp. $Z^-(M)$) by $h^+_t = \pi^* g + t <,>^+_v, t \neq 0, <,>^+_v$ being the restriction of the Lorenz metric to the fibres $S^2_1(1)$ (resp. $h^-_t = \pi^* g + t <,>^-_v$, $t \neq 0, <,>^-_v$ being the restriction of the Lorenz metric to the fibres $S^2_1(-1)$). It is easy to check that h^+ (resp. h^-) is compatible with both $\mathbb{P}_1, \mathbb{P}_2$ (resp. $\mathbb{J}_1, \mathbb{J}_2$) such that $(Z^+(M), h^+_t, \mathbb{P}_{1,2})$ become an almost para-Hermitian manifold (resp. $(Z^-(M), h^-_t, \mathbb{J}_{1,2})$ become an almost Hermitian manifold).

The almost paracomplex structures $\mathbb{P}_1, \mathbb{P}_2$ and the neutral metrics h^+_t on the reflector space of a 4-dimensional manifold with a neutral metric g are investigated in [38]. They show that the almost paracomplex structure $\mathbb{P}_2$ is never integrable while the almost paracomplex structure $\mathbb{P}_1$ is integrable if and only if the neutral metric g is self dual. They also prove that the neutral metric h^+_t on the reflector space is Einstein if and only if g is self-dual Einstein and either $ts = 12$ or $ts = 6$.

Almost Hermitian geometry of $(Z^-(M), h^-_t, \mathbb{J}_{1,2})$ is investigated in [18]. Their calculations are completely applicable to the almost para-Hermitian geometry of $(Z^+(M), h^+_t, \mathbb{P}_{1,2})$. In terms of almost para-Hermitian geometry of $(Z^+(M), h^+_t, \mathbb{P}_{1,2})$ Theorem 1 and Theorem 2 in [18] read as follows

Theorem 6.14. *On the reflector space $(Z^+(M))$ of a para quaternionic Kähler manifold of dimension $4n \geq 8$ we have:*

i) The almost paracomplex structure $\mathbb{P}_1$ is integrable and the Lee form of the para-Hermitian structure $(\mathbb{P}_1, h^+_t)$ is zero. The structure $(\mathbb{P}_1, h^+_t)$ is para-Kähler iff $ts = 4n(n+2)$;

ii) The almost paracomplex structure $\mathbb{P}_2$ is never integrable and the Lee form of the almost para-Hermitian structure $(\mathbb{P}_2, h^+_t)$ is zero. The structure $(\mathbb{P}_2, h^+_t)$ is strict nearly para-Kähler if and only if $ts = 2n(n+2)$ and strict almost para-Kähler if and only if $ts = -4n(n+2)$.

Theorem 6.15. *On the reflector space $(Z^+(M))$ of an oriented 4-dimensional manifold M with a neutral metric g we have the following:*

i) The almost para-Hermitian structure $(\mathbb{P}_1, h_t^+)$ has zero Lee form if and only if the metric g is self-dual. It is para-Kähler if and only if the metric g is Einstein self-dual and $ts = 12$;

ii) The Lee form of the almost para-Hermitian structure $(\mathbb{P}_2, h_t^+)$ is zero. The structure $(\mathbb{P}_2, h_t^+)$ is strict nearly para-Kähler if and only if the metric g is self-dual Einstein and $ts = 6$ and strict almost para-Kähler if and only if $ts = -12$.

Remark 6.16. On a para quaternionic manifold of dimension $4n \geq 8$ we may construct the almost paracomplex structure $\mathbb{P}_1$ on the reflector space Z^+ and the almost complex structure $\mathbb{J}_1$ on the twistor space Z^- using the horizontal distribution generated by a torsion-free connection instead of the horizontal distribution of the Levi-Civita connection. In that case, we find an analogue of the result of S.Salamon [52, 53, 54], (proved also independently by L. Berard-Bergery, unpublished, see [16]). Namely, we have

Theorem 6.17. *On a para quaternionic manifold of dimension $4n \geq 8$ the almost paracomplex structure $\mathbb{P}_1$ on Z^+ and the almost complex structure $\mathbb{J}_1$ on Z^- are always integrable*

We sketch a proof which is completely similar to the proof in the case of quaternionic manifold presented in [16]. Denote by R the curvature of a torsion-free connection ∇. Let $S = xJ_1 + yJ_2 + zJ_3$ be either an almost paracomplex structure or an almost complex structure compatible with the given structure i.e. the triple (x, y, z) satisfies either $x^2 + y^2 - z^2 = 1$ or $x^2 + y^2 - z^2 = -1$. Denote by

$$S(R)(X,Y) = [R(SX,SY),S] - S[R(SX,Y),S] - S[R(X,SY),S] + S^2[R(X,Y),S].$$

In view of the analogy with the proof in the quaternionic case presented in [16], 14.72-14.74 the result will follow if $S(R) = 0$. The last identity can be checked in the exactly same way as it is done in [16], Lemma 14.74 using (6.1) instead of formulas 14.39 in [16].

6.4. **Examples of nearly para-Kähler and almost para-Kähler manifolds.** Theorem 6.14 helps to find examples of nearly para-Kähler and almost para-Kähler manifolds. We note that the sign of the scalar curvature (if it is not zero) is not a restriction since the metric (-g) have scalar curvature with opposite sign. Hence, taking the reflector space of any neutral (anti) self-dual Einstein manifold with non-zero scalar curvature in dimension four and any quaternionic para-Kähler manifold in dimension $4n \geq 8$ we can find a real number t to get nearly para-Kähler and almost para-Kähler structure on it.

(1) The pseudo-sphere S^3_3 is endowed with an almost paracomplex structure [44] and it is shown in [13] that there exists a nearly para-Kähler structure on S^3_3 induced from the

so-called *second kind Cayley numbers* (see [44]) in $\mathbb{R}^4_3$. The structure is Einstein with non-zero scalar curvature, in fact the metric is the standard neutral metric on S^3_3 inherited from $\mathbb{R}^4_3$.

(2) Start with one of the following 4-dimensional neutral self-dual Einstein spaces $(S^2_2 = SO^+(2,3)/GL^+(2,\mathbb{R}), can)$, $(\mathbb{C}P^{1,1} = (SU(2,1)/(SO(1,1).U(1)), can)$, or $(SL(3,\mathbb{R})/GL^+(2,\mathbb{R}), c.Kill|_{sl(3,\mathbb{R})})$, where c is a suitable constant and $Kill|_{sl(3,\mathbb{R})}$ is the restriction of the Killing form of $sl(3,\mathbb{R})$ to the homogeneous space $SL(3,\mathbb{R})/GL^+(2,\mathbb{R}$. The corresponding reflector spaces are $SO^+(2,3)/GL^+(2,\mathbb{R})$, $SU(2,1)/(SO(1,1).U(1))$, $SL(3,\mathbb{R})/(\mathbb{R}^+ \times \mathbb{R}^+ \cup \mathbb{R}^- \times \mathbb{R}^-)$, respectively [42]. These homogeneous spaces admit homogeneous strict nearly para-Kähler structure of non-zero scalar curvature as well as homogeneous strict almost para-Kähler structure according to Theorem 6.15

(3) Non-homogeneous examples arise from the non-(locally) homogeneous neutral self-dual Einstein space of non-zero scalar curvature described in [20]. Its reflector space admits strict nearly para-Kähler structure of non-zero scalar curvature as well as strict almost para-Kähler structure due to Theorem 6.15

To the best of the author's knowledge there are no known examples of Ricci flat 6-dimensional Nearly para-Kähler manifolds.

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