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ELECTROMAGNETIC SPATIAL COHERENCE WAVELETS

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Abstract

The recently introduced concept of spatial coherence wavelets is generalized for describing the propagation of electromagnetic fields in the free space. For this aim, the spatial coherence wavelet tensor is introduced as an elementary amount, in terms of which the formerly known quantities for this domain can be expressed. It allows analyzing the relationship between the spatial coherence properties and the polarization state of the electromagnetic wave. This approach is completely consistent with the recently introduced unified theory of coherence and polarization for random electromagnetic beams, but it provides a further insight about the causal relationship between the polarization states at different planes along the propagation path.

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1. INTRODUCTION

Spatial coherence properties of scalar optical fields and their propagation have been fairly described by the second-order spatial coherence theory [1, 2]. The cornerstone of the theory is the cross-spectral density, which was introduced by Wolf. It can be expressed as the superposition of spatial coherence wavelets, a more elementary concept recently introduced [3, 4]. In terms of such wavelets the role of the spatial coherence in interference, diffraction, imaging process and radiometry has been discussed in the literature [5-7].

New developments in optics, as for instance the nano-optics and micro-structured materials, pointed out the relevance of the polarization of the optical field. The Stokes parameters [1] provide the most classical treatment of polarization, which predates the Maxwell's electromagnetic theory. Jones matrices [1] provide an alternative methodology in the framework of the electromagnetic theory. However, such approaches have the limitation of providing only information about the correlation between Cartesian components of the electric field vector at one point in the space-time reference frame, which has been recognised as an important drawback of these methodologies [8, 9].

Recent investigations indicate that the changes of polarization along the light propagation should be properly described by taking into account the spatial coherence properties of the optical field. This fact has propelled the extension of the scalar theory of coherence to the electromagnetic domain, in such a way that the electromagnetic theory of optical coherence has become a field of intense research in the last years [8-20].

In this context, we propose the electromagnetic spatial coherence wavelets as an elementary quantity for describing the unified propagation of power, correlation and polarization of optical fields. It should be specified by a set of tensors that propagate together. Each tensor should be determined by the correlation between the Cartesian components of the electric and the magnetic field vectors at pairs of points on specific planes.

In this paper we are specifically concerned with the electric component of the electromagnetic spatial coherence wavelets. The extension of our proposal to the magnetic fields is straightforward. We will analyze the relationship between its components and the polarization state of the electromagnetic wave. From this standpoint, the polarization of light acquires a non-local meaning, in counter part to the local significance of its classical version, a feature also analyzed by Gori [9].

The results show that our approach is completely consistent with the recently introduced unified theory of coherence and polarization for random electromagnetic beams and all the new knowledge supported on it [8,15,18-20], but it provides additional insight about the causal relationship between the polarization states at different planes along the propagation path.

2. ELECTROMAGNETIC SPATIAL COHERENCE WAVELETS

Let us represent an electromagnetic wave in paraxial propagation in free space, by means of a Cartesian coordinate system with the z-axis parallel to the direction of propagation. The aperture plane A and the observation plane are placed perpendicular to the z-axis, at a distance

z each other, as depicted in Figure 1. Thus, its electric field vectors at each point on those planes can be decomposed into x- and y-components.

We also use centre and difference coordinates on both planes, as defined in Refs. [3, 4], which are labelled as $(\mathbf{r}'_A, \mathbf{r}'_D)$ and $(\mathbf{r}_A, \mathbf{r}_D)$ respectively. The vectors $\mathbf{r}'_A \pm \frac{\mathbf{r}'_D}{2}$ and $\mathbf{r}_A \pm \frac{\mathbf{r}_D}{2}$ denote the positions of pairs of points on those planes (Figure1).

The electromagnetic wave can be represented in terms of spatial coherence wavelets [3, 4] by applying this concept to the components of both the electric (**E**) and the magnetic (**H**) field vectors. Equation (1) denotes the canonical form of the wavelets for a wave of frequency ν, wavelength λ and wave number $k = \frac{2\pi}{\lambda}$, i.e.:

$$\mathcal{W}_{lm}^{AB}\left(\mathbf{r}_A + \frac{\mathbf{r}_D}{2}, \mathbf{r}_A - \frac{\mathbf{r}_D}{2}, \mathbf{r}'_A; \nu\right) = \mathcal{S}_{lm}^{AB}(\mathbf{r}_A, \mathbf{r}'_A; \nu)\ e^{-i\frac{k}{z}\mathbf{r}_D \cdot \mathbf{r}'_A}, \tag{1}$$

where the superscripts AB point out the involved field vectors and the subscripts l and m stand for x and y, specifying the Cartesian components of these field vectors.

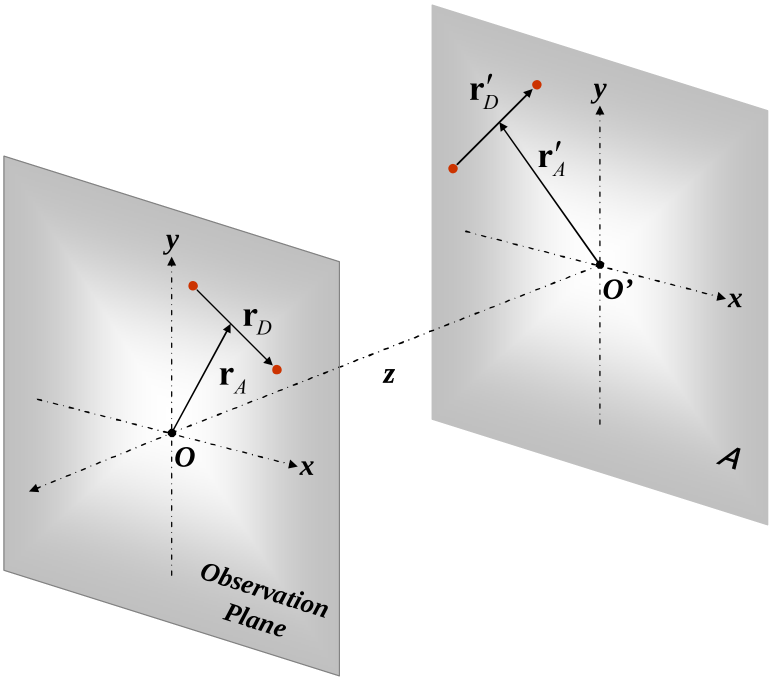

Fig. 1: Cartesian co-ordinate axes and centre and difference coordinates

Equation (1) determines a set of four tensors with four elements, which can be represented by 2x2 matrices of the form $\mathcal{W}\left(\mathbf{r}_A + \frac{\mathbf{r}_D}{2}, \mathbf{r}_A - \frac{\mathbf{r}_D}{2}, \mathbf{r}'_A; \nu\right) = \begin{bmatrix} \mathcal{W}_{xx}^{AB} & \mathcal{W}_{xy}^{AB} \\ \mathcal{W}_{yx}^{AB} & \mathcal{W}_{yy}^{AB} \end{bmatrix}$. This tensor set defines the *electromagnetic spatial coherence wavelet*, whose electric component, *the electric spatial coherence wavelet tensor*, is labelled by $A=B=E$. Similarly, the magnetic component (*the magnetic spatial coherence wavelet tensor*) is labelled by $A=B=H$, and the combinations ($A=E$, $B=H$) and ($A=H$, $B=E$) provide *mixed spatial coherence wavelet tensors*. Furthermore,

$$\mathsf{S}_{lm}^{AB}(\mathbf{r}_A\ ,\mathbf{r}'_A;\nu)=\int_A W_{lm}^{AB}\left(\mathbf{r}'_A+\frac{\mathbf{r}'_D}{2}\,,\mathbf{r}'_A-\frac{\mathbf{r}'_D}{2};\nu\right)e^{i\frac{k}{z}(\mathbf{r}'_A-\mathbf{r}_A)\cdot\mathbf{r}'_D}\ d^2r'_D\ , \tag{2}$$

denotes the corresponding marginal power spectrum [3, 4], with $W_{lm}^{AB}\left(\mathbf{r}'_A+\frac{\mathbf{r}'_D}{2}\,,\mathbf{r}'_A-\frac{\mathbf{r}'_D}{2};\nu\right)$ the elements of the (electric, magnetic or mixed) cross-spectral density tensor $\mathbf{W}\left(\mathbf{r}'_A+\frac{\mathbf{r}'_D}{2}\,,\mathbf{r}'_A-\frac{\mathbf{r}'_D}{2};\nu\right)$ [2] at the A plane. Consequently, equation (2) determines *the electric, magnetic and mixed marginal power spectrum tensors*, each one with four elements that can be arranged as $\mathsf{S}(\mathbf{r}_A,\mathbf{r}'_A;\nu)=\begin{bmatrix}\mathsf{S}_{xx}^{AB} & \mathsf{S}_{xy}^{AB}\\ \mathsf{S}_{yx}^{AB} & \mathsf{S}_{yy}^{AB}\end{bmatrix}$ in matrix notation. It is useful to express equation (2) in matrix notation as

$$\mathsf{S}(\mathbf{r}_A,\mathbf{r}'_A;\nu)=\int_A \mathbf{W}\left(\mathbf{r}'_A+\frac{\mathbf{r}'_D}{2}\,,\mathbf{r}'_A-\frac{\mathbf{r}'_D}{2};\nu\right)e^{i\frac{k}{z}(\mathbf{r}'_A-\mathbf{r}_A)\cdot\mathbf{r}'_D}\ d^2r'_D\ ,$$

a rule that we will follow for all the tensor integrals below. Equations (1) and (2) constitute the support of our analysis.

According to the second-order spatial coherence theory [1, 2], the propagation of the spatial coherence properties of the electromagnetic wave, from the A plane to the observation plane in Fresnel-Fraunhofer domain (i.e. paraxial approach), is described by the expression

$$\begin{aligned}\mathbf{W}\left(\mathbf{r}_A+\frac{\mathbf{r}_D}{2}\,,\mathbf{r}_A-\frac{\mathbf{r}_D}{2};\nu\right)=\left(\frac{1}{\lambda\,z}\right)^2 e^{i\frac{k}{z}\mathbf{r}_A\cdot\mathbf{r}_D}\iint_{A\,A}\mathbf{W}\left(\mathbf{r}'_A+\frac{\mathbf{r}'_D}{2}\,,\mathbf{r}'_A-\frac{\mathbf{r}'_D}{2};\nu\right)e^{i\frac{k}{z}\mathbf{r}'_A\cdot\mathbf{r}'_D}\\ e^{-i\frac{k}{z}(\mathbf{r}_A\cdot\mathbf{r}'_D+\mathbf{r}_D\cdot\mathbf{r}'_A)}\ d^2r'_A\ d^2r'_D\end{aligned} \ . \tag{3}$$

On making use of equations (1) and (2) on (3), it yields

$$\begin{aligned}\mathbf{W}\left(\mathbf{r}_A+\frac{\mathbf{r}_D}{2}\,,\mathbf{r}_A-\frac{\mathbf{r}_D}{2};\nu\right)&=\left(\frac{1}{\lambda\,z}\right)^2 e^{i\frac{k}{z}\mathbf{r}_A\cdot\mathbf{r}_D}\int_A \mathsf{W}\left(\mathbf{r}_A+\frac{\mathbf{r}_D}{2}\,,\mathbf{r}_A-\frac{\mathbf{r}_D}{2}\,,\mathbf{r}'_A\ ;\nu\right)\ d^2r'_A\\ &=\left(\frac{1}{\lambda\,z}\right)^2 e^{i\frac{k}{z}\mathbf{r}_A\cdot\mathbf{r}_D}\int_A \mathsf{S}(\mathbf{r}_A,\mathbf{r}'_A;\nu)\ e^{-i\frac{k}{z}\mathbf{r}_D\cdot\mathbf{r}'_A}\ \ d^2r'_A\end{aligned} \ . \tag{4}$$

By evaluating equation (4) for $\mathbf{r}_D=0$ we obtain the *electric, magnetic and mixed power spectrum tensors* $\mathbf{S}(\mathbf{r}_A;\nu)$ [2], i.e.

$$\mathbf{S}(\mathbf{r}_A;\nu)=\left(\frac{1}{\lambda\, z}\right)^2\int_A \mathbf{S}(\mathbf{r}_A\ ,\mathbf{r}'_A;\nu)\,d^2r'_A\ , \tag{5}$$

whose traces give the corresponding power distributions on the observation plane, that is

$$S(\mathbf{r}_A;\nu)=tr[\mathbf{S}(\mathbf{r}_A;\nu)]=\left(\frac{1}{\lambda\, z}\right)^2\int_A tr[\mathbf{S}(\mathbf{r}_A\ ,\mathbf{r}'_A;\nu)]\,d^2r'_A\ . \tag{6}$$

Equations (4) to (6) mean that the electromagnetic spatial coherence wavelets can be interpreted as the elementary vehicles for transferring correlation and power properties of the electromagnetic wavefront from the A plane to the observation plane [4]. In fact, the cross-spectral density tensors result from a particular superposition of the corresponding spatial coherence wavelet tensors, as specified in equation (4). In a similar fashion, the superposition of the marginal power spectrum tensors, defined in equation (5), give the corresponding power spectrum tensors.

In the following, we will only be concerned with the electric tensors for analyzing the polarization properties of the electromagnetic wave. For this reason we drop out the superscripts of the tensor elements.

3. THE ELECTRIC SPATIAL COHERENCE WAVELET TENSOR

The elements of the electric cross-spectral density tensor at the A plane and at frequency ν are defined by the correlations [2]

$$W_{lm}\left(\mathbf{r}'_A+\frac{\mathbf{r}'_D}{2}\,,\mathbf{r}'_A-\frac{\mathbf{r}'_D}{2};\nu\right)\delta(\nu-\nu')=\left\langle E_l\left(\mathbf{r}'_A+\frac{\mathbf{r}'_D}{2},\nu\right)E^*_m\left(\mathbf{r}'_A-\frac{\mathbf{r}'_D}{2},\nu'\right)\right\rangle,$$

where $\delta(\nu-\nu')$ stands for the Dirac's delta function [21], the angular brackets denote ensemble average, understood in the sense of the second-order coherence theory [2], $(E_l\ ,E_m)$ are Cartesian components of the electric field vector at the A plane and the asterisk denotes complex conjugate. On following the Schwartz inequality [21], such tensor elements can be expressed as

$$W_{lm}\left(\mathbf{r}'_A+\frac{\mathbf{r}'_D}{2}\,,\mathbf{r}'_A-\frac{\mathbf{r}'_D}{2};\nu\right)=\left\langle\left|E_l\left(\mathbf{r}'_A+\frac{\mathbf{r}'_D}{2},\nu\right)\right|^2\right\rangle^{\frac{1}{2}}\eta_{lm}\left(\mathbf{r}'_A+\frac{\mathbf{r}'_D}{2},\mathbf{r}'_A-\frac{\mathbf{r}'_D}{2},\nu\right)\left\langle\left|E_m\left(\mathbf{r}'_A-\frac{\mathbf{r}'_D}{2},\nu\right)\right|^2\right\rangle^{\frac{1}{2}},$$

with $\eta_{lm}\left(\mathbf{r}'_A+\frac{\mathbf{r}'_D}{2},\mathbf{r}'_A-\frac{\mathbf{r}'_D}{2},\nu\right)$ the matrix elements of the tensor

$$\boldsymbol{\eta}\left(\mathbf{r}'_A+\frac{\mathbf{r}'_D}{2},\mathbf{r}'_A-\frac{\mathbf{r}'_D}{2},\nu\right)=\begin{bmatrix}\eta_{xx}\left(\mathbf{r}'_A+\frac{\mathbf{r}'_D}{2},\mathbf{r}'_A-\frac{\mathbf{r}'_D}{2},\nu\right) & \eta_{xy}\left(\mathbf{r}'_A+\frac{\mathbf{r}'_D}{2},\mathbf{r}'_A-\frac{\mathbf{r}'_D}{2},\nu\right)\\ \eta_{yx}\left(\mathbf{r}'_A+\frac{\mathbf{r}'_D}{2},\mathbf{r}'_A-\frac{\mathbf{r}'_D}{2},\nu\right) & \eta_{yy}\left(\mathbf{r}'_A+\frac{\mathbf{r}'_D}{2},\mathbf{r}'_A-\frac{\mathbf{r}'_D}{2},\nu\right)\end{bmatrix},$$

which are complex valued, dimensionless and their modulus take real values in the interval [0,1]. Furthermore $\boldsymbol{\eta}\left(\mathbf{r}'_A+\frac{\mathbf{r}'_D}{2},\mathbf{r}'_A-\frac{\mathbf{r}'_D}{2},\nu\right)=\left[\boldsymbol{\eta}^{\mathrm{t}}\left(\mathbf{r}'_A-\frac{\mathbf{r}'_D}{2},\mathbf{r}'_A+\frac{\mathbf{r}'_D}{2},\nu\right)\right]^*$, where the superscript t stands for the transpose tensor. It follows that $\eta_{ll}(\mathbf{r}'_A,\mathbf{r}'_A,\nu)=1$.

The above expressions indicate that the matrix elements $\eta_{lm}\left(\mathbf{r}'_A+\frac{\mathbf{r}'_D}{2},\mathbf{r}'_A-\frac{\mathbf{r}'_D}{2},\nu\right)$ should be considered to be a measure of the degree of correlation that exists between the electric field components $\left\{E_l\left(\mathbf{r}'_A+\frac{\mathbf{r}'_D}{2},\nu\right), E_m\left(\mathbf{r}'_A-\frac{\mathbf{r}'_D}{2},\nu\right)\right\}$ provided by each pair of centres of secondary disturbance onto the A plane.

Accordingly, the correlations between parallel components will be represented by the diagonal matrix elements, i.e. for $l=m$, and those between mutually orthogonal components will be represented by the off-diagonal matrix elements, i.e. $l\neq m$. It is also reasonable to assume that these correlations should be significant within a delimited region around the position $\mathbf{r}'_A$, which is determined by the supports of $\eta_{lm}\left(\mathbf{r}'_A+\frac{\mathbf{r}'_D}{2},\mathbf{r}'_A-\frac{\mathbf{r}'_D}{2},\nu\right)$, i.e. the region, centred at $\mathbf{r}'_A$ of radius $\mathbf{r}'_D$, outside which the values of the matrix elements are neglected.

Then, the electric cross-spectral density tensor at the A plane can be expressed as

$$\mathbf{W}\left(\mathbf{r}'_A+\frac{\mathbf{r}'_D}{2},\mathbf{r}'_A-\frac{\mathbf{r}'_D}{2},\nu\right)=\mathbf{E}\left(\mathbf{r}'_A+\frac{\mathbf{r}'_D}{2},\nu\right)\boldsymbol{\eta}\left(\mathbf{r}'_A+\frac{\mathbf{r}'_D}{2},\mathbf{r}'_A-\frac{\mathbf{r}'_D}{2},\nu\right)\mathbf{E}\left(\mathbf{r}'_A-\frac{\mathbf{r}'_D}{2},\nu\right), \tag{7}$$

with

$$\mathbf{E}\left(\mathbf{r}'_A\pm\frac{\mathbf{r}'_D}{2},\nu\right)=\begin{bmatrix}\left\langle\left|E_x\left(\mathbf{r}'_A\pm\frac{\mathbf{r}'_D}{2},\nu\right)\right|^2\right\rangle^{\frac{1}{2}} & 0\\ 0 & \left\langle\left|E_y\left(\mathbf{r}'_A\pm\frac{\mathbf{r}'_D}{2},\nu\right)\right|^2\right\rangle^{\frac{1}{2}}\end{bmatrix}.$$

Thus, equations (2) and (7) yield

$$\mathbf{S}(\mathbf{r}_A,\mathbf{r}'_A;\nu)=\int_A \mathbf{E}\left(\mathbf{r}'_A+\frac{\mathbf{r}'_D}{2},\nu\right)\boldsymbol{\eta}\left(\mathbf{r}'_A+\frac{\mathbf{r}'_D}{2},\mathbf{r}'_A-\frac{\mathbf{r}'_D}{2},\nu\right)\mathbf{E}\left(\mathbf{r}'_A-\frac{\mathbf{r}'_D}{2},\nu\right)e^{i\frac{k}{z}(\mathbf{r}'_A-\mathbf{r}_A)\cdot\mathbf{r}'_D}\,d^2r'_D\,. \tag{8}$$

It is clear that $\mathbf{W}\left(\mathbf{r}'_A+\frac{\mathbf{r}'_D}{2},\mathbf{r}'_A-\frac{\mathbf{r}'_D}{2},\nu\right)=\left[\mathbf{W}^{t}\left(\mathbf{r}'_A+\frac{\mathbf{r}'_D}{2},\mathbf{r}'_A-\frac{\mathbf{r}'_D}{2},\nu\right)\right]^{*}$ holds. Consequently, the electric marginal power spectrum tensor should be hermitian, i.e. $\mathbf{S}(\mathbf{r}_A\ ,\mathbf{r}'_A;\nu)=\mathbf{S}^{+}(\mathbf{r}_A\ ,\mathbf{r}'_A;\nu)$, where the superscript + stands for the adjoint tensor [21]. For this reason, its diagonal elements are real valued. Furthermore

$$\mathbf{W}\left(\mathbf{r}_A+\frac{\mathbf{r}_D}{2}\ ,\mathbf{r}_A-\frac{\mathbf{r}_D}{2}\ ,\mathbf{r}'_A;\nu\right)=\left[\mathbf{W}^{t}\left(\mathbf{r}_A-\frac{\mathbf{r}_D}{2}\ ,\mathbf{r}_A+\frac{\mathbf{r}_D}{2}\ ,\mathbf{r}'_A;\nu\right)\right]^{*}.$$

It is worth noticing that the elements of the electric marginal power spectrum tensor are related to Wigner distribution functions (WDF), in such a way that the description of electromagnetic waves in terms of the electromagnetic spatial coherence wavelets should be a branch of the Wigner Optics [22, 23]. Indeed, taking into account that $\frac{k}{z}\mathbf{r}'_A\cdot\mathbf{r}'_D=\frac{k}{2\,z}\left(\left|\mathbf{r}'_A+\frac{\mathbf{r}'_D}{2}\right|^2-\left|\mathbf{r}'_A-\frac{\mathbf{r}'_D}{2}\right|^2\right)$ holds, equation (8) can be expressed as

$$\mathbf{S}(\mathbf{r}_A,\mathbf{r}'_A;\nu)=\int_A \mathbf{\Psi}\left(\mathbf{r}'_A+\frac{\mathbf{r}'_D}{2},\nu\right)\boldsymbol{\eta}\left(\mathbf{r}'_A+\frac{\mathbf{r}'_D}{2},\mathbf{r}'_A-\frac{\mathbf{r}'_D}{2},\nu\right)\mathbf{\Psi}^{*}\left(\mathbf{r}'_A-\frac{\mathbf{r}'_D}{2},\nu\right)e^{-i\frac{k}{z}\mathbf{r}_A\cdot\mathbf{r}'_D}\,d^2r'_D\,, \qquad (9)$$

with $\mathbf{\Psi}\left(\mathbf{r}'_A\pm\frac{\mathbf{r}'_D}{2},\nu\right)=\mathbf{E}\left(\mathbf{r}'_A\pm\frac{\mathbf{r}'_D}{2},\nu\right)e^{i\frac{k}{z}\left|\mathbf{r}'_A\pm\frac{\mathbf{r}'_D}{2}\right|^2}$. The phase factor $e^{i\frac{k}{z}\left|\mathbf{r}'_A\pm\frac{\mathbf{r}'_D}{2}\right|^2}$ denotes propagation of the electromagnetic wave in Fresnel domain, but can be approached to one if the propagation occurs in Fraunhofer domain [24].

The vector $\mathbf{r}'_A$ in equation (9) denotes the position of individual points on the A plane, and $\frac{k}{z}\left|\mathbf{r}_A\right|\approx\frac{2\pi}{\lambda}\sin\theta$ holds in the paraxial approach (Fresnel-Fraunhofer domain), where θ is the inclination angle of the straight line from $\mathbf{r}'_A$ to $\mathbf{r}_A$ with respect to the optical axis. The vertex of this angle is located at $\mathbf{r}'_A$ as depicted in Figure 2. So, $\mathbf{r}'_A$ and $\frac{k}{z}\left|\mathbf{r}_A\right|$ can be interpreted as the “space” and “phase” coordinates respectively, in such a way that the electric marginal power spectrum tensor provides a phase-space representation for the electromagnetic wave. Its diagonal elements are generalized WDF [3-7], and its off-diagonal elements are complex functions, whose modulus are the same generalized WDF, but their phases are reversed to each other. These WDFs are enough for describing spatial coherence and polarization properties of the electromagnetic waves, as we discuss below.

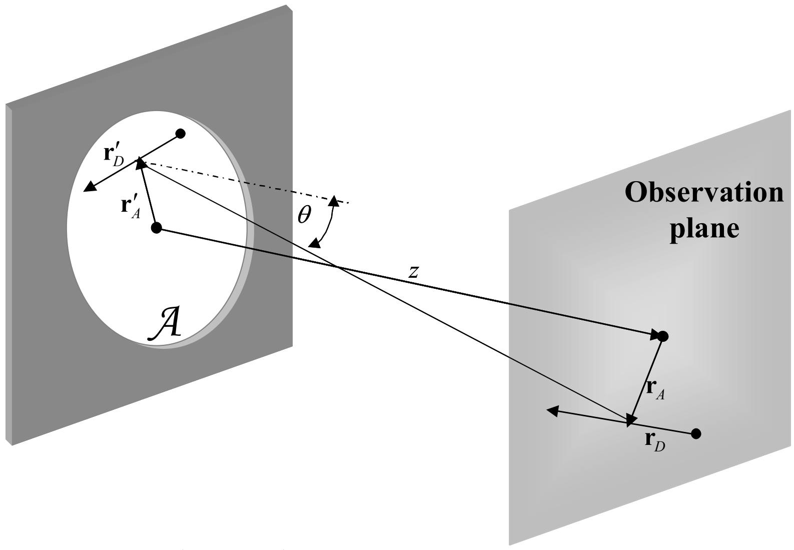

Fig. 2: Illustrating $\mathbf{S}(\mathbf{r}_A\,,\mathbf{r}'_A;\nu)$. The straight line from $\mathbf{r}'_A$ to $\mathbf{r}_A$ makes an angle θ with respect to the optical *z*-axis

4. COMPLETELY POLARIZED AND UNPOLARIZED WAVELETS

For properly characterizing the spatial coherence and polarization of the electromagnetic wave, it is useful to analyze the behaviour of the electric marginal power spectrum tensor $\mathbf{S}(\mathbf{r}_A\,,\mathbf{r}'_A;\nu)$ under arbitrary rotations around the axis $\mathbf{r}'_A \rightarrow \mathbf{r}_A$. This rotation axis is assumed to be parallel to the *z*-axis in paraxial approach.

After rotation by an angle ϑ onto the observation plane, the electric marginal power spectrum tensor becomes $\tilde{\mathbf{S}}(\mathbf{r}_A\,,\mathbf{r}'_A;\nu) = \mathbf{R}^{\mathbf{t}}(\vartheta)\,\mathbf{S}(\mathbf{r}_A\,,\mathbf{r}'_A;\nu)\,\mathbf{R}(\vartheta)$, with $\mathbf{R}(\vartheta) = \begin{bmatrix} \cos\vartheta & sin\,\vartheta \\ -sin\,\vartheta & \cos\vartheta \end{bmatrix}$ and $\mathbf{R}^{\mathbf{t}}(\vartheta)$ its transpose, i.e.

$$\tilde{\mathbf{S}}(\mathbf{r}_A\,,\mathbf{r}'_A;\nu) = \begin{bmatrix} \mathcal{S}_{xx}\cos^2\vartheta + \mathcal{S}_{yy}\sin^2\vartheta - (\mathcal{S}_{xy} + \mathcal{S}_{yx})\sin\vartheta\cos\vartheta & \mathcal{S}_{xy}\cos^2\vartheta - \mathcal{S}_{yx}\sin^2\vartheta + (\mathcal{S}_{xx} - \mathcal{S}_{yy})\sin\vartheta\cos\vartheta \\ \mathcal{S}_{yx}\cos^2\vartheta - \mathcal{S}_{xy}\sin^2\vartheta + (\mathcal{S}_{xx} - \mathcal{S}_{yy})\sin\vartheta\cos\vartheta & \mathcal{S}_{yy}\cos^2\vartheta + \mathcal{S}_{xx}\sin^2\vartheta + (\mathcal{S}_{xy} + \mathcal{S}_{yx})\sin\vartheta\cos\vartheta \end{bmatrix}. \tag{10}$$

As expected, the trace and the determinant of $\mathbf{S}(\mathbf{r}_A\,,\mathbf{r}'_A;\nu)$ remain invariant under the rotation. Now, let us consider the electromagnetic waves with the properties $\mathcal{S}_{xx} = \mathcal{S}_{yy} = \mathcal{S}_0$ and $\mathcal{S}_{xy} = \mathcal{S}_{yx} = 0$. Their electric marginal power spectrum tensors will be invariant under arbitrary rotations and take the form

$$\tilde{\mathbf{S}}(\mathbf{r}_A\,,\mathbf{r}'_A;\nu) = \mathbf{S}(\mathbf{r}_A\,,\mathbf{r}'_A;\nu) = \mathcal{S}_0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{11}$$

and the corresponding electric spatial coherence wavelet tensor [equation (1)] will be

$$\mathbf{W}\left(\mathbf{r}_A+\frac{\mathbf{r}_D}{2},\mathbf{r}_A-\frac{\mathbf{r}_D}{2},\mathbf{r}'_A;\nu\right)=\mathcal{S}_0\, e^{-i\frac{k}{z}\mathbf{r}_D\cdot\mathbf{r}'_A}\begin{bmatrix}1&0\\0&1\end{bmatrix}. \tag{12}$$

From equations (2), (4), (11) and (12) it follows that the electric cross-spectral density tensors of such waves, at both the A plane and the observation plane, are proportional to the identity matrix and remain invariant under arbitrary rotations on such planes. Furthermore, according to equations (7) and (8), it means that

$$\boldsymbol{\eta}\left(\mathbf{r}'_A+\frac{\mathbf{r}'_D}{2},\mathbf{r}'_A-\frac{\mathbf{r}'_D}{2},\nu\right)=\eta_0\left(\mathbf{r}'_A+\frac{\mathbf{r}'_D}{2},\mathbf{r}'_A-\frac{\mathbf{r}'_D}{2},\nu\right)\begin{bmatrix}1&0\\0&1\end{bmatrix}. \tag{13}$$

Thus, electric field vectors of electromagnetic waves, which fulfil the conditions $\mathcal{S}_{xx}=\mathcal{S}_{yy}=\mathcal{S}_0$ and $\mathcal{S}_{xy}=\mathcal{S}_{yx}=0$, exhibit the following properties:

i) their parallel components are equally correlated and
ii) their mutually orthogonal components are uncorrelated

on any pair of points at both the A plane and the observation plane, irrespective of a particular choice of the (*x, y*) and (*x*', *y*') axes.

Such properties characterize the behaviour of completely unpolarized electromagnetic waves, as reported in Refs. [1, 2]. Consequently, equation (12) defines the *completely unpolarized* electric spatial coherence wavelet tensor.

However, such electromagnetic waves can exhibit an arbitrary state of spatial coherence, because there are no a priori restrictions on $\eta_0\left(\mathbf{r}'_A+\frac{\mathbf{r}'_D}{2},\mathbf{r}'_A-\frac{\mathbf{r}'_D}{2},\nu\right)$. Indeed, equations (4), (8) and (12) yield

$$\mathbf{W}\left(\mathbf{r}_A+\frac{\mathbf{r}_D}{2},\mathbf{r}_A-\frac{\mathbf{r}_D}{2};\nu\right)=W_0\left(\mathbf{r}_A+\frac{\mathbf{r}_D}{2},\mathbf{r}_A-\frac{\mathbf{r}_D}{2};\nu\right)\begin{bmatrix}1&0\\0&1\end{bmatrix}, \tag{14}$$

with

$$W_0\left(\mathbf{r}_A+\frac{\mathbf{r}_D}{2},\mathbf{r}_A-\frac{\mathbf{r}_D}{2};\nu\right)=\left(\frac{1}{\lambda z}\right)^2 e^{i\frac{k}{z}\mathbf{r}_A\cdot\mathbf{r}_D}\iint_{A\,A}\left\langle\left|E_0\left(\mathbf{r}'_A+\frac{\mathbf{r}'_D}{2},\nu\right)\right|^2\right\rangle\left\langle\left|E_0\left(\mathbf{r}'_A-\frac{\mathbf{r}'_D}{2},\nu\right)\right|^2\right\rangle \\ \eta_0\left(\mathbf{r}'_A+\frac{\mathbf{r}'_D}{2},\mathbf{r}'_A-\frac{\mathbf{r}'_D}{2},\nu\right)e^{i\frac{k}{z}\mathbf{r}'_A\cdot\mathbf{r}'_D}\, e^{-i\frac{k}{z}(\mathbf{r}'_A\cdot\mathbf{r}_D+\mathbf{r}_A\cdot\mathbf{r}'_D)}\, d^2r'_A\, d^2r'_D\ , \tag{15}$$

on account of the condition $\boldsymbol{W}_{xx}\left(\mathbf{r}'_A+\frac{\mathbf{r}'_D}{2},\mathbf{r}'_A-\frac{\mathbf{r}'_D}{2},v\right)=\boldsymbol{W}_{yy}\left(\mathbf{r}'_A+\frac{\mathbf{r}'_D}{2},\mathbf{r}'_A-\frac{\mathbf{r}'_D}{2},v\right)=$ $\boldsymbol{W}_0\left(\mathbf{r}'_A+\frac{\mathbf{r}'_D}{2},\mathbf{r}'_A-\frac{\mathbf{r}'_D}{2},v\right)$ and then $\left\langle\left|E_l\left(\mathbf{r}'_A\pm\frac{\mathbf{r}'_D}{2},v\right)\right|^2\right\rangle=\left\langle\left|E_0\left(\mathbf{r}'_A\pm\frac{\mathbf{r}'_D}{2},v\right)\right|^2\right\rangle$. It means that the electromagnetic wave remains completely unpolarized at the observation plane but can change its spatial coherence properties along the propagation, as establish by the second-order spatial coherence theory. In particular, if the wave is fully spatially incoherent at the A plane, i.e. $\boldsymbol{\eta}\left(\mathbf{r}'_A+\frac{\mathbf{r}'_D}{2},\mathbf{r}'_A-\frac{\mathbf{r}'_D}{2},v\right)=\lambda\ z\ \delta(\mathbf{r}'_D)\begin{bmatrix}1&0\\0&1\end{bmatrix}$, its electric marginal power spectrum tensor reduces to $\mathbf{S}(\mathbf{r}_A,\mathbf{r}'_A;v)=\lambda\ z\left\langle\left|E_0\left(\mathbf{r}'_A,v\right)\right|^2\right\rangle\begin{bmatrix}1&0\\0&1\end{bmatrix}$. The corresponding electric spatial coherence wavelet tensor takes the form

$$\mathbf{W}\left(\mathbf{r}_A+\frac{\mathbf{r}_D}{2},\mathbf{r}_A-\frac{\mathbf{r}_D}{2},\mathbf{r}'_A;v\right)=\lambda\ z\left\langle\left|E_0\left(\mathbf{r}'_A,v\right)\right|^2\right\rangle e^{-i\frac{k}{z}\mathbf{r}_D\cdot\mathbf{r}'_A}\begin{bmatrix}1&0\\0&1\end{bmatrix}.$$

Consequently, equation (4) yields

$$\mathbf{W}\left(\mathbf{r}_A+\frac{\mathbf{r}_D}{2},\mathbf{r}_A-\frac{\mathbf{r}_D}{2};v\right)=\frac{1}{\lambda\ z}e^{i\frac{k}{z}\mathbf{r}_A\cdot\mathbf{r}_D}\int_A\left\langle\left|E_0\left(\mathbf{r}'_A,v\right)\right|^2\right\rangle e^{-i\frac{k}{z}\mathbf{r}_D\cdot\mathbf{r}'_A}\ d^2r'_A\begin{bmatrix}1&0\\0&1\end{bmatrix}. \tag{16}$$

Therefore a completely unpolarized and fully spatially incoherent electromagnetic wave cannot acquire a specific state of polarization by simply propagating in free space, but it can gain spatial coherence according to the coefficient of the identity matrix in equation (16), as established by the Van Cittert – Zernike theorem [1, 2]. Equation (16) can be regarded as the Van Citter – Zernike theorem for completely unpolarized and fully spatially incoherent sources.

Equation (15) also means that the interference capability of completely unpolarized electromagnetic waves will depend on the correlation between the parallel components of their electric field vectors at pair of points $\mathbf{r}'_A\pm\frac{\mathbf{r}'_D}{2}$ on the A plane, i.e. on the morphology and support of $\eta_0\left(\mathbf{r}'_A+\frac{\mathbf{r}'_D}{2},\mathbf{r}'_A-\frac{\mathbf{r}'_D}{2},v\right)$. Indeed, equation (6) yields

$$S(\mathbf{r}_A;v)=\frac{2}{\lambda\ z}\int_A\left\langle\left|E_0\left(\mathbf{r}'_A,v\right)\right|^2\right\rangle d^2r'_A \tag{17}$$

for the power spectrum on the observation plane provided by a fully spatially incoherent wave. It is apparent that such a wave is not able to produce interference patterns at the observation plane. However, if $\eta_0\left(\mathbf{r}'_A+\frac{\mathbf{r}'_D}{2},\mathbf{r}'_A-\frac{\mathbf{r}'_D}{2},v\right)$ is not delta-like, we obtain

$$S(\mathbf{r}_A;\nu)=\frac{2}{\lambda\, z}\int_A\left\langle\left|E_0\left(\mathbf{r}'_A,\nu\right)\right|^2\right\rangle d^2r'_A+4\left(\frac{1}{\lambda\, z}\right)^2\int_A\int_{\substack{A\\ \mathbf{r}'_D\neq 0}}\left\langle\left|E_0\left(\mathbf{r}'_A+\frac{\mathbf{r}'_D}{2},\nu\right)\right|^2\right\rangle^{\frac{1}{2}}\left\langle\left|E_0\left(\mathbf{r}'_A-\frac{\mathbf{r}'_D}{2},\nu\right)\right|^2\right\rangle^{\frac{1}{2}}$$

$$\left|\eta_0\left(\mathbf{r}'_A+\frac{\mathbf{r}'_D}{2},\mathbf{r}'_A-\frac{\mathbf{r}'_D}{2},\nu\right)\right|\cos\left[\frac{k}{z}\left(\mathbf{r}'_A-\mathbf{r}_A\right)\cdot\mathbf{r}'_D+\beta_0\right]d^2r'_D\, d^2r'_A \tag{18}$$

for the power distribution on the observation plane, by introducing the dimensionless function $\lambda\, z\, \delta(\mathbf{r}'_D)+\left[1-\lambda\, z\, \delta(\mathbf{r}'_D)\right]$, with $\delta(\mathbf{r}'_D)$ the Dirac's delta function, in the integral of equation (6) and applying the swept condition described in Refs. [3, 4]. In equation (18) $\eta_0\left(\mathbf{r}'_A+\frac{\mathbf{r}'_D}{2},\mathbf{r}'_A-\frac{\mathbf{r}'_D}{2},\nu\right)=\left|\eta_0\left(\mathbf{r}'_A+\frac{\mathbf{r}'_D}{2},\mathbf{r}'_A-\frac{\mathbf{r}'_D}{2},\nu\right)\right|e^{i\,\beta_0\left(\mathbf{r}'_A+\frac{\mathbf{r}'_D}{2},\mathbf{r}'_A-\frac{\mathbf{r}'_D}{2},\nu\right)}$ holds.

Equation (18) reveals cosine-like modulations on the power distribution, due to the interference of contributions associated to the parallel electric components at pair of points $\mathbf{r}'_A\pm\frac{\mathbf{r}'_D}{2}$, within the support of $\eta_0\left(\mathbf{r}'_A+\frac{\mathbf{r}'_D}{2},\mathbf{r}'_A-\frac{\mathbf{r}'_D}{2},\nu\right)$. Conventionally, the interference fringes will be orthogonal to the separation vectors $\mathbf{r}'_D$ and their periods will be given by $\frac{\lambda\, z}{\left|\mathbf{r}'_D\right|}$. Equation (18) also describes the power spectrum provided by a spatially partially coherent electromagnetic wave, with a complex degree of spatial coherence $\eta_0\left(\mathbf{r}'_A+\frac{\mathbf{r}'_D}{2},\mathbf{r}'_A-\frac{\mathbf{r}'_D}{2},\nu\right)$, at the observation plane. For this reason, the tensor defined in equation (13) can be called *the electric spatial coherence degree tensor* of completely unpolarized electromagnetic waves.

It becomes $\boldsymbol{\eta}\left(\mathbf{r}'_A+\frac{\mathbf{r}'_D}{2},\mathbf{r}'_A-\frac{\mathbf{r}'_D}{2},\nu\right)=\mathbf{I}=\begin{bmatrix}1&0\\0&1\end{bmatrix}$ for fully spatially coherent electromagnetic waves, which provides the highest contrasted interference modulations, according to equation (18). Therefore, completely unpolarized beams might be spatially fully coherent, as was recently pointed out by Wolf [8].

A second extreme case concerns electromagnetic waves with the properties $\mathsf{S}_{xx}\neq\mathsf{S}_{yy}$ but real valued and $\mathsf{S}_{xy}=\mathsf{S}^*_{yx}=\mathsf{S}^{\frac{1}{2}}_{xx}\,\mathsf{S}^{\frac{1}{2}}_{yy}\,e^{i\,\beta}$, with β constant. The electric marginal power spectrum tensors of such waves are not rotation-invariant, as apparent in the following expressions:

$$\mathsf{S}(\mathbf{r}_A\,,\mathbf{r}'_A;\nu)=\begin{bmatrix}\mathsf{S}_{xx} & \mathsf{S}^{\frac{1}{2}}_{xx}\,\mathsf{S}^{\frac{1}{2}}_{yy}\,e^{i\,\beta}\\ \mathsf{S}^{\frac{1}{2}}_{xx}\,\mathsf{S}^{\frac{1}{2}}_{yy}\,e^{-i\,\beta} & \mathsf{S}_{yy}\end{bmatrix}. \tag{19}$$

and

$$\tilde{\mathbf{S}}(\mathbf{r}_A\,,\,\mathbf{r}'_A;\nu)=\begin{bmatrix} \mathcal{S}_{xx}\cos^2\vartheta+\mathcal{S}_{yy}\sin^2\vartheta-2\,\mathcal{S}_{xx}^{\frac{1}{2}}\,\mathcal{S}_{yy}^{\frac{1}{2}}\cos\beta\sin\vartheta\cos\vartheta & \mathcal{S}_{xx}^{\frac{1}{2}}\,\mathcal{S}_{yy}^{\frac{1}{2}}\left(e^{i\beta}\cos^2\vartheta-e^{-i\beta}\sin^2\vartheta\right)+\left(\mathcal{S}_{xx}-\mathcal{S}_{yy}\right)\sin\vartheta\cos\vartheta \\ \mathcal{S}_{xx}^{\frac{1}{2}}\,\mathcal{S}_{yy}^{\frac{1}{2}}\left(e^{-i\beta}\cos^2\vartheta-e^{i\beta}\sin^2\vartheta\right)+\left(\mathcal{S}_{xx}-\mathcal{S}_{yy}\right)\sin\vartheta\cos\vartheta & \mathcal{S}_{yy}\cos^2\vartheta+\mathcal{S}_{xx}\sin^2\vartheta+2\,\mathcal{S}_{xx}^{\frac{1}{2}}\,\mathcal{S}_{yy}^{\frac{1}{2}}\cos\beta\sin\vartheta\cos\vartheta \end{bmatrix} \tag{20}$$

respectively. Equations (19) and (20) can be obtained if $\boldsymbol{\eta}\left(\mathbf{r}'_A+\frac{\mathbf{r}'_D}{2},\mathbf{r}'_A-\frac{\mathbf{r}'_D}{2},\nu\right)=\begin{bmatrix}1 & e^{i\beta}\\ e^{-i\beta} & 1\end{bmatrix}$ and $\mathcal{S}_{ll}(\mathbf{r}_A,\mathbf{r}'_A;\nu)=\int_A\left\langle\left|E_l\left(\mathbf{r}'_A+\frac{\mathbf{r}'_D}{2},\nu\right)\right|^2\right\rangle^{\frac{1}{2}}\left\langle\left|E_l\left(\mathbf{r}'_A-\frac{\mathbf{r}'_D}{2},\nu\right)\right|^2\right\rangle^{\frac{1}{2}}e^{i\frac{k}{z}(\mathbf{r}'_A-\mathbf{r}_A)\cdot\mathbf{r}'_D}\,d^2r'_D$. The above expressions describe a spatially fully coherent and completely polarized electromagnetic wave, as can be verified by comparing them with the tensors for a plane, monochromatic electromagnetic wave, which propagates along the z-axis. Consequently,

$$\mathbf{W}\left(\mathbf{r}_A+\frac{\mathbf{r}_D}{2}\,,\,\mathbf{r}_A-\frac{\mathbf{r}_D}{2}\,,\,\mathbf{r}'_A;\nu\right)=\begin{bmatrix}\mathcal{S}_{xx} & \mathcal{S}_{xx}^{\frac{1}{2}}\,\mathcal{S}_{yy}^{\frac{1}{2}}\,e^{i\beta}\\ \mathcal{S}_{xx}^{\frac{1}{2}}\,\mathcal{S}_{yy}^{\frac{1}{2}}\,e^{-i\beta} & \mathcal{S}_{yy}\end{bmatrix}e^{-i\frac{k}{z}\mathbf{r}_D\cdot\mathbf{r}'_A} \tag{21}$$

defines the *completely polarized and fully spatially coherent* electric spatial coherence wavelet tensor.

The above development yields $\det[\mathbf{S}(\mathbf{r}_A\,,\,\mathbf{r}'_A;\nu)]=\mathcal{S}_0^2$ for unpolarized electromagnetic spatial coherent wavelets (independently from their spatial coherent state) and $\det[\mathbf{S}(\mathbf{r}_A\,,\,\mathbf{r}'_A;\nu)]=0$ for the completely polarized and fully spatially coherent wavelet. This feature is useful for the discussion below.

5. THE POLARIZATION PARAMETER

According to the former section, completely unpolarized electromagnetic spatial coherence wavelets satisfy the condition $\eta_{xy}\left(\mathbf{r}'_A + \frac{\mathbf{r}'_D}{2}, \mathbf{r}'_A - \frac{\mathbf{r}'_D}{2}; \nu\right) = 0$ whereas $\left|\eta_{xy}\left(\mathbf{r}'_A + \frac{\mathbf{r}'_D}{2}, \mathbf{r}'_A - \frac{\mathbf{r}'_D}{2}; \nu\right)\right| = 1$ holds for completely polarized ones. Then, it is completely reasonable to regard that partially polarized electromagnetic spatial coherence wavelets should achieve the condition $0 < \left|\eta_{xy}\left(\mathbf{r}'_A + \frac{\mathbf{r}'_D}{2}, \mathbf{r}'_A - \frac{\mathbf{r}'_D}{2}; \nu\right)\right| < 1$. Consequently, a *polarization parameter* should be defined for estimating how much the wavelets are polarized.

It could be performed by showing that the electric spatial coherence wavelet tensor, or more specifically the electric marginal power spectrum tensor, can be uniquely expressed in terms of a completely unpolarized component and a polarized component, i.e. $\mathbf{S}(\mathbf{r}_A, \mathbf{r}'_A; \nu) = \mathbf{S}_{unpol}(\mathbf{r}_A, \mathbf{r}'_A; \nu) + \mathbf{S}_{pol}(\mathbf{r}_A, \mathbf{r}'_A; \nu)$ with the property that $\det[\mathbf{S}_{pol}(\mathbf{r}_A, \mathbf{r}'_A; \nu)] = 0$. Expressed in matrix notation,

$$\mathbf{S}(\mathbf{r}_A, \mathbf{r}'_A; \nu) = \begin{bmatrix} S_{xx} & S_{xy} \\ S_{yx} & S_{yy} \end{bmatrix} = S_0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} S_{xx}^{(pol)} & S_{xy} \\ S_{yx} & S_{yy}^{(pol)} \end{bmatrix} \tag{22}$$

and $S_{xx}^{(pol)} S_{yy}^{(pol)} - S_{xy} S_{yx} = (S_{xx} - S_0)(S_{yy} - S_0) - S_{xy} S_{yx} = 0$, so that, $\det[\mathbf{S}(\mathbf{r}_A, \mathbf{r}'_A; \nu)] = S_0^2$. Therefore,

$$S_0^2 - S_0 (S_{xx} + S_{yy}) + S_{xx} S_{yy} = S_0^2 - S_0\, tr[\mathbf{S}(\mathbf{r}_A, \mathbf{r}'_A; \nu)] + \det[\mathbf{S}(\mathbf{r}_A, \mathbf{r}'_A; \nu)] = 0, \tag{23}$$

whose solution for S_0 is

$$S_0^{(\pm)} = \frac{1}{2}\left\{ tr[\mathbf{S}(\mathbf{r}_A, \mathbf{r}'_A; \nu)] \pm \left(tr^2[\mathbf{S}(\mathbf{r}_A, \mathbf{r}'_A; \nu)] - 4 \det[\mathbf{S}(\mathbf{r}_A, \mathbf{r}'_A; \nu)] \right)^{\frac{1}{2}} \right\}. \tag{24}$$

Consequently,

$$S_{xx(\pm)}^{(pol)} = \frac{1}{2}\left\{ S_{xx} - S_{yy} \pm \left(tr^2[\mathbf{S}(\mathbf{r}_A, \mathbf{r}'_A; \nu)] - 4 \det[\mathbf{S}(\mathbf{r}_A, \mathbf{r}'_A; \nu)] \right)^{\frac{1}{2}} \right\} \tag{25a}$$

and

$$S_{yy(\pm)}^{(pol)} = \frac{1}{2}\left\{ S_{yy} - S_{xx} \pm \left(tr^2[\mathbf{S}(\mathbf{r}_A, \mathbf{r}'_A; \nu)] - 4 \det[\mathbf{S}(\mathbf{r}_A, \mathbf{r}'_A; \nu)] \right)^{\frac{1}{2}} \right\}. \tag{25b}$$

To perform an appropriate choice of S_0, $S_{xx}^{(pol)}$ and $S_{yy}^{(pol)}$ let us take into account that they should be locally real valued and positive definite, i.e. $S_0 \geq 0$, $S_{xx}^{(pol)} \geq 0$ and $S_{yy}^{(pol)} \geq 0$ for

$\mathbf{r}'_D = 0$, because $\mathcal{S}_{ll}(\mathbf{r}_A, \mathbf{r}'_A;\nu)\big|_{\mathbf{r}'_D=0} = \lambda\, z \left\langle \left| E_l(\mathbf{r}'_A,\nu)\right|^2 \right\rangle \geq 0$. As a consequence, the relationship [8]

$$\left(tr^2\left[\mathbf{S}(\mathbf{r}_A, \mathbf{r}'_A;\nu)\right] - 4\det\left[\mathbf{S}(\mathbf{r}_A, \mathbf{r}'_A;\nu)\right]\right)^{\frac{1}{2}} = \left[\left(\mathcal{S}_{xx} - \mathcal{S}_{yy}\right)^2 + 4\left|\mathcal{S}_{xy}\right|^2\right]^{\frac{1}{2}} \geq \left|\mathcal{S}_{xx} - \mathcal{S}_{yy}\right|$$

holds at least for $\mathbf{r}'_D = 0$. It is fulfilled if the values of $\mathcal{S}_0^{(-)}$, $\mathcal{S}_{xx(+)}^{(pol)}$ and $\mathcal{S}_{yy(+)}^{(pol)}$ are chosen. Therefore, the electric marginal power spectrum tensor can be univocally expressed by replacing them in equation (22). The above results also yield

$$tr\left[\mathbf{S}_{pol}(\mathbf{r}_A, \mathbf{r}'_A;\nu)\right] = \left(tr^2\left[\mathbf{S}(\mathbf{r}_A, \mathbf{r}'_A;\nu)\right] - 4\det\left[\mathbf{S}(\mathbf{r}_A, \mathbf{r}'_A;\nu)\right]\right)^{\frac{1}{2}}. \qquad (26)$$

It represents the electric energy density of the polarized component of the electromagnetic spatial coherence wavelet that propagates from $\mathbf{r}'_A$ to $\mathbf{r}_A$. Such an amount of energy can be cosine-like modulated by contributions from the set of pairs of centres of secondary disturbance delimited by the complex degree of spatial coherence tensor around $\mathbf{r}'_A$.

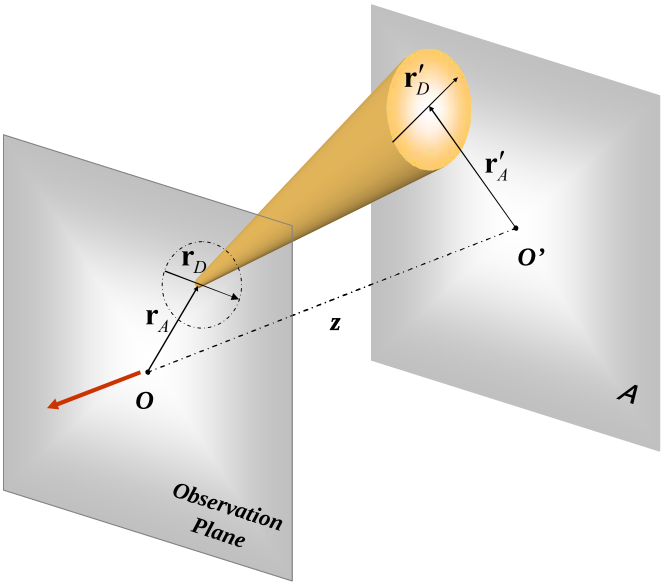

Fig. 3: Illustrating the non-local character of the polarization parameter

Thus, we define the *polarization parameter* of the electromagnetic wave in a similar fashion as the degree of polarization in Refs. [2, 8], i.e.:

$$\mathbf{P}(\mathbf{r}_A, \mathbf{r}'_A;\nu) = \frac{tr\left[\mathbf{S}_{pol}(\mathbf{r}_A, \mathbf{r}'_A;\nu)\right]}{tr\left[\mathbf{S}(\mathbf{r}_A, \mathbf{r}'_A;\nu)\right]} = \frac{tr\left[\mathbf{S}_{coh}^{(pol)}(\mathbf{r}_A, \mathbf{r}'_A;\nu)\right]}{tr\left[\mathbf{S}(\mathbf{r}_A, \mathbf{r}'_A;\nu)\right]} = \sqrt{1 - \frac{4\det\left[\mathbf{S}(\mathbf{r}_A, \mathbf{r}'_A;\nu)\right]}{tr^2\left[\mathbf{S}(\mathbf{r}_A, \mathbf{r}'_A;\nu)\right]}}. \qquad (27)$$

On account of equations (1) and (8), $\mathbf{P}(\mathbf{r}_A, \mathbf{r}'_A;\nu)$ describes how much the electromagnetic spatial coherence wavelet, that propagates from $\mathbf{r}'_A$ to $\mathbf{r}_A$, is polarized, and additional insight can be addressed from it. According to the meaning of the spatial coherence wavelets [3, 7] $\mathbf{P}(\mathbf{r}_A, \mathbf{r}'_A;\nu)$ can be considered as a descriptor of a causal relationship between the aperture and the observation plane, which estimates the effect of the correlation properties of the electric field vector within the region $\mathbf{r}'_A \pm \frac{\mathbf{r}'_D}{2}$, given by the tensor $\boldsymbol{\eta}\left(\mathbf{r}'_A + \frac{\mathbf{r}'_D}{2}, \mathbf{r}'_A - \frac{\mathbf{r}'_D}{2}, \nu\right)$, onto the polarization state within the vicinity $\mathbf{r}_A \pm \frac{\mathbf{r}_D}{2}$ specified by the electric wavelet tensor $\mathbf{W}\left(\mathbf{r}_A + \frac{\mathbf{r}_D}{2}, \mathbf{r}_A - \frac{\mathbf{r}_D}{2}, \mathbf{r}'_A; \nu\right) = \mathbf{S}(\mathbf{r}_A, \mathbf{r}'_A; \nu)\; e^{-i\frac{k}{z}\mathbf{r}_D \cdot \mathbf{r}'_A}$, as depicted in Figure 3.

Indeed, $\mathbf{P}(\mathbf{r}_A , \mathbf{r}'_A; \nu)$ is a real valued and positive definite parameter that fulfills the condition $0 \le \mathbf{P}(\mathbf{r}_A , \mathbf{r}'_A; \nu) \le 1$, which yields $tr[\mathbf{S}(\mathbf{r}_A , \mathbf{r}'_A; \nu)] \ge 2\sqrt{\det[\mathbf{S}(\mathbf{r}_A , \mathbf{r}'_A; \nu)]}$. Then, $tr[\mathbf{S}(\mathbf{r}_A , \mathbf{r}'_A; \nu)] = 2\sqrt{\det[\mathbf{S}(\mathbf{r}_A , \mathbf{r}'_A; \nu)]}$ implies $\mathbf{P}(\mathbf{r}_A , \mathbf{r}'_A; \nu) = 0$, i.e. the electromagnetic spatial coherence wavelets should be completely unpolarized. It is fulfilled if $\boldsymbol{S}_{xx} = \boldsymbol{S}_{yy} = \boldsymbol{S}_0$ and $\boldsymbol{S}_{xy} = 0$, so that $\mathbf{W}\left(\mathbf{r}_A + \frac{\mathbf{r}_D}{2}, \mathbf{r}_A - \frac{\mathbf{r}_D}{2}, \mathbf{r}'_A; \nu\right) = \boldsymbol{S}_0(\mathbf{r}_A , \mathbf{r}'_A; \nu) e^{-i\frac{k}{z}\mathbf{r}_D \cdot \mathbf{r}'_A} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Expressed differently, there is no specific polarization state within $\mathbf{r}_A \pm \frac{\mathbf{r}_D}{2}$ because the mutually orthogonal components of the electric field vector at any pair of points within the vicinity $\mathbf{r}'_A \pm \frac{\mathbf{r}'_D}{2}$ are uncorrelated.

If $\det[\mathbf{S}(\mathbf{r}_A , \mathbf{r}'_A; \nu)] = 0$ then $\mathbf{P}(\mathbf{r}_A , \mathbf{r}'_A; \nu) = 1$ holds, i.e. the electromagnetic spatial coherence wavelets should be completely polarized. Such waves exhibit the property $\boldsymbol{S}_{xx}\, \boldsymbol{S}_{yy} = \left|\boldsymbol{S}_{xy}\right|^2$, and equation (21) describes the fully spatially coherent case. This property establishes the correlation between mutually orthogonal components of the electric field vector at any pair of points within $\mathbf{r}'_A \pm \frac{\mathbf{r}'_D}{2}$ required to obtain a well-defined polarization state within $\mathbf{r}_A \pm \frac{\mathbf{r}_D}{2}$. In the same sense, the wavelets will be partially polarized if $0 < \mathbf{P}(\mathbf{r}_A , \mathbf{r}'_A; \nu) < 1$.

The feature that this causal relationship involves surroundings of points on each plane suggests the existence of *polarization domains*, that is, regions on a plane across which the polarization state remains unchanged, and therefore the polarization parameter takes the same value for all points within the particular domain. They are determined by superimposing the contributions of all the electromagnetic spatial coherence wavelets to each surrounding area $\mathbf{r}_A \pm \frac{\mathbf{r}_D}{2}$. So, if all of them are completely polarized in the same polarization state, this state of polarization should characterize the area $\mathbf{r}_A \pm \frac{\mathbf{r}_D}{2}$.

Thus, the polarization domain will be arbitrary great if the above condition holds for every vicinity $\mathbf{r}_A \pm \frac{\mathbf{r}_D}{2}$. However, because of the superposition of contributions, the polarization domains can become arbitrary small. In this case, the wave will be completely polarized at all points $\mathbf{r}_A$ but the state of polarization will change point-to-point. For example, it occurs if

$$\boldsymbol{\eta}\left(\mathbf{r}'_A + \frac{\mathbf{r}'_D}{2}, \mathbf{r}'_A - \frac{\mathbf{r}'_D}{2}, \nu\right) = \lambda\, z \begin{bmatrix} \eta_{xx} & \eta_{xy}(\mathbf{r}'_A)\,\delta(\mathbf{r}'_D) \\ \eta_{yx}(\mathbf{r}'_A)\,\delta(\mathbf{r}'_D) & \eta_{yy} \end{bmatrix}.$$

Thus, the electric spatial coherence wavelet tensor will take the form

$$\mathsf{W}\left(\mathbf{r}_A + \frac{\mathbf{r}_D}{2}, \mathbf{r}_A - \frac{\mathbf{r}_D}{2}, \mathbf{r}'_A; \nu\right) = \lambda\, z\, e^{-i\frac{k}{z}\mathbf{r}_D\cdot\mathbf{r}'_A},$$

$$\begin{bmatrix} \mathcal{S}_{xx}(\mathbf{r}_A, \mathbf{r}'_A; \nu) & \left\langle \left| E_x(\mathbf{r}'_A, \nu) \right|^2 \right\rangle^{\frac{1}{2}} \eta_{xy}(\mathbf{r}'_A, \nu) \left\langle \left| E_y(\mathbf{r}'_A, \nu) \right|^2 \right\rangle^{\frac{1}{2}} \\ \left\langle \left| E_y(\mathbf{r}'_A, \nu) \right|^2 \right\rangle^{\frac{1}{2}} \eta_{yx}(\mathbf{r}'_A, \nu) \left\langle \left| E_x(\mathbf{r}'_A, \nu) \right|^2 \right\rangle^{\frac{1}{2}} & \mathcal{S}_{yy}(\mathbf{r}_A, \mathbf{r}'_A; \nu) \end{bmatrix}$$

with

$$\mathcal{S}_{ll}(\mathbf{r}_A, \mathbf{r}'_A; \nu) = \int_A \left\langle \left| E_l\left(\mathbf{r}'_A + \frac{\mathbf{r}'_D}{2}, \nu\right) \right|^2 \right\rangle^{\frac{1}{2}} \eta_{ll}\left(\mathbf{r}'_A + \frac{\mathbf{r}'_D}{2}, \mathbf{r}'_A - \frac{\mathbf{r}'_D}{2}, \nu\right) \left\langle \left| E_l\left(\mathbf{r}'_A - \frac{\mathbf{r}'_D}{2}, \nu\right) \right|^2 \right\rangle^{\frac{1}{2}} e^{i\frac{k}{z}(\mathbf{r}'_A - \mathbf{r}_A)\cdot\mathbf{r}'_D}\, d^2 r'_D$$

and the polarization parameter will be given by

$$\mathsf{P}(\mathbf{r}_A, \mathbf{r}'_A; \nu) = \sqrt{1 - \frac{4\,\mathcal{S}_{xx}\,\mathcal{S}_{yy}}{\left[\mathcal{S}_{xx}(\mathbf{r}_A, \mathbf{r}'_A; \nu) + \mathcal{S}_{yy}(\mathbf{r}_A, \mathbf{r}'_A; \nu)\right]^2} \left[1 - \frac{\left\langle \left| E_x(\mathbf{r}'_A, \nu) \right|^2 \right\rangle \left\langle \left| E_y(\mathbf{r}'_A, \nu) \right|^2 \right\rangle}{\mathcal{S}_{xx}\,\mathcal{S}_{yy}} \left| \eta_{xy}(\mathbf{r}'_A, \nu) \right|^2 \right]} > 0$$

on account that $\eta_{xy}(\mathbf{r}'_A, \nu) = \eta^*_{yx}(\mathbf{r}'_A, \nu)$. This result means that those kinds of waves can change their polarization domains, and then their polarization state, with propagation in free space. It is not the case of a wave without polarization domains as we showed in the former section.

Fully spatially incoherent electromagnetic waves with arbitrary small polarization domains can also change their polarization states as they propagate. Indeed, $\boldsymbol{\eta}\left(\mathbf{r}'_A + \frac{\mathbf{r}'_D}{2}, \mathbf{r}'_A - \frac{\mathbf{r}'_D}{2}, \nu\right) = \lambda\, z\, \delta(\mathbf{r}'_D) \begin{bmatrix} 1 & \eta_{xy} \\ \eta_{yx} & 1 \end{bmatrix}$ holds in this case, so that

$$\mathsf{W}\left(\mathbf{r}_A + \frac{\mathbf{r}_D}{2}, \mathbf{r}_A - \frac{\mathbf{r}_D}{2}, \mathbf{r}'_A; \nu\right) = \lambda\, z\, e^{-i\frac{k}{z}\mathbf{r}_D\cdot\mathbf{r}'_A},$$

$$\begin{bmatrix} \left\langle \left| E_x(\mathbf{r}'_A, \nu) \right|^2 \right\rangle & \left\langle \left| E_x(\mathbf{r}'_A, \nu) \right|^2 \right\rangle^{\frac{1}{2}} \eta_{xy}(\mathbf{r}'_A, \nu) \left\langle \left| E_y(\mathbf{r}'_A, \nu) \right|^2 \right\rangle^{\frac{1}{2}} \\ \left\langle \left| E_x(\mathbf{r}'_A, \nu) \right|^2 \right\rangle^{\frac{1}{2}} \eta_{yx}(\mathbf{r}'_A, \nu) \left\langle \left| E_y(\mathbf{r}'_A, \nu) \right|^2 \right\rangle^{\frac{1}{2}} & \left\langle \left| E_y(\mathbf{r}'_A, \nu) \right|^2 \right\rangle \end{bmatrix}$$

and $\mathbf{P}(\mathbf{r}_A\,,\mathbf{r}'_A;\nu)=\sqrt{1-\dfrac{4\left\langle\left|E_x(\mathbf{r}'_A,\nu)\right|^2\right\rangle\left\langle\left|E_y(\mathbf{r}'_A,\nu)\right|^2\right\rangle\left[1-\left|\eta_{xy}(\mathbf{r}'_A,\nu)\right|^2\right]}{\left[\left\langle\left|E_x(\mathbf{r}'_A,\nu)\right|^2\right\rangle+\left\langle\left|E_y(\mathbf{r}'_A,\nu)\right|^2\right\rangle\right]^2}}$. It follows straightforwardly that $\mathbf{P}(\mathbf{r}_A\,,\mathbf{r}'_A;\nu)=\left|\eta_{xy}(\mathbf{r}'_A,\nu)\right|$ if $\left\langle\left|E_x(\mathbf{r}'_A,\nu)\right|^2\right\rangle=\left\langle\left|E_y(\mathbf{r}'_A,\nu)\right|^2\right\rangle$, i.e. the modulus of the off-diagonal elements of the tensor $\boldsymbol{\eta}\left(\mathbf{r}'_A+\dfrac{\mathbf{r}'_D}{2},\mathbf{r}'_A-\dfrac{\mathbf{r}'_D}{2},\nu\right)$ provide a measurement of the polarization parameter in this case.

It is useful to regard the electrical cross-spectral density tensor for this kind of electromagnetic waves, because it provides a certain generalization of the Van Cittert – Zernike theorem. To this aim, we express that tensor at the observation plane as

$$\mathbf{W}\left(\mathbf{r}_A+\frac{\mathbf{r}_D}{2}\,,\mathbf{r}_A-\frac{\mathbf{r}_D}{2};\nu\right)=\begin{bmatrix}W_{xx} & 0\\ 0 & W_{xx}\end{bmatrix}+\begin{bmatrix}0 & W_{xy}\\ W_{yx} & 0\end{bmatrix}, \qquad (28)$$

where

$$W_{ll}\left(\mathbf{r}_A+\frac{\mathbf{r}_D}{2}\,,\mathbf{r}_A-\frac{\mathbf{r}_D}{2};\nu\right)=\frac{1}{\lambda\,z}\,e^{i\frac{k}{z}\mathbf{r}_A\cdot\mathbf{r}_D}\int_A\left\langle\left|E_l(\mathbf{r}'_A,\nu)\right|^2\right\rangle e^{-i\frac{k}{z}\mathbf{r}_D\cdot\mathbf{r}'_A}\,d^2r'_A \qquad (29)$$

and

$$W_{lm}\left(\mathbf{r}_A+\frac{\mathbf{r}_D}{2}\,,\mathbf{r}_A-\frac{\mathbf{r}_D}{2};\nu\right)=\frac{1}{\lambda\,z}\,e^{i\frac{k}{z}\mathbf{r}_A\cdot\mathbf{r}_D}\int_A\left\langle\left|E_l(\mathbf{r}'_A,\nu)\right|^2\right\rangle^{\frac{1}{2}}\eta_{lm}(\mathbf{r}'_A,\nu)\left\langle\left|E_m(\mathbf{r}'_A,\nu)\right|^2\right\rangle^{\frac{1}{2}}e^{-i\frac{k}{z}\mathbf{r}_D\cdot\mathbf{r}'_A}\,d^2r'_A\,, \qquad (30)$$

according to equation (4). It is apparent that the first matrix in equation (28) describes the spatial coherence rising of the wave, without changing its polarization state, as it propagates in free space. So, this term is corresponding to the classical Van Cittert – Zernike theorem. Furthermore, from the above reasoning, it follows that the second matrix is responsible for the changes in the polarization state of the wave along its propagation. Therefore, equations (28) to (30) constitute a generalization of the Van Cittert – Zernike theorem, which describes the changes in both spatial coherence and polarization of an electromagnetic wave emitted by a fully spatially incoherent and strictly locally polarized source. It is an alternative analysis to that fairly reported by Gori and collaborators [11].

If the result of the superposition of contributions at the neighbourhood $\mathbf{r}_A\pm\dfrac{\mathbf{r}_D}{2}$ is completely unpolarized, there is no polarization domain associated to this vicinity. This can occur for any surrounding, in such a way that the electromagnetic wave will not exhibit polarization domains. Thus, equation (22) points out that an electromagnetic wave in any state of polarization can be understood as the result of the superposition of a wave with specific polarization domains and a wave without domains. The second wave introduces the randomness in the domains of the first wave that characterizes the partial polarization.

Finally, it is worth remarking that a fully spatially incoherent electromagnetic wave might be fully polarized. It can be confirmed by passing natural light through a linear polarizer, attached at the A plane. Its transmission matrix is defined as [2] $[\mathbf{T}_{\mathsf{LP}}(\theta)] = \begin{bmatrix} \cos^2\theta & \cos\theta \sin\theta \\ \cos\theta \sin\theta & \sin^2\theta \end{bmatrix}$, with θ the angle between the transmission axis of the polarizer and the x'-axis.

Now, $\boldsymbol{\eta}\left(\mathbf{r}'_A + \frac{\mathbf{r}'_D}{2}, \mathbf{r}'_A - \frac{\mathbf{r}'_D}{2}, \nu\right) = \lambda\, z\, \delta(\mathbf{r}'_D) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ holds for natural light, so that the corresponding electric marginal power spectrum tensor [equation (8)] takes the form

$$\mathbf{S}(\mathbf{r}_A, \mathbf{r}'_A; \nu) = \lambda\, z \begin{bmatrix} \left\langle \left| E_x(\mathbf{r}'_A, \nu) \right|^2 \right\rangle & 0 \\ 0 & \left\langle \left| E_y(\mathbf{r}'_A, \nu) \right|^2 \right\rangle \end{bmatrix}.$$

After passing through the polarizer, it becomes

$$\mathbf{T}_{\mathsf{LP}}(\theta)\, \mathbf{S}(\mathbf{r}_A\ , \mathbf{r}'_A; \nu) = \lambda\, z \begin{bmatrix} \left\langle \left| E_x(\mathbf{r}'_A, \nu) \right|^2 \right\rangle \cos^2\theta & \left\langle \left| E_x(\mathbf{r}'_A, \nu) \right|^2 \right\rangle \sin\theta \cos\theta \\ \left\langle \left| E_y(\mathbf{r}'_A, \nu) \right|^2 \right\rangle \sin\theta \cos\theta & \left\langle \left| E_y(\mathbf{r}'_A, \nu) \right|^2 \right\rangle \sin^2\theta \end{bmatrix}$$

and its trace and determinant are given by

$$tr[\mathbf{T}_{\mathsf{LP}}(\theta)\, \mathbf{S}(\mathbf{r}_A\ , \mathbf{r}'_A; \nu)] = \lambda\, z \left[\left\langle \left| E_x(\mathbf{r}'_A, \nu) \right|^2 \right\rangle \cos^2\theta + \left\langle \left| E_y(\mathbf{r}'_A, \nu) \right|^2 \right\rangle \sin^2\theta \right]$$

and

$$\det[\mathbf{T}_{\mathsf{LP}}(\theta)\, \mathbf{S}(\mathbf{r}_A\ , \mathbf{r}'_A; \nu)] = 0$$

respectively. Following equation (6), the power distribution at the observation plane behind the polarizer will be given by

$$S(\mathbf{r}_A; \nu) = \left\langle w_x(\nu) \right\rangle \cos^2\theta + \left\langle w_y(\nu) \right\rangle \sin^2\theta\,,$$

with $\left\langle w_l(\nu) \right\rangle = \frac{1}{\lambda\, z} \int_A \left\langle \left| E_l(\mathbf{r}'_A, \nu) \right|^2 \right\rangle d^2 r'_A$ the average energy density of the l-component of the electric field vector. The power distribution at the observation plane will be given by the addition of the energy fractions provided by the projections of the electric field components along the transmission axis of the polarizer. This power distribution is corresponding to a fully spatially incoherent electromagnetic wave. However, according to equation (27), this wave is fully polarized, hence $\mathbf{P}(\mathbf{r}_A\ , \mathbf{r}'_A; \nu) = 1$ because $\det[\mathbf{T}_{\mathsf{LP}}(\theta)\, \mathbf{S}(\mathbf{r}_A\ , \mathbf{r}'_A; \nu)] = 0$.

CONCLUSION

The recently introduced scalar concept of the spatial coherence wavelet is extended to the electromagnetic domain by means of bringing the spatial coherence wavelet tensor in the framework of the electromagnetic coherence theory. The mathematical properties of this tensor and its physical meaning are discussed throughout the text, allowing us to express in terms of this tensor quantity, the fundamental characteristics of the propagation of the electromagnetic waves in the free space.

This alternative approach to the study of the propagation of the electromagnetic waves, not only reproduces successfully the previous results, recently reported in the literature, but additionally it provides further insight about the causal relationship between the polarization states at different planes along the propagation path.

In this context, we introduced the polarization domains, defined as regions on the observation plane with a specific state of polarization, i.e. the polarization parameter will take a fixed value within each region. The changes in the state and in the parameter of polarization that an electromagnetic wave undergoes by propagation in the free space [18, 20] can be completely understood within the framework of the polarization domains.

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