Images of
Bose-Einstein condensates

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PRA 67, 033608 (2003)
cond-mat/0503328

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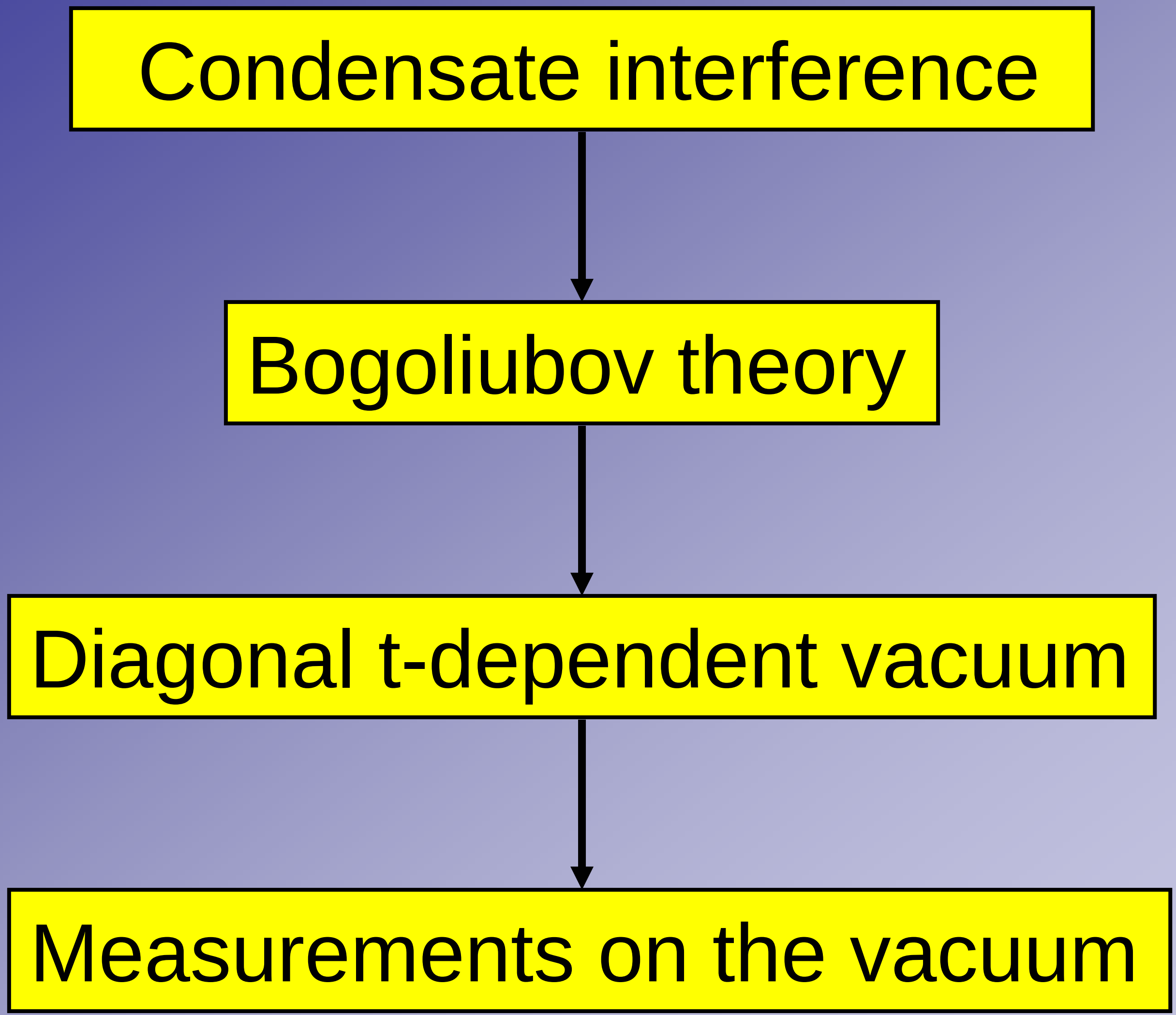


Condensate interference
Bogoliubov theory
Diagonal t-dependent vacuum
Measurements on the vacuum

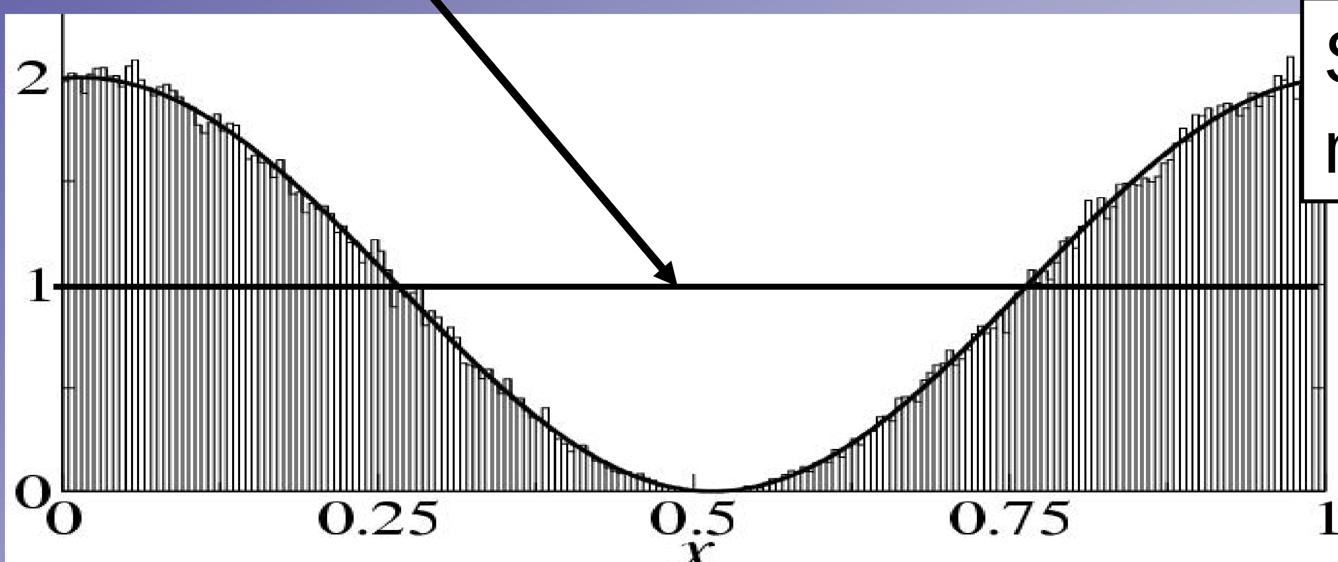
Condensate interference

Javanainen & Yoo, PRL 76, 161 (1996)

Fock state

$$\left|\frac{N}{2},\frac{N}{2}\right\rangle \quad \text{in} \quad e^{+i\pi x} \quad \text{and} \quad e^{-i\pi x}$$

$$\rho(x) = \left\langle \Psi^{+}(x)\Psi(x)\right\rangle = 1$$



Single density measurement

Condensate interference

In experiment

$$P(x_1, x_2, \ldots, x_N) \propto$$

$$\left\langle \tfrac{N}{2}, \tfrac{N}{2} \right| \hat{\Psi}^+(x_1)\hat{\Psi}^+(x_2)\ldots\hat{\Psi}^+(x_N)\hat{\Psi}(x_N)\ldots\hat{\Psi}(x_2)\hat{\Psi}(x_1) \left| \tfrac{N}{2}, \tfrac{N}{2} \right\rangle$$

In computer experiment

$$P(x_1) \propto \left\langle \tfrac{N}{2}, \tfrac{N}{2} \right| \hat{\Psi}^+(x_1)\hat{\Psi}(x_1) \left| \tfrac{N}{2}, \tfrac{N}{2} \right\rangle$$

$$\vdots$$

annihilation of atoms

$$P(x_{k+1} \mid x_k \ldots x_1) \propto$$

$$\left[\left\langle \tfrac{N}{2}, \tfrac{N}{2} \right| \hat{\Psi}^+(x_1)\ldots\hat{\Psi}^+(x_k)\right] \hat{\Psi}^+(x_{k+1})\hat{\Psi}(x_{k+1}) \left[\hat{\Psi}(x_k)\ldots\hat{\Psi}(x_1)\left| \tfrac{N}{2}, \tfrac{N}{2} \right\rangle\right]$$

Condensate interference

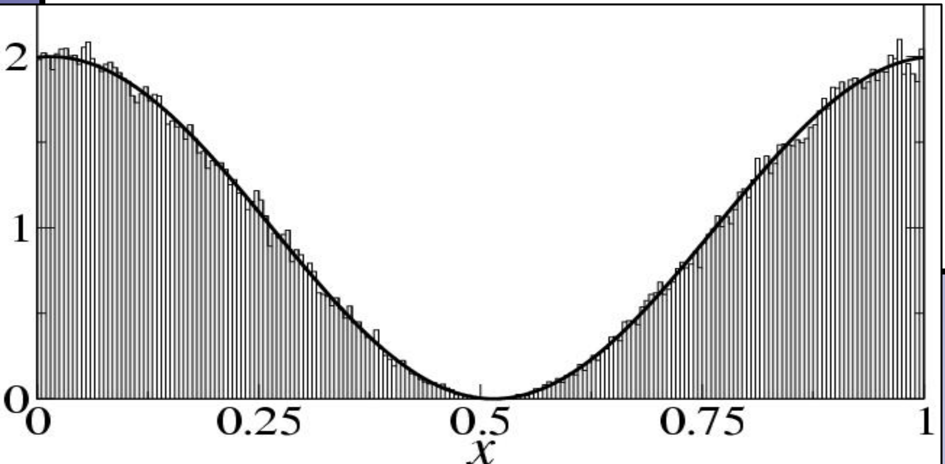
$$\left|\tfrac{N}{2},\tfrac{N}{2}\right\rangle \propto \int_0^{2\pi} d\varphi \, \frac{1}{\sqrt{2\pi}} \left| \mathrm{N}: \; e^{+i\frac{\varphi}{2}} e^{i\pi x} + e^{-i\frac{\varphi}{2}} e^{-i\pi x} \right\rangle \propto$$

$$\int_0^{2\pi} d\varphi \, \frac{1}{\sqrt{2\pi}} \left| \mathrm{N}: \; \cos(\pi x + \tfrac{\varphi}{2}) \right\rangle$$

$$\hat{\Psi}(x_k)\ldots\hat{\Psi}(x_1) \left|\tfrac{N}{2},\tfrac{N}{2}\right\rangle = \int_0^{2\pi} d\varphi \;\; \psi(\varphi,\varphi_0) \left| \mathrm{N}: \cos(\pi x + \tfrac{\varphi}{2}) \right\rangle$$

$$\varphi \approx \varphi_0(x_1,\ldots,x_k)$$

$$\sigma_\varphi \propto \frac{1}{\sqrt{k}}$$



Phase ``collapse'' by density measurement

Quantum Bogoliubov theory

Hamiltonian

$$\hat{H} = \int \mathrm{dx} \left[\tfrac{1}{2} \partial_x \hat{\Psi}^+ \partial_x \hat{\Psi} + V(x) \hat{\Psi}^+ \hat{\Psi} - \mu \hat{\Psi}^+ \hat{\Psi} + \tfrac{1}{2} g \hat{\Psi}^+ \hat{\Psi}^+ \hat{\Psi} \hat{\Psi} \right]$$

Small fluctuations

$$\hat{\Psi}(x) = \sqrt{N} \phi_0(x) + \hat{\psi}(x)$$

$$\mu \, \phi_0 = \left[- \tfrac{1}{2} \partial_x^2 + V(x) + g \, \phi_0^* \phi_0 \right] \phi_0 \equiv H_{\mathrm{GP}} \, \phi_0$$

$$\hat{H}_2 = \int dx \, \left(\hat{\psi}^+, -\hat{\psi} \right) L \begin{pmatrix} \hat{\psi} \\ \hat{\psi}^+ \end{pmatrix}$$

Expansion

with $$L = \begin{pmatrix} H_{\mathrm{GP}} + g \, \phi_0^* \phi_0 & g \, \phi_0 \phi_0 \\ - g \, \phi_0^* \phi_0^* & - H_{\mathrm{GP}} - g \, \phi_0^* \phi_0 \end{pmatrix}$$

Bogoliubov transformation

$$\hat{\psi}(x) = \sum_{m=1}^{\infty} \hat{b}_m u_m(x) + \hat{b}_m^+ v_m^*(x)$$

$$\omega_m \begin{pmatrix} u_m \\ v_m \end{pmatrix} = L \begin{pmatrix} u_m \\ v_m \end{pmatrix}$$

particles <-> quasiparticles

$$\hat{b}_m = \langle u_m | \hat{\psi} \rangle - \langle v_m | \hat{\psi}^+ \rangle$$

Bogoliubov Hamiltonian

$$\hat{H}_2 = \sum_{m=1}^{\infty} \omega_m \hat{b}_m^+ \hat{b}_m$$

Bogoliubov vacuum

$$\hat{b}_m |0_b\rangle = 0$$

Time-dependent Bogoliubov theory

Condensate in the ground state $= |0_b\rangle$

↓ $V(x,t)$

Condensate in an excited state $= |0_{b(t)}\rangle$

$$\hat{\Psi}(x) = \sqrt{N}\phi_0(x) + \hat{\psi}(x)$$

$$i\partial_t\,\phi_0 = \left[-\tfrac{1}{2}\partial_x^2 - \mu + g\phi_0^*\phi_0 + V(x,t)\right]\phi_0$$

Time-dependent Bogoliubov theory

$$\hat{\psi} = \sum_{m=1}^{\infty} \hat{b}_m \, u_m(t) + \hat{b}_m^{+} \, v_m^{*}(t)$$

$$|0_b\rangle \xrightarrow{t} |0_b\rangle$$

Ansatz

Excited state =
t-dependent vacuum

$$i\partial_t \begin{pmatrix} u_m \\ v_m \end{pmatrix} = L(t) \begin{pmatrix} u_m \\ v_m \end{pmatrix}$$

Solution

$$\hat{b}_m(t) = \langle u_m(t) | \hat{\psi} \rangle - \langle v_m(t) | \hat{\psi}^{+} \rangle$$

$$\hat{b}_m(t) \, |0_{b(t)}\rangle = 0$$

N-conserving theory

Girardeau & Arnowitt, PR 113, 755 (1959)
Castin & Dum, PRA 57, 3008 (1998)

$$\hat{b}_m = \left\langle u_m(t) \middle| \hat{\psi} \right\rangle - \left\langle v_m(t) \middle| \hat{\psi}^+ \right\rangle \rightarrow$$

$$\hat{b}_m = \left\langle u_m(t) \middle| \hat{\psi} \right\rangle \frac{\hat{a}_0^+(t)}{\sqrt{N}} - \left\langle v_m(t) \middle| \hat{\psi}^+ \right\rangle \frac{\hat{a}_0(t)}{\sqrt{N}}$$

N-conserving operator

$$\hat{b}_m(t) \left| 0_{b(t)} \right\rangle = 0$$

Vacuum is N-particle state

Diagonal dynamical vacuum

We claim

$$\left|0_{b(t)}\right\rangle \propto \left(a_0^+ a_0^+ + \sum_{\alpha=1}^{\infty} \lambda_\alpha a_\alpha^+ a_\alpha^+ \right)^{\frac{N}{2}} \left|0\right\rangle$$

$\phi_0(t)$ – condensate wave function

$\lambda_\alpha(t) \in [0,1)$

$\phi_\alpha(t)$ – orthonormal basis

D & Sacha, PRA 67, 033608 (2003)
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Proof (constructive)

$$\left\langle 0_{b(t)} \left| \hat{\psi}^{+}(x)\hat{\psi}(y) \right| 0_{b(t)} \right\rangle = \sum_{\alpha=1}^{\infty} dN_{\alpha} \; \Phi_{\alpha}^{*}(t,x) \; \Phi_{\alpha}(t,y)$$

$$\left\langle 0_{b(t)} \left| \hat{\psi}(x)\hat{\psi}(y) \right| 0_{b(t)} \right\rangle =$$

$$\sum_{\alpha=1}^{\infty} e^{2\,i\,\Delta_{\alpha}} \sqrt{dN_{\alpha}(1+dN_{\alpha})} \; \Phi_{\alpha}(t,x) \; \Phi_{\alpha}(t,y)$$

$$\lambda_{\alpha}(t) = \sqrt{\frac{dN_{\alpha}}{1+dN_{\alpha}}} \in [0,1)$$

$$\phi_{\alpha}(t,x) = \Phi_{\alpha}(t,x) \; e^{i\,\Delta_{\alpha}}$$

$$u_m(t) \; v_m(t) \to \lambda_{\alpha}(t) \; \phi_{\alpha}(t)$$

q - representation

$$\left(a_0^+ a_0^+ + \sum_\alpha \lambda_\alpha a_\alpha^+ a_\alpha^+ \right)^{\frac{N}{2}} \left|0\right\rangle \approx$$

$$\int dq \quad \exp\left(-\sum_\alpha \left(\frac{1-\lambda_\alpha}{2\lambda_\alpha}\right) q_\alpha^2 \right) \quad \left| N : \phi_0 + \frac{1}{\sqrt{N}} \sum_\alpha q_\alpha \phi_\alpha \right\rangle$$

Real coordinates q

q - representation

$$\int dq \quad \exp\left(-\sum_\alpha \left(\frac{1-\lambda_\alpha}{2\lambda_\alpha}\right) q_\alpha^2 \right) \left| N : \phi_0 + \frac{1}{\sqrt{N}} \sum_\alpha q_\alpha \phi_\alpha \right\rangle$$

$$\text{For} \quad dN_\alpha = \frac{\lambda_\alpha^2}{1-\lambda_\alpha^2} >> 1$$

$$\langle q | q' \rangle \; = \; e^{-\frac{1}{2}\sum_\alpha (q_\alpha - q'_\alpha)^2} \approx \; \delta(q - q')$$

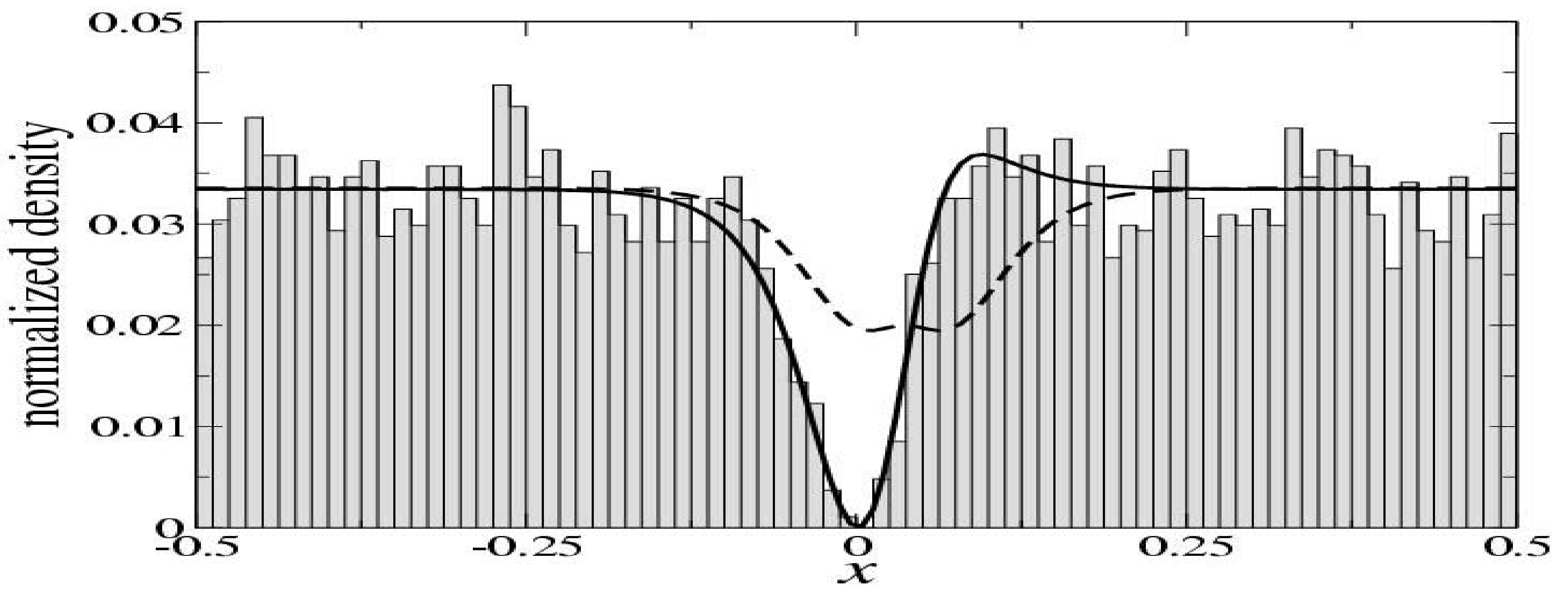
$$P(q) \; \approx \; e^{-\sum_\alpha \left(\frac{1-\lambda_\alpha}{\lambda_\alpha}\right) q_\alpha^2} \; \approx \; e^{-\frac{1}{2}\sum_\alpha \frac{q_\alpha^2}{dN_\alpha}}$$

Probability for q

Density measurement

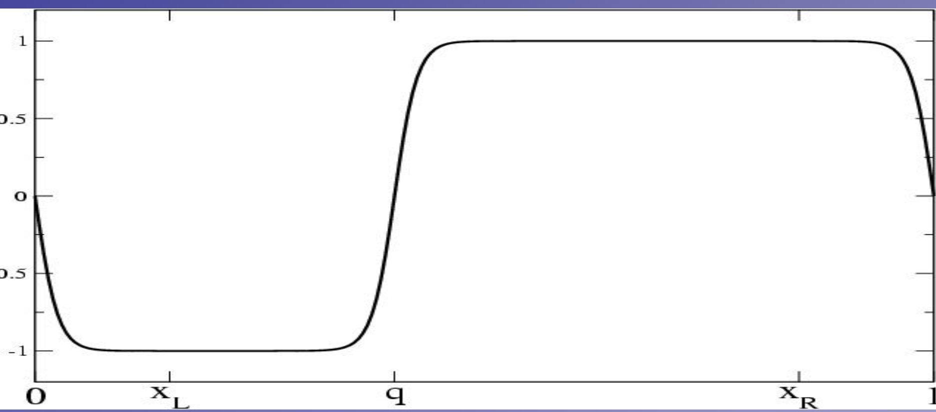
$$P(q) \propto \prod_{\alpha} \exp\left(-\frac{1}{2}\frac{q_{\alpha}^{2}}{dN_{\alpha}}\right) \rightarrow$$

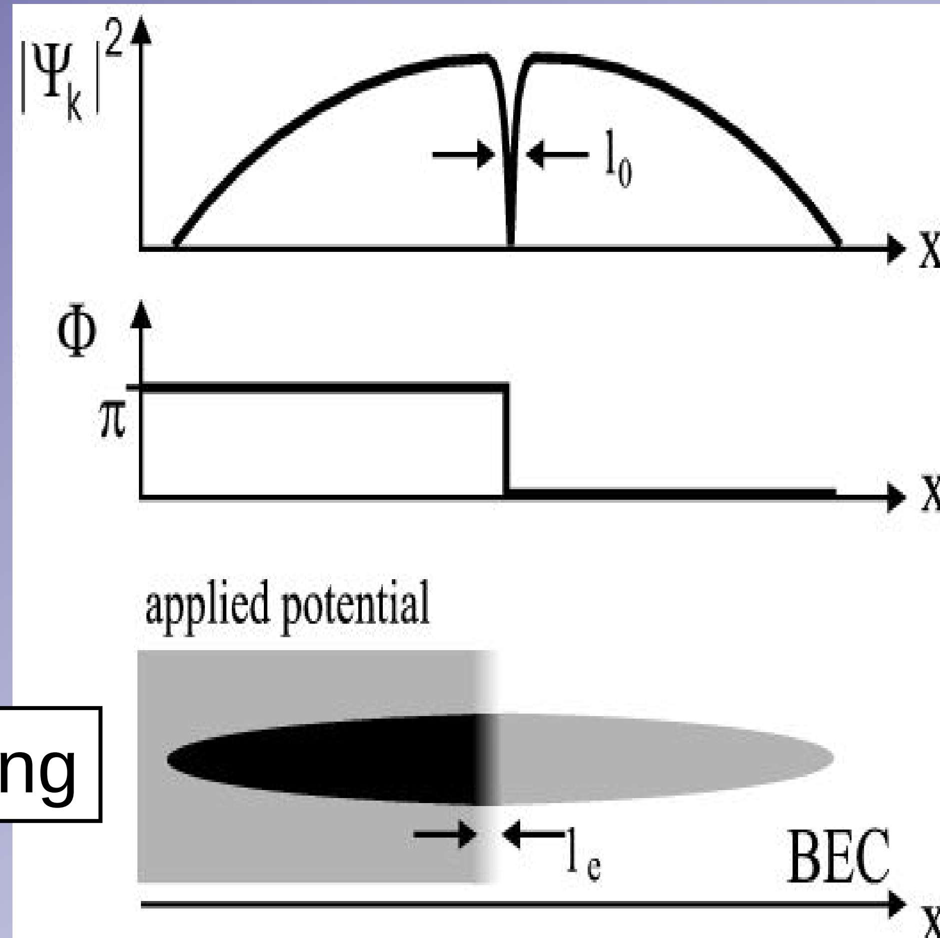
$$\rho(x\,|\,q) = N\left|\phi_0(x) + \frac{1}{\sqrt{N}}\sum_{\alpha} q_{\alpha}\phi_{\alpha}(x)\right|^2$$



Dark soliton

$$i\partial_t \phi_0 = -\tfrac{1}{2}\partial_x^2 \phi_0 + \tfrac{1}{2}x^2\phi_0 + g|\phi_0|^2\phi_0 + V(t,x)\phi_0$$



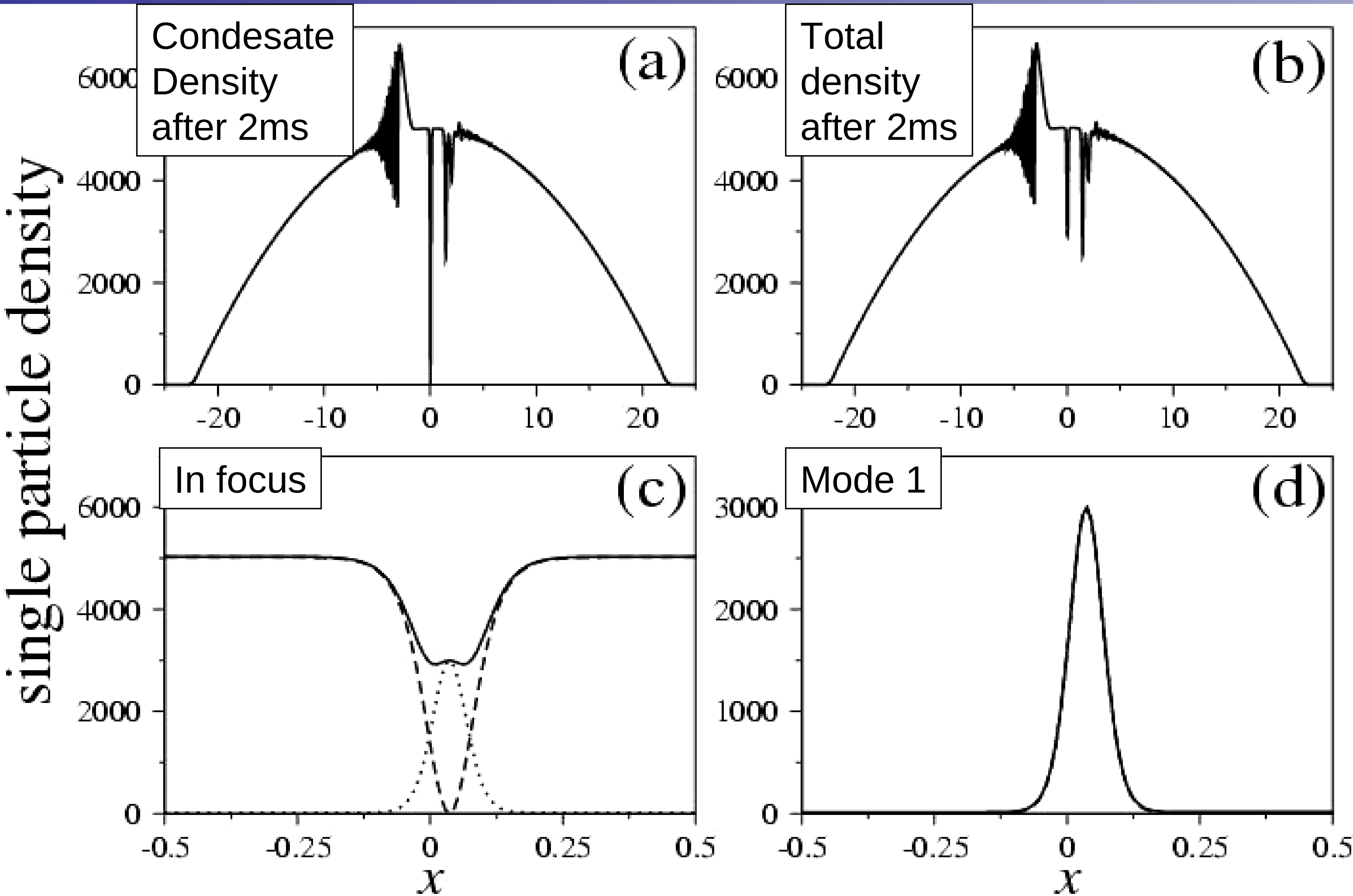


Phase imprinting

Organizers

Burger et al., PRL 83, 5198 (1999)

Phase imprinting in Bogoliubov theory

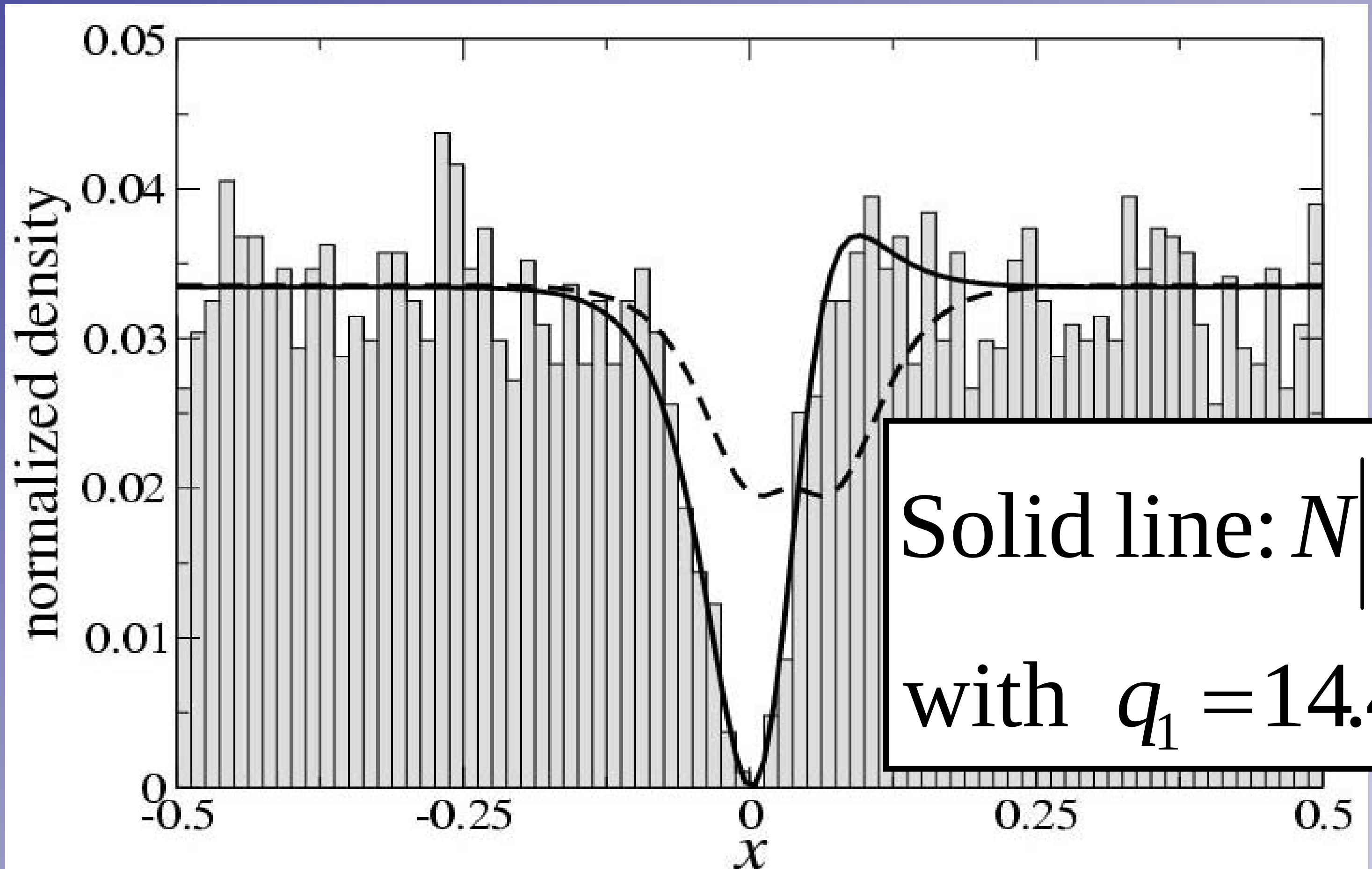


Density measurement

$$\left|0_b\right\rangle \propto \left(a_0^+ a_0^+ + \lambda_1 a_1^+ a_1^+\right)^{\frac{N}{2}} \left|0\right\rangle$$

Truncated vacuum

$$N = 1.5\times 10^5 \qquad dN_1 \approx 273$$



Solid line: $N\left|\phi_0 + \frac{q_1}{\sqrt{N}}\phi_1\right|^2$

with $q_1 = 14.48$

Conclusion

$$\left|0_b\right\rangle \propto \left(a_0^+ a_0^+ + \sum_{\alpha} \lambda_\alpha a_\alpha^+ a_\alpha^+ \right)^{\frac{N}{2}} \left|0\right\rangle$$

Condensate in thermal state

$V(x,t)$

Condensate in ``excited thermal state''

$$\widehat{\rho} = \int d^2 b_L d^2 b_R \quad e^{-\frac{1}{2} b_L^* b_L - \frac{1}{2} b_R^* b_R + b_L^* e^{-\beta\omega} b_R} \left| N : b_L \right\rangle \left\langle N : b_R \right|$$

$$\left. \begin{array}{l} \mathrm{Tr}\ \widehat{\rho}\ \widehat{\psi}^+(x)\widehat{\psi}(y) \\ \mathrm{Tr}\ \widehat{\rho}\ \ \widehat{\psi}(x)\widehat{\psi}(y) \end{array} \right\} \Rightarrow \phi(x|q) = \sqrt{\mathrm{N}}\,\phi_0(t,x) + \sum_{\alpha=1}^{\infty} q_\alpha \phi_\alpha(t,x)$$

$\phi_\alpha(t,x)$ are NOT orthogonal